AKCE Int. J. Graphs Comb., 10, No. 1 (2013), pp. 1-14

HISTs of triangulations on surfaces

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Communicated by: Wayne Goddard
Received 17 May 2012; accepted 30 September 2012

Abstract

A spanning tree with no vertices of degree two of a graph is called a *homeomorphically irreducible spanning tree* (or a *HIST*) of the graph. In [4], Ellingham has proposed a conjecture that every triangulation on a closed surface with sufficiently large representativity has a HIST. In this paper, we solve Ellingham's conjecture. Moreover we prove that every triangulation on the Möbius band or the Klein bottle has a HIST.

Keywords: homeomorphically irreducible spanning tree(HIST), triangulation, surface.

2010 Mathematics Subject Classification: 05C10.

1. Introduction

Let G be a graph and let H be a subgraph of G. If H contains all vertices of G, then it is called a *spanning subgraph* of G. (If a spanning subgraph H of G is a tree, then it is called a *spanning tree* of G.) A number of fundamental problems in graph theory ask whether a graph has a particular type of spanning subgraph. For example, in the Hamiltonian path problem, we seek a spanning tree with all but two vertices of degree two. In this paper, we search "homeomorphically irreducible spanning trees", a class antithetical to Hamiltonian paths.

A graph is said to be *homeomorphically irreducible* if it has no vertices of degree two. Let T be a spanning tree of a graph G. If T has no vertices of degree two, then T is called a *homeomorphically irreducible spanning tree* (or a *HIST*) of G. For example, the octahedral graph has a HIST with two vertices of degree three and four vertices of degree one. See Figure 1. Joffe has constructed infinite families of 4-regular, 3-connected planar graphs that have no HISTs [8]. Albertson, Berman, Hutchinson, and Thomassen have shown that it is NP-complete to decide whether a graph contains a HIST [1].

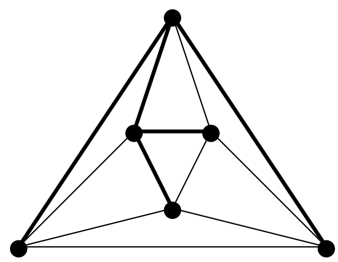

Figure 1: A HIST on the octahedral graph.

A *map* means a fixed embedding of a simple graph on a surface. A *triangulation* on a surface is a map on the surface with each face triangular. A *near triangulation* R is a 2-connected map on the plane with boundary cycle of length at least three such that each face of R is triangular other than the outer face. Hill conjectured that every planar triangulation other than the complete graph K_3 with three vertices has a HIST [7]. Malkevitch extended Hill's conjecture to near triangulations [9]. For their conjectures, Albertson, Berman, Hutchinson, and Thomassen have proved the following.

Theorem 1.1. [1] *Every near triangulation with at least four vertices has a HIST.*

Moreover, they extended Hill's conjecture to all triangulations on any surface.

Conjecture 1.2. [1] *Every triangulation on any surface with at least four vertices has a HIST.*

A *triangulation on the annulus* is a 2-connected map on the plane with two distinguished faces bounded by disjoint cycles C_1 and C_2 such that the length of C_1 (resp., C_2) is at least three and that all other faces are triangular. For triangulations on the annulus, Davidow, Hutchinson and Huneke have proved the following.

Theorem 1.3. [3] *Every triangulation on the annulus has a HIST.*

They also proved that every triangulation on the torus has a HIST [3]. In [6], Fiedler, Huneke, Richter and Robertson have proved that every triangulation on the projective plane has a near triangulation as a spanning subgraph. By their result and Theorem 1.1, it has already been proved that every triangulation on the projective plane has a HIST. Following these results, we consider HISTs of triangulations on surfaces in this paper. For a simple notation, we call a triangulation on the Möbius band a *Möbius triangulation*, throughout the paper. In this paper, we prove the following theorems.

Theorem 1.4. *Every Möbius triangulation has a HIST.*

Theorem 1.5. *Every triangulation on the Klein bottle has a HIST.*

Let G be a triangulation on the projective plane. Then, a triangulation obtained from G by removing the interior of a face of G is a Möbius triangulation. Therefore, by Theorem 1.4, we have an alternative proof of the following.

Corollary 1.6. *Every triangulation on the projective plane has a HIST.*

Let G be a map on a surface F^2 other than the sphere. A simple closed curve l on F^2 is said to be *essential* if l bounds no 2-cell on F^2. The *representativity* of G, denoted by $r(G)$, is the minimum number of intersecting points of G and l, where l ranges over all essential simple closed curves on F^2. We say that G is *r-representative* if $r(G) \geq r$.

In [4], Ellingham has conjectured that every triangulation on a closed surface with sufficiently large representativity has a HIST, which is weaker than Conjecture 1.2. (In [4], he also claimed that Theorem 1.5 has already been proved. However, we confirmed that the proof by Ellingham requires the representativity at least four [5], and hence, his proof is wrong.) The following is our main result, which gives an answer for Ellingham's conjecture and a partial answer for Conjecture 1.2.

Theorem 1.7. *For any closed surface F^2 other than the sphere, there exists a positive integer $N(F^2)$ such that every $N(F^2)$-representative triangulation on F^2 has a HIST.*

In the proof of Theorem 1.7, a key notion is a *k-holed triangulation*, which is a 2-connected map on the plane with k distinguished faces bounded by pairwise disjoint cycles $C_1, \cdots, C_k$ such that the length of C_i is at least three for $i = 1, \cdots, k$ and that all other faces are triangular. In a k-holed triangulation G, each C_i is called a *boundary cycle* of G, and each face bounded by C_i is called a *hole* of G. Remark that we do not regard holes of G as faces of G throughout the paper. By using "planarizing cycles" ideas [11, 12], we can prove that "Every triangulation G on a closed surface has a k-holed triangulation for some k as a spanning subgraph if $r(G)$ is sufficiently large". Since a near triangulation (resp., a triangulation on the annulus) is a 1-holed triangulation (resp., a 2-holed triangulation), it has been proved that every k-holed triangulation G has a HIST when $k = 1, 2$ by Theorems 1.1 and 1.3. (Actually, when $k = 1$, we must assume that G has at least four vertices.) So, we consider whether a k-holed triangulation has a HIST when $k \geq 3$.

Let G be a k-holed triangulation and let $C_1, \cdots, C_k$ be distinct k boundary cycles of G. Let C be a cycle of G. If both regions separated by C contain at least one hole of G, then C is called an *essential cycle* of G.

Theorem 1.8. *Let G be a k-holed triangulation with at least four vertices, where k is a positive integer. If G has no essential cycles whose length is less than 7, then G has a HIST.*

In [1, 3], Theorems 1.1 and 1.3 are proved by induction of the number of edges. In order to prove Theorem 1.8, we use induction of the number of holes on k-holed triangulations in addition to the previous methods. In Section 2, we prove some results containing the essence in the proof of Theorems 1.1 and 1.3. In Section 3, we deal with Möbius triangulations and triangulations on the Klein bottle, i.e., we prove Theorems 1.4 and 1.5. In Section 4, we deal with k-holed triangulations, i.e., we prove Theorem 1.8. In Section 5, we prove our main result (Theorem 1.7).

2. Minimality of k-holed triangulations

In this section, we prove a lemma and a proposition to prove our results. Let G be a k-holed triangulation with distinct boundary cycles $C_1, \cdots, C_k$. If a face f of G has at least one edge contained in C_i, where $1 \leq i \leq k$, then it is called a *face on* C_i. In k-holed triangulations, there exist four kinds of faces on each boundary cycle of G. Let $f = xyz$ be a face on C_i such that xy is contained in C_i. If z is not contained in any boundary cycle of G, then f is called a *trivial face*. If z is contained in C_i and xz (or yz) is also contained in C_i, then f is called a *leaf face*. Suppose that z is contained in a boundary cycle of G but f is not a leaf face. Then there are two possibilities, one is that a graph obtained from G by removing x, y and z is connected, the other is that a graph obtained from G by removing x, y and z is not connected. The former case, f is called a *crossing face*, and the latter case, f is called a *disconnecting face*. A vertex v of a crossing face f of G is called a *head* if edges of f incident to v are not contained in any boundary cycle of G. Moreover, let f be a disconnecting face on C_i with boundary cycle C. We obtain two cycles D and D' with exactly one common vertex v from the symmetric difference between edges of C and C_i. Let G_1 and G_2 be a l-holed triangulation and m-holed triangulation bounded by D and D', respectively, such that the common vertex of G_1 and G_2 is only v, and that $l + m = k + 1$. Then, we call each of G_1 and G_2 a *separated triangulation* of f (note that a separated triangulation of f does not contain the other). Moreover, if we take the *boundary cycle* C of a Möbius triangulation so that C is the boundary cycle of the Möbius band, then we also define these four kinds of faces in Möbius triangulations. For faces on C_i, we consider three kinds of transformations, as follows. Let xyz be a trivial face such that the edge xy is contained in C_i. We call removing xy an *edge deletion* of xy. Let f_1 and f_2 be two adjacent leaf faces and let x, y be vertices of f_1, f_2 with degree two, respectively. We call removing x and y a (2,2)-*deletion* of x, y. On the other hand, let $f_1 = xyz$ be a leaf face, where the degree of y is two and let $f_2 = xzy'$ be a disconnecting face such that xy' is contained in C_i. We call removing two vertices x and y a (2,3)-*deletion* of x, y. We also define these three kinds of deletions in Möbius triangulations. For these deletions, we prove the following.

Lemma 2.1. *Let G be a k-holed triangulation or a Möbius triangulation and let G' be a k-holed triangulation or a Möbius triangulation obtained from G by a sequence of edge deletions,* $(2,2)$*-deletions or* $(2,3)$*-deletions. If G' has a HIST, then G has a HIST.*

Proof. We may suppose that G' is obtained from G by one operation of an edge deletion, a $(2,2)$-deletion or a $(2,3)$-deletion. For each operation, we should prove that if G' has a HIST, then G has a HIST. If G' is obtained from G by an edge deletion, then, by the assumption of the lemma, G' has a HIST which is required one in G. If G' is obtained from G by a (2,2)-deletion or a (2,3)-deletion of x, y, then let z be a common neighbor of x and y. By the assumption of the lemma, we can find a HIST H' of G'. Moreover, $H' \cup \{xz, yz\}$ is a HIST of G. □

A triangulation G on a surface is said to be *minimal*, if G has no faces such that we can apply edge deletions, $(2,2)$-deletions or $(2,3)$-deletions. By Lemma 2.1, it suffices to show that minimal triangulations on a surface have a HIST in order to prove our results. For the minimality of G, we prove the following.

Proposition 2.2. *A minimal k-holed triangulation and a minimal Möbius triangulation have neither trivial faces, two adjacent leaf faces nor a disconnecting face such that at least one separated triangulation of it is a near triangulation.*

Proof. Let G be a minimal triangulation on a surface. (When G is a minimal Möbius triangulation, we can prove the proposition by the same arguments.) It is obvious that G has neither trivial faces nor two adjacent leaf faces. (Otherwise, we can apply an edge deletion or a $(2,2)$-deletion to G.) So, we should prove that G has no disconnecting face such that at least one separated triangulation of it is a near triangulation. For a contradiction, suppose that G has a disconnecting face f whose separated triangulation R is a near triangulation and that f is chosen such that R has as few faces as possible. Note that there are no crossing faces in R since f is an obstruction. By the minimality of G, neither trivial faces nor two adjacent leaf faces are contained in R. Moreover, by the minimality of R, there are no disconnecting faces in R. Therefore, in R, there is exactly one leaf face of G, and hence, we can apply a $(2,3)$-deletion to G, a contradiction. □

3. Cases for Möbius band and Klein bottle

In this section, we prove Theorems 1.4 and 1.5. First, we prove Theorem 1.4.

Proof of Theorem 1.4. Let M be a Möbius triangulation. By Lemma 2.1, we may suppose that M is minimal and hence M has neither trivial faces, two adjacent leaf faces nor disconnecting faces by Proposition 2.2. This implies that each face on the boundary cycle of M is a crossing face or a leaf face. So, M has a crossing face f_1 with the head v_1. Since M has no two adjacent leaf faces, M has a crossing face f_2 with the head v_2 such that v_1 is contained in f_2 but v_1 is not v_2. Let R be a near triangulation or an edge between f_1 and f_2 containing the edge v_1v_2, where the interior of R contains neither f_1 nor f_2. We choose f_1 so that R has as few faces as possible, and hence R has no crossing faces. Let u_1 (resp., u_2) be the vertex of f_1 (resp., f_2) which is not a head and not contained in R. Moreover, let w_1 be the vertex of f_1 which is neither v_1 nor u_1.

If R is an edge (i.e., $v_2 = w_1$), let M' be a near triangulation with at least four vertices obtained from M by removing v_1v_2, v_1u_2 and v_2u_1 (note that v_1, v_2, u_1 and u_2 are distinct vertices, otherwise M has a multiple edge).

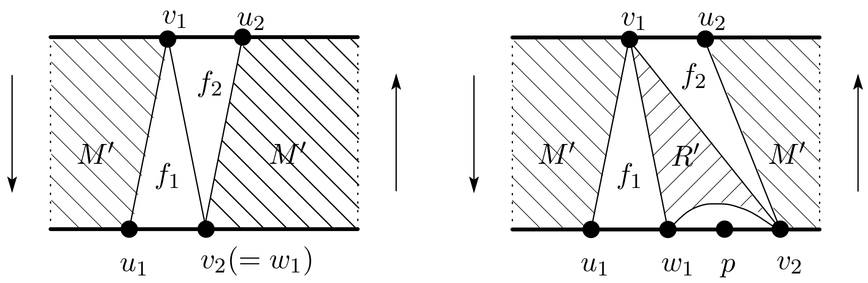

Figure 2: Two faces f_1 and f_2 of M.

See the left of Figure 2. By Theorem 1.1, M' has a HIST which is a required one in M.

Otherwise (i.e., R is a near triangulation), let M' be a near triangulation with at least four vertices obtained from M by removing v_1u_2, v_1v_2 and all vertices and edges of R other than v_1, v_2. By Theorem 1.1, M' has a HIST H'. So, we prove that we obtain a HIST of M by adding suitable edges of R to H'. By Proposition 2.2 and the minimality of M and R, R has a leaf face f containing v_2 and w_1. Let p be the vertex of f whose degree is two and let R' be the near triangulation bounded by $v_1w_1v_2$. See the right of Figure 2. Let R'' be the map obtained from R' by removing v_1. If R'' is a near triangulation with at least four vertices, then, by Theorem 1.1, we can find a HIST H'' of R''. Moreover, $H' \cup H'' \cup pv_2$ is a HIST of M. If we cannot apply Theorem 1.1 to R'', then R'' is an edge w_1v_2 or a near triangulation with three vertices w_1, v_2, r, where r is a vertex of R' adjacent to w_1, v_2 and v_1. In this case, $H' \cup \{pv_2, w_1v_2\}$ $(\cup\{rv_2\})$ is a HIST of M. □

Next, we prove Theorem 1.5 by using Theorem 1.4. In order to do so, we prove the following.

Lemma 3.1. *Every triangulation on the Klein bottle has a Möbius triangulation as a spanning subgraph.*

Proof. Let G be a triangulation on the Klein bottle. The theorem proved by Sulanke [10] guarantees that G has a cycle separating G into two Möbius triangulations. Let C be a cycle separating G into two Möbius triangulations M_1 and M_2 such that M_1 contains as many vertices as possible. Then, we prove that M_1 is a spanning subgraph of G.

If G has a vertex v not in M_1, then there are three paths (in G therefore in M_2) from v to vertices on C such that they are disjoint other than v since G is 3-connected. As observed in [6], some pair of these paths would allow us to find a larger Möbius triangulation containing M_1, a contradiction. □

Proof of Theorem 1.5. Let G be a triangulation on the Klein bottle. By Lemma 3.1, G has a Möbius triangulation M as a spanning subgraph. By Theorem 1.4, M has a HIST, which is a required one in G. □

4. Cases for k-holed triangulations

In this section, we prove Theorem 1.8. In order to do so, we give some definitions, and prove a lemma and a proposition. Let G be a k-holed triangulation and let f_1 and f_2 be two distinct crossing faces of G with distinct heads. Suppose that the map between f_1 and f_2 is a near triangulation or an edge. Then, $f_1 \cup f_2$ is called a *good structure* of G. Moreover, we call a near triangulation (or an edge) between f_1 and f_2 an *interval map* of $f_1 \cup f_2$, where the interval map contains neither f_1 nor f_2. Let v_1 and v_2 be heads of f_1 and f_2, respectively. When v_1 and v_2 are contained in different boundary cycles of G, there are three possibilities of good structures depending on the number of common vertices of f_1 and f_2. See (a), (b), (c) of Figure 3. (In Figure 3, painted regions denote holes of G, heavy lines denote boundary cycles of G, and shaded regions denote near triangulations between f_1 and f_2.) When v_1 and v_2 are contained in the same boundary cycle of G, there are two possibilities of good structures depending on whether f_1 and f_2 have a common vertex. See (d), (e) of Figure 3. So, there are five possibilities of good structures. If the union of f_1, f_2 and the interval map of $f_1 \cup f_2$ contains no good structures other than $f_1 \cup f_2$, then we call that $f_1 \cup f_2$ is a *minimal* good structure. Note that if a k-holed triangulation has a good structure, then there exists the minimal one in the union of the good structure and the interval map of it.

Suppose that $f_1 \cup f_2$ is a minimal good structure of G. Let R be the interval map of $f_1 \cup f_2$. Let v_1 and v_2 be heads of f_1 and f_2, respectively, and let u_1 (resp., u_2) be a vertex of f_1 (resp., f_2) which is not a head and not contained in R. Moreover, let w_1 (resp., w_2) be the vertex of f_1 (resp., f_2) which is neither v_1 (resp., v_2) nor u_1 (resp., u_2). We obtain a $(k-1)$-holed triangulation G' from G by removing edges u_1w_1, u_2w_2 and all vertices and edges of R other than v_1 and v_2. Then, G' is called a *good* $(k-1)$*-holed triangulation* obtained from G. Note that, by the definition of G', if G has no essential cycles of length less than l, then G' also has no essential cycles of length less than l. For good structures, we prove the following.

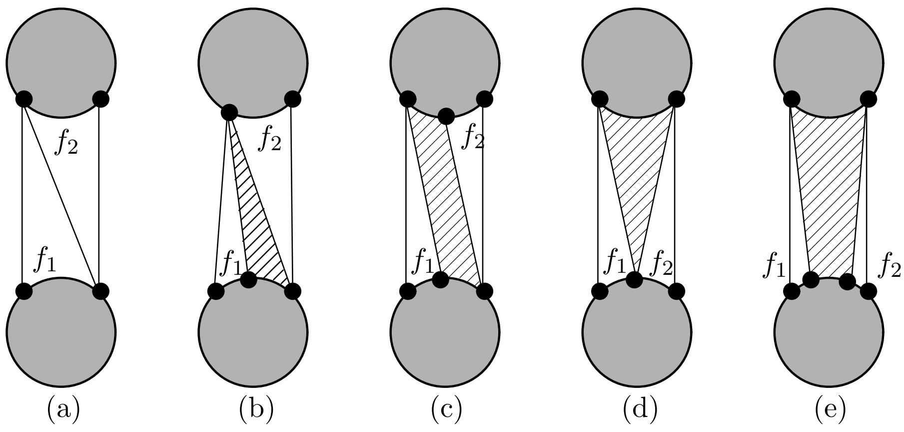

Figure 3: Five good structures of G.

Lemma 4.1. *Let $\mathcal{G}$ be a class of triangulations with holes that is closed under edge deletions,* (2,2)*-deletions,* (2,3)*-deletions, and under the removal of interval maps R of minimal good structures $f_1 \cup f_2$, as described above. Let G be a k-holed triangulation in $\mathcal{G}$. If every l-holed triangulation in $\mathcal{G}$ such that $l < k$ has a HIST, then G has a HIST.*

Proof. Note that each deletion (i.e., an edge deletion, a (2,2)-deletion and a (2,3)-deletion) does not delete crossing faces, that is, they break no good structure of G. Therefore, we may suppose that G is minimal by Lemma 2.1. Let $f_1 = v_1u_1w_1$ and $f_2 = v_2u_2w_2$ be crossing faces of a minimal good structure of G and let R be the interval map of $f_1 \cup f_2$. (We suppose that each of v_1, v_2 is a head and each of w_1, w_2 is contained in R.) Let G' be the good $(k-1)$-holed triangulation obtained from G by removing edges u_1w_1, u_2w_2 and all vertices and edges of R other than v_1 and v_2. By the assumption of the lemma, G' has a HIST H'. If R is an edge (i.e., the case (a) of Figure 3), H' is also a HIST of G. So, we prove that when R is a near triangulation, we obtain a HIST of G by adding suitable edges of R to H'.

First, we consider the case when v_1, v_2 are contained in different boundary cycles of G and f_1, f_2 have exactly one common vertex (i.e., the case (b) of Figure 3). We may suppose that $v_1 = w_2$. By the minimality of G and $f_1 \cup f_2$, we can use the same arguments in the proof of Theorem 1.4, and hence, we can add suitable edges of R to H'. Therefore, G has a HIST.

Secondly, we consider the case when v_1, v_2 are contained in different boundary cycles of G and f_1, f_2 have no common vertices (i.e., the case (c) of Figure 3). By Proposition 2.2 and the minimality of G and $f_1 \cup f_2$, there are two leaf faces f_a and f_b in R such that both w_1 (resp., w_2) and v_2 (resp., v_1) are contained in f_a (resp., f_b). Let p (resp., q) be the vertex of f_a (resp., f_b) whose degree is two. Let R' be the near triangulation bounded by $v_1w_1v_2w_2$. See the left of Figure 4. If R' has an edge v_1v_2, then let R'_1 and R'_2 be near triangulations bounded by $v_1w_1v_2$ and $v_1v_2w_2$, respectively. See the right of Figure 4. Let R''_1 be the map obtained from R'_1 by removing v_1. If R''_1 is a near triangulation with at least four vertices, then by Theorem 1.1, we can find a HIST H''_1 of R''_1 and we can add H''_1 to H' by using pv_2. Even if we cannot apply Theorem 1.1 (i.e., the underlying graph of R''_1 is K_2 or K_3), we can add suitable edges of R''_1 and pv_2 to H'. Similarly, for the map R''_2 obtained from R'_2 by removing v_2, we can add a HIST or suitable edges of R''_2 to H' by using the edge qv_1. Therefore, G has a HIST. Moreover, when R' has an edge w_1w_2, we can prove that G has a HIST by arguments similar to the case when R' has an edge v_1v_2.

So, we suppose that R' has neither v_1v_2 nor w_1w_2. In this case, let $N(v_1)$ be a set of vertices adjacent to v_1 in R'. Let R'' be the near triangulation obtained from R' by removing v_1. See the left of Figure 5. We prove that there exists a vertex $x \in N(v_1)$ such that the map obtained from R'' by removing x is a near triangulation. If R'' is 3-connected, then we can chose any vertex of $N(v_1)$ as x. So, we may suppose that R'' is 2-connected, and hence, R'' has an edge yz such that yz separates R'' into two near triangulations S_1, S_2 and that S_1 has as few vertices as possible. See the right of Figure 5. (Although Figure 5 shows the case when $z = v_2$, we can deal with the case when both y and z are

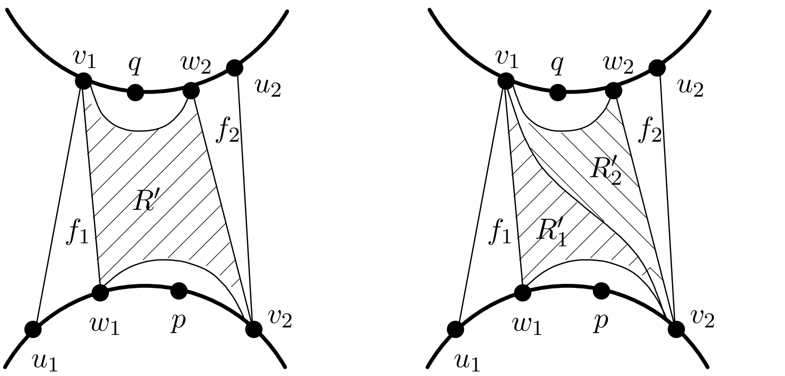

Figure 4: The near triangulation R' in the case (c) (left) and near triangulations R'_1 and R'_2 in the case (c) (right).

contained in $N(v_1)$ by the same argument.) By the minimality of S_1, S_1 is 3-connected (or a near triangulation with 3-vertices). Since R' (therefore R'') has no w_1w_2, S_1 has a vertex of $N(v_1)$ other than y, z. Therefore, we can find x in S_1. Let R''' be a near triangulation obtained from R'' by removing x. If R''' has at least four vertices, then, by Theorem 1.1, R''' has a HIST H''' and $H' \cup H''' \cup \{pv_2, qv_1, xv_1\}$ be a HIST of G. Otherwise (i.e., R''' is a near triangulation with three vertices), we can add suitable edges of R''' to H' by using pv_2. Furthermore, by adding xv_1 and qv_1, we can find a HIST of G.

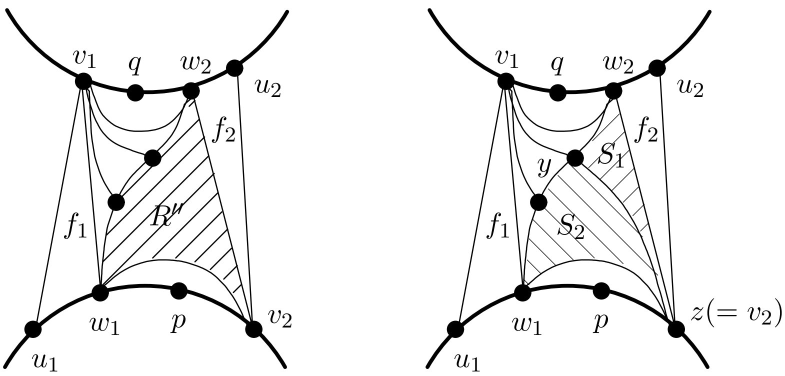

Figure 5: The near triangulation R'' in the case (c) (left) and near triangulations S_1 and S_2 in the case (c) (right).

Finally, we consider the case when v_1, v_2 are contained in the same boundary cycle C_i (i.e., the cases (d) and (e) of Figure 3). By Proposition 2.2 and the minimality of G and $f_1 \cup f_2$, R has exactly one leaf face f on C_i with a vertex p of degree two. (When w_1 and w_2 are distinct vertices, R has a leaf face other than f which contains w_1 and w_2.) Let R' be the near triangulation obtained from R by removing p. See Figure 6. Let R'' be the map obtained from R' by removing v_1. (If w_1 and w_2 are distinct vertices and R has an edge v_1w_2, we suppose that R'' be a map obtained from R' by removing v_2.) Note that R'' is a near triangulation or an edge. By above arguments, we can add a HIST or suitable edges of R'' to H' by using an edge pv_1 or pv_2. Therefore, G has a HIST. □

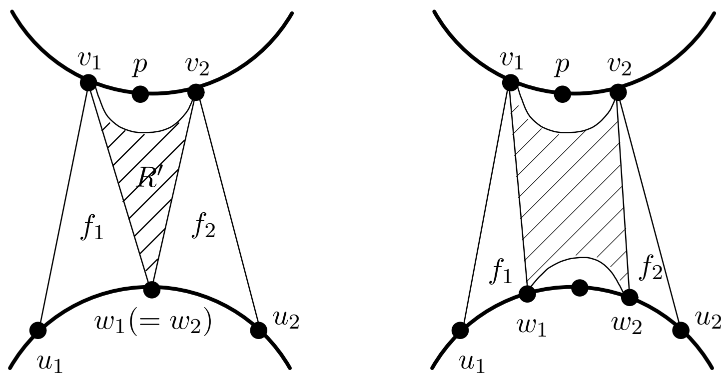

Figure 6: The interval map in the case (d) (left) and the interval map in the case (e) (right).

Let G be a minimal k-holed triangulation with distinct boundary cycles $C_1, \cdots, C_k$. By Proposition 2.2, if G has a disconnecting face f, then each separated triangulation of f contains at least one hole. Let f and f' be distinct two disconnecting faces on C_i. If f has a separated triangulation R such that the holes contained in R are the same as those in a separated triangulation of f', then f and f' are said to be *equivalent*. Otherwise, f and f' are said to be *distinguished*. Similarly, for two crossing faces on C_i, they are said to be *equivalent* if their heads are the same vertex. Otherwise (i.e., their heads are distinct), they are said to be *distinguished*. Note that if f is a disconnecting face on C_i and f' is a crossing face on C_i, then we call that f and f' are not equivalent.

Let $\mathrm{dis}(C_i)$ be the number of pairwise distinguished disconnecting faces on C_i and let $\mathrm{cros}(C_i)$ be the number of pairwise distinguished crossing faces on C_i. Note that $\mathrm{dis}(C_i)$ (resp., $\mathrm{cros}(C_i)$) is equal to the number of equivalence class of disconnecting faces (resp., crossing faces) on C_i.

Proposition 4.2. *Let G be a minimal k-holed triangulation with distinct boundary cycles $C_1, \cdots, C_k$, where $k \geq 2$. If G has no essential cycle of length less than 7, then for any C_i with $1 \leq i \leq k$, we have that* $\mathrm{dis}(C_i) + \mathrm{cros}(C_i) \geq 3$.

Proof. We should prove that, for some i, if $\mathrm{dis}(C_i) + \mathrm{cros}(C_i) < 3$, then G has an essential cycle of length less than 7. By Proposition 2.2, C_i has neither trivial faces nor two adjacent leaf faces. Let $F_1, F_2, \cdots, F_n$ be a partition of the disconnecting faces and the crossing faces on C_i such that any two faces in the same set are equivalent and that any two faces in different sets are not equivalent. Since $\mathrm{dis}(C_i) + \mathrm{cros}(C_i) < 3$, n is at most two.

If each of F_1 and F_2 contains at least two crossing faces and C_i has two leaf faces between F_1 and F_2, there exists an essential cycle C of length 6 (note that this is the worst case). See Figure 7. (In Figure 7, strong lines denote edges on C and painted regions denote holes of G.)

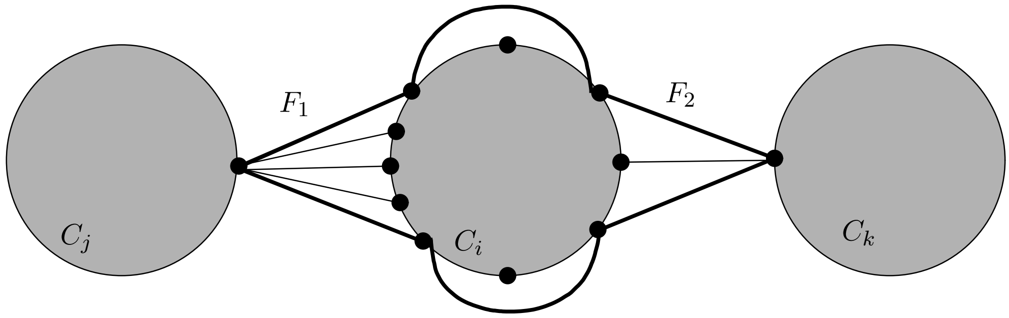

Figure 7: An essential cycle of length 6 around C_i.

It is obvious that if $n \leq 1$, then G has an essential cycle whose length less than 6. If F_1 (or F_2) contains either disconnecting faces or exactly one crossing face, then we can find an essential cycle C' whose length less than 6 since C' pass through exactly one edge of F_1. Moreover, if C_i has no leaf face between F_1 and F_2, we can find an essential cycle whose length less than 6. □

Now we should prove Theorem 1.8.

Proof of Theorem 1.8 (By induction of the number of holes of G.) If $k = 1, 2$, then G has a HIST by Theorems 1.1 and 1.3. So, we suppose that $k \geq 3$. By Lemma 2.1, we may suppose that G is minimal. Note that G has no essential cycles of length less than 7 if G is obtained from a k-holed triangulation with no essential cycles of length less than 7 by a sequence of the three deletions (i.e., an edge deletion, a (2,2)-deletion and a (2,3)-deletion).

First, we construct a plane digraph $D(G)$ from G as follows.

(a) For each hole of G with boundary cycle C_i, we put a vertex v_i in $D(G)$.

(b) For each equivalence class of disconnecting faces on C_i, we put a directed loop incident to v_i in $D(G)$.

(c) For each equivalence class of crossing faces on C_i whose heads are contained in C_j, we put a directed edge from v_i to v_j in $D(G)$.

Note that even if G has many disconnecting faces (resp., crossing faces) in an equivalence class we put exactly one directed loop (resp., directed edge) corresponding to the equivalence class. Moreover, we do not care the direction of the loop. Therefore, in $D(G)$, we position the ends of edges around a vertex so that their order matches that of the corresponding equivalence classes of faces in G.

Since G is a graph on the plane, we can construct $D(G)$ as a plane digraph. Note that, for any vertex v_i of $D(G)$, the out degree of v_i is at least three by Proposition 4.2 and the definition of $D(G)$.

Next, we prove that $D(G)$ has a 2-gon, where an *n-gon* is a face of $D(G)$ bounded by a cycle (not necessarily a directed cycle) of length n. By Proposition 2.2, G has no disconnecting face such that at least one separated triangulation of it is a near triangulation, and hence, $D(G)$ has no 1-gon.

Claim 1. *Every connected component of $D(G)$ obtained from a minimal k-holed triangulation G has no 1-gon.*

Proof. For a contradiction, suppose that $D(G)$ has at least one connected component with a 1-gon, say P. By Proposition 2.2, G has no disconnecting face such that at least one separated triangulation of it is a near triangulation. This implies that, in the interior of the 1-gon, there are some connected components other than the component containing the 1-gon. We choose P so that the interior of P has no 1-gon of any connected component of $D(G)$. Let A be the connected component of $D(G)$ containing P. We prove that at least one component in the interior of P is connected to A and hence, P is not 1-gon of $D(G)$.

Let f be a disconnecting face on C_i of G one of whose separated triangulation is corresponding to P and components in the interior of P in $D(G)$. We choose f so that the separated triangulation R of f contains no crossing face equivalent to f. Let v and u be vertices of f contained in R. Since G has no multiple edges, G has no leaf face containing both v and u. By the assumption of f, there are no disconnecting faces on C_i in R. By Proposition 2.2 and the minimality of G, R has neither trivial faces nor two adjacent leaf faces. So, R has a crossing face on C_i. This implies that, in $D(G)$, at least one connected component in the interior of P is connected to A, contrary to that P is a 1-gon of A.

Let $D'(G)$ be a connected component of $D(G)$. By Claim 1, $D'(G)$ has no 1-gon. This implies that $D'(G)$ has at least two vertices. Let V, E and F be the number of vertices, edges, and faces of $D'(G)$, respectively. Since $D(G)$ is a digraph on the plane, $D'(G)$ is also a digraph on the plane. We prove that $D'(G)$ has at least two 2-gons. For a contradiction, we suppose that $D'(G)$ has at most one 2-gon. By Euler's formula, we have

$$V - E + F = 2 \tag{1}$$

By double counting, we have

$$2E \geq 3(F-1) + 2 \tag{2}$$

From the equation (1) and the inequality (2), we have

$$3V - 5 \geq E \tag{3}$$

However, since the out degree of any vertex of $D(G)$ (therefore $D'(G)$) is at least three, we have

$$3V \leq E \tag{4}$$

It is a contradiction.

So, $D'(G)$ has at least two 2-gons. If the 2-gon of $D'(G)$ is not a 2-gon of $D(G)$, then we can find other connected component $D''(G)$ of $D(G)$ in the 2-gon. Repeating the above arguments, we can find a 2-gon of $D''(G)$ other than the outer face of $D''(G)$. Therefore $D(G)$ has a 2-gon.

Finally, we prove that G has a HIST. By the rule (b), no 2-gon is bounded by two loops in $D(G)$. Moreover, by the rule (c), the edges of a 2-gon in $D(G)$ are not equivalent crossing faces in G. This means that G has two crossing faces with distinct heads such that the map between them is either an edge or a near triangulation, i.e., G has a good structure. So, we can find a minimal good structure in G, and hence, we obtain a good $(k-1)$-holed triangulation G'. By the definition of good $(k-1)$-holed triangulations, G' has no essential cycles of length less than 7. Therefore, by the induction hypothesis, G' has a HIST, and hence, G has a HIST by Lemma 4.1 taking $\mathcal{G}$ to be the class of triangulations with no essential cycle of length less than 7. □

5. Cases for triangulations with large representativity

In this section, we prove our main result (Theorem 1.7) by using Theorem 1.8. In order to do so, we need a lemma connecting triangulations on closed surfaces and k-holed triangulations. By the "planarizing cycles" ideas (see Theorem 3.3 in [11] and Theorem 4.3 in [12]), we obtain the following lemma.

Lemma 5.1. *For any closed surface F^2 other than the sphere, there exists a positive integer $N(F^2)$ such that every $N(F^2)$-representative triangulation on F^2 has a k-holed triangulation for some k with no essential cycle whose length is less than $N(F^2)$ as a spanning subgraph.* □

Now we should prove Theorem 1.7.

Proof of Theorem 1.7. Suppose $r = \max\{N(F_2), 7\}$, where $N(F^2)$ is the number in Lemma 5.1. Let G be an r-representative triangulation on a closed surface F^2. By Lemma 5.1 and the assumption of G, G has a k-holed triangulation G' for some k with no essential cycles of length less than 7 as a spanning subgraph. By Theorem 1.8, G' has a HIST, which is a required one in G. □

Acknowledgement

Recently, Chen, Ren and Shan [2] have solved Conjecture 1.2 independently. They use "locally connected graphs" in order to solve a problem on triangulations on surfaces. On

the other hand, we use very typical method for such kind of problem i.e., to cut open a triangulation into a plane graph and find a HIST. Therefore, our result is meaningful in the point of the proof technique.

We thank Wayne Goddard for a helpful advice to publish this manuscript. Without his help, we could not publish this manuscript.

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