AKCE Int. J. Graphs Comb., 10, No. 1 (2013), pp. 15-28

On the existence of (k, l)-kernels in digraphs with a given circumference

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Communicated by: S. Arumugam
Received 26 February 2011; accepted 29 October 2012

Abstract

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively.

A (k, l)-kernel N of D is a k-independent (if $u, v \in N$ then $d(u, v), d(v, u) \geq k$) and l-absorbent (if $u \in V(D) - N$ then there exists $v \in N$ such that $d(u, v) \leq l$) set of vertices. A k-kernel is a $(k, k-1)$-kernel. For a strong digraph D, a set $S \subset V(D)$ is a separator if $D \setminus S$ is not strong, D is σ-strong if $|V(D)| \geq \sigma + 1$ and has no separator with less than σ vertices. A digraph D is locally in(out)-semicomplete if whenever $(v, u), (w, u) \in A(D)$ ($(u, v), (u, w) \in A(D)$), then $(v, w) \in A(D)$ or $(w, v) \in A(D)$. A digraph D is k-quasi-transitive if the existence of a directed path $(v_0, v_1, \ldots, v_k)$ in D implies that $(v_0, v_k) \in A(D)$ or $(v_k, v_0) \in A(D)$. In a digraph D which has at least one directed cycle, the length of a longest directed cycle is called its circumference.

We propose the following conjecture, if D is a digraph with circumference l, then D has a l-kernel. This conjecture is proved for two families of digraphs and a partial result is obtained for a third family. In this article we prove that if D is a σ-strong digraph with circumference l, then D has a $(k, (l-1) + (l-\sigma) \left\lfloor \frac{k-2}{\sigma} \right\rfloor)$-kernel for every $k \geq 2$. Also, that if D is a locally in/out-semicomplete digraph such that, for a fixed integer $l \geq 1$, $(u, v) \in A(D)$ implies $d(v, u) \leq l$, then D has a (k, l)-kernel for every $k \geq 2$. As a consequence of this theorems we have that every $(l-1)$-strong digraph with circumference l and every locally out-semicomplete digraph with circumference l have an l-kernel, and every locally in-semicomplete digraph with circumference l has an l-solution. Also, we prove that every k-quasi-transitive digraph with circumference $l \leq k$ has an n-kernel for every $n \geq k$.

Keywords: digraph, kernel, (k,l)-kernel, k-kernel, k-kings, locally in-semicomplete, locally out-semicomplete, circumference, strong digraph.

2010 Mathematics Subject Classification: 05C20.

1. Introduction

In this work, $D = (V(D), A(D))$ will denote a finite digraph without loops or multiple arcs in the same direction, with vertex set $V(D)$ and arc set $A(D)$. For general concepts and notation we refer the reader to [1], [4] and [8], particularly we will use the notation of [8] for walks, if $\mathscr{C} = (x_0, x_1, \ldots, x_n)$ is a walk and $i < j$ then $x_i \mathscr{C} x_j$ will denote the subwalk of $\mathscr{C}$ $(x_i, x_{i+1}, \ldots, x_{j-1}, x_j)$. Union of walks will be denoted by concatenation or with $\cup$. We say that a vertex u reaches a vertex v in D if a directed uv-directed path (a path with initial vertex u and terminal vertex v) exists in D. An arc $(u,v) \in A(D)$ is called *asymmetrical* (resp. symmetrical) if $(v,u) \notin A(D)$ (resp. $(v,u) \in A(D)$).

A digraph is *strongly connected* (or strong) if for every $u, v \in V(D)$, there exists a uv-directed path, *i.e.*, a directed path with initial vertex u and terminal vertex v. A *strong component* (or component) of D is a maximal strong subdigraph of D. The *condensation* of D is the digraph $D^\star$ with $V(D^\star)$ equal to the set of all strong components of D, and $(S,T) \in A(D^\star)$ if and only if there is an ST-arc in D. Clearly $D^\star$ is an acyclic digraph (a digraph without directed cycles), and thus, it has both vertices of out-degree equal to zero and vertices of in-degree equal to zero. A *terminal component* of D is a strong component T of D such that $d^+_{D^\star}(T) = 0$. An *initial component* of D is a strong component S of D such that $d^-_{D^\star}(S) = 0$.

In [24], von Neumann and Morgenstern introduce the concept of solution and kernel of a digraph in the context of game theory, the former defined as an independent and dominating subset of $V(D)$ and the latter dually defined to be an independent and absorbing subset of $V(D)$. Since then, a lot of research have been done in the subject, principally because of the Strong Perfect Graph Conjecture proposed by Berge and proved by Chudnovsky, *et al* [7]; the paper [5] is a very nice survey about kernels and its relation to perfect graphs and game theory.

In this paper our main interest is a generalization of the concept of kernel (and dually a solution) of a digraph, the (k,l)-kernel of a digraph, which is a k-independent and l-absorbent subset of $V(D)$. It is clear that under this notion a kernel in the usual sense is a $(2,1)$-kernel, and inspired in the relation between k and l in this usual definition of kernel, the concept of k-kernel is defined to be a $(k, k-1)$-kernel (thus a kernel is a 2-kernel). This generalization was introduced by Kwaśnik and Borowiecki in [18], and studied initially by Kwaśnik [19, 17], and later by Galeana-Sánchez, *et al* [9, 10, 16, 11], and Włoch and Włoch, *et al* [6, 22, 23, 25, 26], principally. Recently some work have been done by Galeana-Sánchez and Hernández-Cruz studying some families of digraphs which have k-kernel for given values of k or sufficient conditions for the existence of k-kernels in digraphs [12, 13, 14].

In [21], Richardson proved that every digraph without odd cycles has a 2-kernel, as a particular case of this result we can observe that if a digraph D has circumference 2, then D has a 2-kernel. As a matter of fact, it is easy to prove that if a digraph D has circumference 2, then D has a k-kernel for every $k \geq 2$. A short proof of this fact is given in the following section. From this, we conjecture that if D is a digraph with circumference

l, then D has a l-kernel. Also, a stronger version of this conjecture is proposed, if D is a digraph with circumference l, then D has a k-kernel for every $k \geq l$. As we noted before, the strong version of the conjecture is true for $l = 2$. The aim of this work is to prove this conjecture true for some families of digraphs.

In Section 2, we introduce a new definition of (k,l)-semikernel of a digraph as a set S, k-independent and such that, for every vertex $v \in V(D) \setminus S$, $d(S,v) \leq k-1$ implies $d(v,S) \leq l$, improving the existing definition in the sense that, analogous to classical results concerning semikernels could be obtained. We use this new definition to work with specific families of digraphs, namely σ-strong digraphs and locally in/out-semicomplete digraphs (defined in the abstract).

In Section 3, we prove that if D is a σ-strong digraph with circumference l then $\sigma \leq l-1$ and D has a $(k, (l-1) + (l-\sigma)\left\lfloor \frac{k-2}{\sigma} \right\rfloor)$-kernel for every integer $k \geq 2$. In Section 4 we prove that if D is a locally out(in)-semicomplete digraph with circumference $l+1$, then D has a (k,l)-kernel(solution) for every integer $k \geq 2$. In Section 5, we prove that if D is a k-quasi-transitive digraph with circumference $l \leq k$, then D has an n-kernel for every $n \geq k$.

2. Preliminary results

The notion of semikernel of a digraph proved to be very useful in the study of the kernel problem, principally because of the lemma due to Neumann-Lara [20] asserting that a digraph D is kernel perfect (every induced subdigraph of D has a kernel) if and only if every induced subdigraph of D has a nonempty semikernel. In [12], Galeana-Sánchez and Hernández-Cruz proposed a generalization of this concept, the k-semikernel of a digraph, but a more general concept can be defined and can be used to prove an analogous version of the aforementioned result due to Neumann-Lara. A (k,l)-semikernel of D is defined next.

Definition 2.1. *Let D be a digraph. A subset $S \subseteq V(D)$ will be a (k,l)-semi-kernel of D if:*

(i) S is k-independent.

(ii) For every vertex $v \in V(D) \setminus S$, $d(S,v) \leq k-1$ implies $d(v,S) \leq l$.

The condition (ii) will be often referred as “the second (k,l)-semi-kernel condition”. A $(k,k-1)$-semi-kernel will be simply called a k-kernel. There was a previous attempt to define a (k,l)-semikernel of a digraph due to Kucharska and Kwaśnik in [17] which was also used by Galeana-Sánchez and Gómez-Aiza in [11], and though the definition they proposed worked well in the context they were using it, it was not possible to find an analogous result to the one of Neumann-Lara with that definition. It is worth observing that under either definition of (k,l)-semikernel, the k-semikernels remain the same. A version for k-kernels of the next lemma can be found in [12], together with a brief explanation about the motives for the (seemingly very strong) hypothesis that we need to prove it.

Lemma 2.2. *Let D be a digraph such that $\{v\}$ is a (k,l)-semi-kernel for every vertex $v \in V(D)$, then D has a (k,l)-kernel.*

Proof. Since every vertex in D is a (k,l)-semi-kernel, then D has at least one non-empty (k,l)-semi-kernel and thus we can consider a $(\subseteq)$maximal (k,l)-semi-kernel of D, namely $S \subseteq V(D)$. If S is l-absorbent then S is a (k,l)-kernel of D, so let us assume that S is not l-absorbent, therefore there must exist a vertex $v \in V(D) \setminus S$ such that $d(v,S) > l$. Let us observe that $d(S,v) > k-1$ because, by the second condition of (k,l)-semi-kernel, $d(S,v) \leq k-1$ implies that $d(v,S) \leq l$ but v is not l-absorbed by S. We will consider two cases.

Case 1. If $k-1 \leq l$, then $k-1 \leq l < d(v,S)$, so, in view that $d(S,v) > k-1$, we have that $S' = S \cup \{v\}$ is a k-independent set. Moreover, if $u \in V(D)$ is such that there exists an $S'u$-directed path $\mathscr{C}$ of length less than or equal to $k-1$ then, since S is a (k,l)-semi-kernel, if $\mathscr{C}$ is a Su-directed path, then there exists an uS-directed path of length less than or equal to $k-1$, but this path is also a uS'-directed path; and since $\{v\}$ is also a (k,l)-semi-kernel, then if $\mathscr{C}$ is a vu-directed path, this implies that there exists a uv-directed path of length less than or equal to $k-1$, which is also a uS'-directed path, and then S' is a (k,l)-semi-kernel properly containing S which contradicts the election of S as a maximal (k,l)-semi-kernel.

Case 2. If $l < k-1$, then we can assume that $d(v,S) \leq k-1$, otherwise $S \cup \{v\}$ would be k-independent and we can proceed as in Case 1. So, since $\{v\}$ is a (k,l)-semi-kernel, then $d(S,v) \leq l < k-1$ which results in a contradiction.

In both cases a contradiction arises from the assumption that S is not l-absorbent, so S must be l-absorbent and hence the desired (k,l)-kernel. □

The following lemma is a very simple one, but it will help us to prove some results to propose a conjecture about the relation between the circumference of a digraph and the existence of certain k-kernels. It also inspired the results of the next section.

Lemma 2.3. *Let D be a strong digraph with circumference l. If $(u,v) \in A(D)$, then $d(v,u) \leq l-1$.*

Proof. Let $(u,v) \in A(D)$ and $\mathscr{C} = (v = v_0, v_1, \ldots, v_n = u)$ a vu-directed path (which exists because D is strong). It is clear that $\mathscr{C} \cup (u,v)$ is a directed cycle in D and therefore $\ell(\mathscr{C} \cup (u,v)) = n+1 \leq l$, this implies that $n \leq l-1$. □

The following theorem was mentioned in the Introduction as the inspiration for the conjecture proposed in this work.

Theorem 2.4. *Let D be a digraph with circumference 2, then D has a k-kernel for every $k \geq 2$.*

Proof. It can be easily observed that if D has circumference 2, then every strong component of D is a symmetrical digraph. Otherwise, let $(u, v) \in A(D)$ be an asymmetrical arc in a strong component of D. Since u and v are in the same component, then there is a vu-directed path $\mathscr{C}$ of length at least 2 in D. But $\mathscr{C} \cup (u, v)$ is a directed cycle of length greater than or equal to 3. It follows from this observation that the underlying graph of every strong component of D is a tree.

The existence of a k-kernel can now be proved by induction on $|V(D)|$. If $|V(D)| = 1$, the result is obtained trivially. If not, let v be a leaf in the underlying graph of an initial strong component S of D. So, v has only one (in and out) neighbor in D. By the Induction Hypothesis, $D - v$ has a k-kernel N. If v is $(k-1)$-absorbed by N in D, then N is a kernel for D. Otherwise, we have two cases. If $N \cap V(S) = \varnothing$, then, since S is an initial component, there are not NS-directed paths in D. Thus, $N \cup \{v\}$ is a k-independent (because v is a leaf of the underlying graph of S), $(k-1)$-absorbent set in D, and thus a k-kernel. If $N \cap V(S) \neq \varnothing$, then we affirm that there are not Nv-directed path of length less than or equal to $(k-1)$ in D. This is because, since S is an initial component, the only vertices of N that can reach v are also in S, but S is a symmetrical digraph. Hence, the existence of an Nv-directed path of length less than or equal to $(k-1)$ in D impies the existence of a vN-directed path of length less than or equal to $(k-1)$ in D, but v is not $(k-1)$-absorbed by N. Once again, v is a leaf in the underlying graph of S, thus $N \cup \{v\}$ is a k-independent, $(k-1)$-absorbent set of D. In both cases the existence of a k-kernel is proved.

The result follows from the principle of mathematical induction. □

3. σ-strongly connected digraphs

It is completely natural to consider a directed version of the concept of σ-connectedness, we define it next.

Definition 3.1. *For a strong digraph D, a set $S \subset V(D)$ is a separator (or separating set) if $D \setminus S$ is not strong. A digraph D is σ-strongly connected (or σ-strong) if $|V(D)| \geq \sigma + 1$ and has no separator with less than σ vertices.*

As a first observation, let us notice that if D is a σ-strong digraph with circumference l, then $l \geq \sigma + 1$. To prove this, let $\mathscr{C}$ be a longest cycle in D. If $|V(\mathscr{C})| \leq \sigma$, then fix an arc (x, y) in $\mathscr{C}$ and delete all vertices of $\mathscr{C} - \{x, y\}$ and the arc (x, y). The resulting digraph is strongly connected (since $|V(\mathscr{C})| \leq \sigma$), so there is an xy-path of length at least 2 in D. Thus, a cycle longer than $\mathscr{C}$ can be constructed in D, contradicting the choice of $\mathscr{C}$.

The degree of strong connectivity of a digraph has consequences on the distances between its vertices. The next couple of lemmas show this relation.

Lemma 3.2. *Let D be a σ-strong digraph with circumference l, $k \geq 2$ a fixed integer and $\mathscr{C} = (x_0, x_1 \ldots, x_m)$ a directed path of length m. If $m = q\sigma + r$ where q and r are given by the division algorithm, then:*

(i) If $r = 0$, then $d(x_m, x_0) \leq (l-\sigma)q$.

(ii) If $r > 0$, then $d(x_m, x_0) \leq (l-r) + (l-\sigma)\left\lfloor \frac{m-1}{\sigma} \right\rfloor$.

Proof. For (i) we have that $m = q\sigma$ and we will proceed by induction on q. If $q = 0$, then $\mathscr{C} = (x_0)$ and there is nothing to prove, so let us suppose that $q \geq 1$. Let us consider the set $S = \{x_{(q-1)\sigma+1}, x_{(q-1)\sigma+2}, \ldots, x_{q\sigma-1}\}$, it is clear that $|S| = \sigma - 1$ and hence $D \setminus S$ is strong, so there exists an $x_m x_{(q-1)\sigma}$-directed path in $D \setminus S$, namely $\mathscr{D}$ which is clearly internally disjoint with $\mathscr{E} = (x_{(q-1)\sigma}, x_{(q-1)\sigma+1}, \ldots, x_m)$, so $\mathscr{D} \cup \mathscr{E}$ is a directed cycle in D. But recalling that D has circumference l, we have that $\ell(\mathscr{D} \cup \mathscr{E}) = \ell(\mathscr{D}) + \ell(\mathscr{E}) = \ell(\mathscr{D}) + \sigma \leq l$, and thence $\ell(\mathscr{D}) \leq l - \sigma$. By induction hypothesis there exists an $x_{(q-1)\sigma}x_0$-directed path $\mathscr{D}'$ of length less than or equal to $(l-\sigma)(q-1)$, so $\mathscr{D} \cup \mathscr{D}'$ is an $x_m x_0$-directed path of length less than or equal to $(l-\sigma)q$. The desired result now follows from the Principle of Mathematical Induction.

For (ii) we have that $m = q\sigma + r$ with $0 < r < k$, so $q = \left\lfloor \frac{m-1}{\sigma} \right\rfloor$. If we consider the set $S = \{x_{q\sigma+1}, x_{q\sigma+2}, \ldots, x_{q\sigma+(r-1)}\}$ with cardinality $|S| = r - 1 \leq \sigma - 1$ we can observe that there is an $x_m x_{q\sigma}$-directed path $\mathscr{D}$ in $D \setminus S$ because D is σ-strong. The directed path $\mathscr{D}$ is internally disjoint with the directed path $\mathscr{E} = (x_{q\sigma}, x_{q\sigma+1}, \ldots, x_m)$, therefore $\mathscr{D} \cup \mathscr{E}$ is a directed cycle in D and thus, $\ell(\mathscr{D} \cup \mathscr{E}) = \ell(\mathscr{D}) + \ell(\mathscr{E}) = \ell(\mathscr{D}) + r \leq l$, so $\ell(\mathscr{D}) \leq l - r$. By (i) we know that $d(x_{q\sigma}, x_0) \leq (l-\sigma)q$ and we have just proved that $d(x_m, x_{q\sigma}) \leq l - r$, by the triangle inequality we can conclude that $d(x_m, x_0) \leq (l-r) + (l-\sigma)q = (l-r) + (l-\sigma)\left\lfloor \frac{m-1}{\sigma} \right\rfloor$. □

Lemma 3.3. *Let D be a σ-strong digraph with circumference l, then for every $v \in V(D)$, $\{v\}$ is a $(k, (l-1) + (l-\sigma)\left\lfloor \frac{k-2}{\sigma} \right\rfloor)$-semi-kernel for every integer $k \geq 2$.*

Proof. Let us recall that $\sigma \leq l - 1$. Let $k \geq 2$ and $v \in V(D)$ be fixed and let $\mathscr{C} = (v = x_0, x_1, \ldots, x_m)$ be a vx_m-directed path of length $m \leq k - 1$. In virtue of Lemma 3.2, $d(x_m, v) \leq (l-1) + (l-\sigma)\left\lfloor \frac{m-1}{\sigma} \right\rfloor \leq (l-1) + (l-\sigma)\left\lfloor \frac{k-2}{\sigma} \right\rfloor$ and then $\{v\}$ fulfills the second $(k, (l-1) + (l-\sigma)\left\lfloor \frac{k-2}{\sigma} \right\rfloor)$-semi-kernel condition. □

The principal theorem of the section is proved next. It explores what kind of (k,l)-kernels exists with given values of circumference and strong connectivity.

Theorem 3.4. *Let D be a σ-strong digraph with circumference l. Then D has a $(k, (l-1) + (l-\sigma)\left\lfloor \frac{k-2}{\sigma} \right\rfloor)$-kernel for every integer $k \geq 2$.*

Proof. It follows immediately from Lemmas 2.2 and 3.3. □

We would like to point out a special case of Theorem 3.4, concerning the conjecture proposed above.

Corollary 3.5. *Let D be a $(l-1)$-strong digraph with circumference l, then D has an l-kernel.*

Proof. In virtue of Theorem 3.4, D has an $(l, (l-1) + (l-(l-1)) \left\lfloor \frac{l-2}{l-1} \right\rfloor)$-kernel. But for every integer $l \geq 2$, we have that $\left\lfloor \frac{l-2}{l-1} \right\rfloor = 0$. Thus D has an $(l, l-1)$-kernel. □

4. Locally in/out-semicomplete digraphs

These families of digraphs have been largely studied by Bang-Jensen, *et al.* A very surprising theorem, that is also particularly useful in the study of k-kernels, will be stated after the formal definition of these classes of digraphs.

Definition 4.1. *Let D be a digraph. Then D is:*

- *Locally in-semicomplete if whenever $(v,u),(w,u) \in A(D)$, then $(v,w) \in A(D)$ or $(w,v) \in A(D)$.*
- *Locally out-semicomplete if whenever $(u,v),(u,w) \in A(D)$, then $(v,w) \in A(D)$ or $(w,v) \in A(D)$.*
- *Locally semicomplete if it is both, locally out-semicomplete and locally in-semicomplete.*

And the theorem due to Bang-Jensen, Huang and Prisner is stated next.

Theorem 4.2. [3] *A locally in-semicomplete digraph D of order $n \geq 2$ is Hamiltonian if and only if D is strong.*

Let us observe that the definition of a locally in-semicomplete digraph is equivalent to the fact that for every $v \in V(D)$, $D[\Gamma^-(v)]$ is a semicomplete digraph; analogously for the locally out-semicomplete an locally semicomplete digraphs. Also, we may observe that every directed cycle is a locally in/out-semicomplete digraph. Let us recall that a directed cycle of length ℓ has k-kernel if and only if $\ell \equiv 0 \pmod{k}$, so there is an infinite subfamily of locally in/out-semicomplete strong digraphs that does not have a k-kernel for a fixed k, so it is not surprising that heavy restrictions have to be considered in order to guarantee the existence of k-kernels.

The next lemma will be useful to dualize some results from locally out-semicomplete digraphs to locally in-semicomplete digraphs.

Lemma 4.3. *A digraph D is locally in-semicomplete if and only if $\overleftarrow{D}$ is locally out-semicomplete. As a consequence, a digraph D is locally semicomplete if and only if $\overleftarrow{D}$ is locally semicomplete.*

Proof. It is straightforward. □

The previous lemma also extends Theorem 4.2 to locally out-semicomplete digraphs, so a locally out-semicomplete digraph D of order $n \geq 2$ is Hamiltonian if and only of D is strong. Since we are studying classes of digraphs in which we can find a k-kernel, where k depends on the circumference of the digraph, any condition that we find on this family of digraphs can be easily verified thanks to Theorem 4.2.

The next lemma is just a technical one that will be used to prove the one after it.

Lemma 4.4. *Let $l \geq 1$ be an integer, D a locally out-semicomplete digraph and $(x_0, x_1, \ldots, x_n)$ is a x_0x_n-directed path of length $n \leq l$. If $(x_0, v_0) \in A(D)$ and $(x_n, v_0) \notin A(D)$, then $d(v_0, x_n) \leq l$.*

Proof. If $v_0 = x_i$ for some $1 \leq i \leq n-1$, then $(v_0, x_{i+1}, \ldots, x_n)$ is a v_0x_n-directed path of length less than or equal to $n \leq l$, so let us take for granted that $v_0 \neq x_i$ for all $0 \leq i \leq n$. For each $0 \leq i < n$, if $(x_i, v_0) \in A(D)$ then $(x_{i+1}, v_0) \in A(D)$ or $(v_0, x_{i+1}) \in A(D)$, because $(x_i, x_{i+1}) \in A(D)$ and D is locally out-semicomplete. So, since $(x_0, v_0) \in A(D)$, let us consider the greatest $0 \leq i \leq n$ such that $(x_i, v_0) \in A(D)$. Clearly $i \neq n$, because $(x_n, v_0) \notin A(D)$, and by the choice of i, $(x_{i+1}, v_0) \notin A(D)$, thus, $(v_0, x_{i+1}) \in A(D)$ and $(v_0, x_{i+1}, \ldots, x_n)$ is a v_0x_n-directed path of length less than or equal to $n \leq l$. □

The hypothesis of the following lemma may look a bit odd, but we observed that, to use Lemma 2.2, only the vertices reached at distance one failed to fulfill the second (k,l)-semi-kernel condition. It is easy to observe that for a connected locally out-semicomplete digraph D such that, for a fixed integer $l \geq 1$, whenever $(u,v) \in A(D)$ then $d(u,v) \leq l$, the digraph D results strongly connected, and thus Hamiltonian by Theorem 4.2, so it will always have a one vertex $|V(D)|$-kernel, but a better result can be proved with this hypothesis.

Lemma 4.5. *Let D be a locally out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u,v) \in A(D)$ then $d(v,u) \leq l$. Then $\{v\}$ is a (k,l)-semikernel for every integer $k \geq 2$ and every $v \in V(D)$.*

Proof. Let $(v = v_0, v_1, \ldots, v_m)$ be a vv_m-directed path of length m. We will prove by induction on m that $d(v_m, v) \leq l$. If $m = 1$, then by hypothesis $d(v_m, v) \leq l$. So let us consider the result valid for $m-1$ and let $(v = v_0, v_1, \ldots, v_m)$ be a vv_m-directed path of length m. By induction hypothesis there exists a $v_{m-1}v$-directed path of length less than or equal to l, besides $(v_{m-1}, v_m) \in A(D)$ and since $d(v, v_m) \geq 2$, $(v, v_m) \notin A(D)$, it follows from Lemma 4.4 that $d(v_m, v) \leq l$. □

Theorem 4.6. *Let D be a locally out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u,v) \in A(D)$, then $d(v,u) \leq l$. Then D has a (k,l)-kernel for every integer $k \geq 2$.*

Proof. It follows immediately from Lemmas 2.2 and 4.5. □

In the following corollary let us point out a special case.

Corollary 4.7. *Let D be a locally out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then D has a k-kernel for every integer $k \geq l + 1$.*

The following proposition can be derived from Lemma 2.3 and Theorem 4.6, but it is also a trivial consequence of Theorem 4.2.

Proposition 4.8. *Let D be a locally out-semicompete strong digraph with circumference $l + 1$, then D has a (k, l)-kernel for every integer $k \geq 2$.*

Proof. Since every locally out-semicomplete strong digraph is Hamiltonian, then $l + 1 = |V(D)|$. Trivially, for every vertex $v \in V(D)$, $\{v\}$ is a k-independent, l-absorbent set. □

Now we will obtain analogous results for locally in-semicomplete digraphs by means of dualization.

Lemma 4.9. *Let D be a locally in-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then $\{v\}$ is a (k, l)-semikernel for every integer $k \geq 2$ and every $v \in V(D)$.*

Proof. By Lemma 4.3, $\overleftarrow{D}$ is a locally out-semicomplete digraph, and is straightforward to verify that the hypothesis of Lemma 4.5 hold and hence for every vertex $u \in V(\overleftarrow{D}) = V(D)$, $\{u\}$ is a (k, l)-semikernel for $\overleftarrow{D}$ for every $k \geq 2$. Now, let $v \in V(D)$ be an arbitrary vertex and $\mathscr{C}$ be a vw-directed path in D, then $\mathscr{C}^{-1}$ is a wv-directed path in $\overleftarrow{D}$ and since $\{w\}$ is a (k, l)-semikernel of $\overleftarrow{D}$ for every $k \geq 2$, then there exists a vw-directed path of length less than or equal to l in $\overleftarrow{D}$, namely $\mathscr{D}$. Thus $\mathscr{D}^{-1}$ is a wv-directed path of length less than or equal to l in D. Thence, $\{v\}$ is a (k, l)-semikernel of D for every $k \geq 2$. □

Theorem 4.10. *Let D be a locally out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then D has a (k, l)-kernel for every integer $k \geq 2$.*

Proof. It follows immediately from Lemmas 2.2 and 4.9. □

Corollaries analogous to those of the locally out-semicomplete case can be also obtained.

Corollary 4.11. *Let D be a locally out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then D has a k-kernel for every integer $k \geq l + 1$.*

Proposition 4.12. *Let D be a locally out-semicompete strong digraph with circumference $l + 1$, then D has a (k, l)-kernel for every integer $k \geq 2$.*

It is our desire to find families of digraphs such that circumference k implies the existence of a k-kernel, Propositions 4.8 and 4.12 tell us that locally out/in-semicomplete strong digraphs have this property. But these results follow easily from Theorem 4.2, so we will prove that the result can be improved at least in the locally out-semicomplete case for non-strong digraphs, however, the dualization method in this case is not enough to prove the corresponding locally in-semicomplete case, but a dual result about (k,l)-solutions can be obtained.

The next theorems are due to Bang Jensen and Gutin [2].

Theorem 4.13. *Let D be a locally out-semicomplete digraph and S,T distinct strong components of D. If a vertex $b \in T$ absorbs some vertex in S, then $S \mapsto b$.*

Theorem 4.14. *Let D be a locally in-semicomplete digraph and S,T distinct strong components of D. If a vertex $a \in S$ dominates some vertex in T, then $a \mapsto T$.*

And can be easily generalized as follows.

Lemma 4.15. *Let D be a locally out-semicomplete digraph and S,T distinct strong components of D. If some vertex of S is l-absorbed by a vertex $b \in T$, then $S \overset{l}{\mapsto} b$.*

Proof. By induction on l. Case $l = 1$ is Theorem 4.13, so let us consider S and T distinct strong components of D such that a vertex $b \in T$ l-absorbs some vertex in $a \in S$, then there must exist an ab-directed path $\mathscr{C} = (a = v_0, v_1, \dots, v_n = b)$ of length $n \leq l$. If $n < l$ by induction hypothesis we are done, so let us consider that $n = l$. Let v_j be the first vertex of $\mathscr{C}$ not in A, then for every $i < j$, $v_i \in S$ and v_j absorbs v_{j-1}, and by Theorem 4.13, $A \mapsto v_j$; thus, for every $v \in S$, $(v, v_j, \dots, v_n = b)$ is a vb-directed path of length less than or equal to l, therefore $S \overset{l}{\mapsto} b$. □

Lemma 4.16. *Let D be a locally in-semicomplete digraph and S,T distinct strong components of D. If some vertex in T is l-dominated by a vertex $a \in S$, then $a \overset{l}{\mapsto} T$.*

Proof. The proof is analogous to the one of the previous Lemma. □

Lemma 4.17. *Let D be a locally out-semicomplete digraph, $(y_0, y_1, \dots, y_s)$ a y_0y_s-directed path in D and $x \in V(D)$ a vertex such that $(y_0, x) \in V(D)$ but $(x, y_j) \notin A(D)$ for every $1 \leq j \leq s$, then $(y_j, x) \in A(D)$ for every $0 \leq j \leq s$.*

Proof. By induction on j. For $j = 0$, $(y_0, x) \in A(D)$ by hypothesis. If $(y_j, x) \in A(D)$, since $(y_j, y_{j+1}) \in A(D)$ also, by the locally out-semicomplete hypothesis $(x, y_{j+1}) \in A(D)$ or $(y_{j+1}, x) \in A(D)$, but by hypothesis $(x, y_{j+1}) \notin A(D)$, so $(y_{j+1}, x) \in A(D)$. □

Theorem 4.18. *Let D be a locally out-semicomplete digraph with circumference $l+1$, then D has a (k,l)-kernel for every integer $k \geq 2$.*

Proof. By induction on $|V(D)|$. If $|V(D)| = 1$ the result is obvious, so let us consider the result valid for every digraph with $|V(D)| < n$ and let D be a digraph with $|V(D)| = n$. By Proposition 4.8 we may choose a (k,l)-kernel for every terminal strong component of D; let N_1 be the union of all such (k,l)-kernels. If M is the set of all vertices l-absorbed by N_1, and for some strong component S of D, a vertex $u \in S$ is also in M, then in virtue of Lemma 4.15 $S \subseteq M$, so every strong component of D is either contained in M or in $V(D) \setminus M$. By induction hypothesis there exists a (k,l)-kernel for $D \setminus M$, namely N_2. If N_2 is k-independent in D then clearly $N_1 \cup N_2$ is a k-independent set in D.

Let us assume that N_2 is not k-independent in D to reach a contradiction. We know that N_2 is k-independent in $D \setminus M$, so if $u, v \in N_2$ are such that $d_D(u,v) \leq k-1$, then every uv-directed path in D must have at least one vertex in M. Let $\mathscr{C} = (u = x_0, x_1, \dots, x_r = v)$ be a uv-directed path in D and let w be the last vertex of $\mathscr{C}$ that is in M; it is clear that $u \neq w \neq v$, so $w = x_i$ for some $1 \leq i \leq n-1$ and $x_{i+1} \in V(D) \setminus M$. But $w \in M$ implies that w is l-absorbed by N_1, thus there exists a wN_1-directed path $\mathscr{D} = (w = y_0, y_1, \dots, y_s)$ of length $s \leq l$. Let us observe that $(w, y_1), (w, x_{i+1}) \in A(D)$, and that $(x_{i+1}, y_j) \notin A(D)$ for every $1 \leq j \leq s$, for this would imply that $(x_{i+1}, y_j, y_{j+1}, \dots, y_s)$ is a directed path of length less than or equal to l and hence x_{i+1} would be in M which can not occur; therefore by Lemma 4.17 we have that $(y_s, x_{i+1}) \in A(D)$, which results in a contradiction because $y_s \in N_1$ is a vertex in a terminal component of D. Thence, N_2 is k-independent in D.

As we observed earlier, $N_1 \cup N_2 = N$ is a k-independent set of D, also N_1 is l-absorbent in M and N_2 is l-absorbent in $D \setminus M$, so N is l-absorbent in D, and then is the desired (k,l)-kernel. □

Corollary 4.19. *Let D be a locally out-semicompete digraph with circumference l, then D has a k-kernel for every integer $k \geq l$.*

Although we can not prove analogous results to those of Theorem 4.18 and Corollary 4.19 for locally in-semicomplete digraphs, we can dualize these results by means of Lemma 4.3 to (k,l)-solutions and k-solutions.

Theorem 4.20. *Let D be a locally in-semicomplete digraph with circumference $l+1$, then D has a (k,l)-solution for every integer $k \geq 2$.*

Proof. Let D be a locally in-semicomplete digraph with circumference $l+1$. In virtue of Lemma 4.3 $\overleftarrow{D}$ is a locally out-semicomplete digraphs with circumference $l+1$, and by Theorem 4.19, $\overleftarrow{D}$ has a (k,l)-kernel, namely N. Clearly N is k-independent and l-absorbent in D and thus a (k,l)-solution. □

Corollary 4.21. *Let D be a locally in-semicompete digraph with circumference l, then D has a k-solution for every integer $k \geq l$.*

Let us recall that the problem of determining if a digraph has a k-kernel or not is *NP*-complete, also the problem of finding a longest cycle in a digraph is *NP*-complete, so Theorems 4.18 and 4.20 become more valuable if stated in the next way.

Corollary 4.22. *Let D be a locally out-semicompete digraph with set of strong components $\mathcal{C}$ and $l+1 = \max_{H\in\mathcal{C}} \left|V(H)\right|$, then D has a (k,l)-kernel for every integer $k \geq 2$.*

Proof. In virtue of Theorem 4.2 and the fact that every directed cycle is contained in a single strong component of D, the circumference of D is equal to the greatest of the orders of the strong components of D, so this is merely a different statement for Theorem 4.18. □

Corollary 4.23. *Let D be a locally in-semicompete digraph with set of strong components $\mathcal{C}$ and $l+1 = \max_{H\in\mathcal{C}} \left|V(H)\right|$, then D has a (k,l)-solution for every integer $k \geq 2$.*

In view of the theorems that we have proved, these results are not as best as possible, because if D is a locally out-semicomplete digraph we may have as an hypothesis that whenever $(u,v) \in A(H)$ then $d_H(v,u) \leq l$ for every strong component H of D and we would get a strengthening of Theorems 4.18 and 4.20, but in the form of the two previous corollaries we may decide the existence of a (k,l)-kernel in polynomial time, this is because the only thing we have to do is to find the condensation and calculate the order of the strong components of D, which can be done in polynomial time.

To finish this section, let us observe that some of this results can be extended to k-kings and k-serfs. Considering Lemma 4.5, we may conclude that, for $k \geq l$, any (k,l)-kernel in a locally out-semicomplete digraph such that, whenever $(u,v) \in A(D)$ we have that $d(u,v) \leq l$, consists in a single vertex. Assume the contrary, and for a fixed integer $k \geq 2$, let N be a (k,l)-kernel with more than one vertex in a locally out-semicomplete digraph that fulfills the aforementioned hypothesis, and let $u,v \in N$. It also follows from Lemma 4.5 that $\{u\}$ is a $(d(u,v)+1,l)$-semi-kernel, and then $d(v,u) \leq l \leq k$, contradicting the k-independence of k. Thus, the next result is obtained.

Theorem 4.24. *Let D be a locally in/out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u,v) \in A(D)$, then $d(v,u) \leq l$, then D has a $(l+1)$-serf and a $(l+1)$-king.*

5. k-quasi-transitive digraphs

A digraph D is k-transitive if the existence of a directed path $(v_0, v_1, \ldots, v_k)$ of length k in D implies that $(v_0, v_k) \in A(D)$. A digraph D is k-quasi-transitive if the existence of a directed path $(v_0, v_1, \ldots, v_k)$ of length k in D implies that $(v_0, v_k) \in A(D)$ or $(v_k, v_0) \in A(D)$. Both families were studied by the authors in [15] where the fact that every k-transitive digraph has an n-kernel for every $n \geq k$ is proved. In view of this result we prove the following simple proposition.

Proposition 5.1. *Let D be a k-quasi-transitive digraph. If D does not contain directed cycles of length $k+1$, then D is k-transitive.*

Proof. Let $(v_0, v_1, \ldots, v_k)$ be a directed path in D. If $(v_k, v_0) \in A(D)$, then $(v_0, v_1, \ldots, v_k, v_0)$ is a directed path of lenght $k+1$ in D, which cannot occur. Thus, $(v_0, v_k) \in A(D)$ and D is k-transitive. □

Corollary 5.2. *Let D be a k-quasi-transitive digraph. If D does not contain directed cycles of length $k+1$, then D has an n-kernel for every $n \geq k$.*

As a particular case of this result, we have that if D is a k-quasi-transitive digraph with circumference $l \leq k$, then D has an n-kernel for every $n \geq k$.

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