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Graphs that induce only k-cycles

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Abstract

Every induced cycle is a 3-cycle if and only if the graph is chordal, and this is also equivalent to, in every induced hamiltonian subgraph, every two adjacent vertices (or, equivalently, every two adjacent edges) being adjacent to a third vertex (or edge, respectively). Related characterizations are given, for each $k \geq 3$, of every induced cycle being a k-cycle, involving vertex/edge self-dual conditions holding for all induced hamiltonian subgraphs.

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1. Vertex-edge duality in hamiltonian subgraphs

All graphs considered will be simple—multiple edges and loops are not allowed. A *chord* of a cycle is an edge between nonconsecutive vertices in the cycle, and so an *induced cycle* is a cycle that has no chords. A *hamiltonian subgraph* H is a subgraph that has a spanning cycle C; thus $V(H) = V(C)$ and $E(H) = E(C) \cup \{e : e \text{ is a chord of } C\}$. A *$k$-cycle* is a cycle of *length* $k = |C| = |E(C)| = |V(C)|$, and the *girth* of a graph is the smallest length of a cycle in the graph. Two edges that have exactly one vertex in common are called *adjacent edges*. A graph is 2-*connected* if it has at least three vertices and every two vertices—or, equivalently, every two edges—are in a cycle.

For vertices $v_1, \ldots, v_k$, let $\textsc{ic}[v_1, \ldots, v_k]$ abbreviate that v_i is adjacent to v_{i+1} whenever $1 \leq i < k$ and v_k is adjacent to v_1, but no other pairs v_i, v_j are adjacent. Similarly for edges $e_1, \ldots, e_k$, let $\textsc{ic}[e_1, \ldots, e_k]$ abbreviate that e_i is adjacent to e_{i+1} whenever $1 \leq i < k$ and e_k is adjacent to e_1, but no other pairs e_i, e_j are adjacent. This $\textsc{ic}$ notation reflects that $\textsc{ic}[v_1, \ldots, v_k]$ holds precisely when the listed vertices, in the order listed, form an induced cycle, just as $\textsc{ic}[e_1, \ldots, e_k]$ holds precisely when the listed edges, in the order listed, form an induced cycle *except when* $k = 3$: $\textsc{ic}[e_1, e_2, e_3]$ only means that e_1, e_2, e_3 are pairwise adjacent, so either they form a triangle or they are all incident with a common vertex.

Lemma 1 will contain some very simple, but useful, reformulations of every induced cycle being a k-cycle. Lemmas 2 and 3 contain fundamental machinery that will drive the characterizations in Section 2, for each value of k, of the graphs that induce only k-cycles. Lemma 4 will be used in the proofs of Theorems 7 and 8.

Lemma 1. *The following are equivalent for every graph and every $k \geq 3$:*

(1.1) *Every induced cycle is a k-cycle.*

(1.2) *Every induced 2-connected subgraph contains vertices $v_1, \ldots, v_k$ such that* $\textsc{ic}[v_1, \ldots, v_k]$.

(1.3) *Every induced hamiltonian subgraph contains vertices $v_1, \ldots, v_k$ such that* $\textsc{ic}[v_1, \ldots, v_k]$.

(1.4) *Every induced hamiltonian subgraph has girth k.*

Proof. **(1.1) ⇒ (1.2)**: Suppose G satisfies (1.1) and H is any induced 2-connected subgraph of G. Thus H contains a cycle, and so H contains an induced cycle C. By (1.1), $|C| = k$. Thus H contains vertices $v_1, \ldots, v_k$ such that $\textsc{ic}[v_1, \ldots, v_k]$.

(1.2) ⇒ (1.3): This follows from every hamiltonian subgraph being 2-connected.

(1.3) ⇒ (1.4): Suppose graph G satisfies (1.3) and H is any induced hamiltonian subgraph of G. By (1.3), H contains a k-cycle. For every cycle C in H, the hamiltonian subgraph induced by $V(C)$ also contains a k-cycle by (1.3), and so $|C| \geq k$. Therefore, H has girth k.

(1.4) ⇒ (1.1): Suppose G satisfies (1.4) and C is any induced cycle of G. By (1.4), the hamiltonian subgraph H induced by $V(C)$ has girth k, and so, since C has no chords, $|C| = k$. □

Lemma 2. *The following are equivalent for every graph and $k \geq 3$:*

(2.0v) *For every induced hamiltonian subgraph H, there exist $v_1, \ldots, v_k \in V(H)$ such that* $\textsc{ic}[v_1, \ldots, v_k]$.

(2.0e) *For every induced hamiltonian subgraph H, there exist $e_1, \ldots, e_k \in E(H)$ such that* $\textsc{ic}[e_1, \ldots, e_k]$.

(2.1v) *For every induced hamiltonian subgraph H and every $v_1 \in V(H)$, there exist $v_2, \ldots, v_k \in V(H)$ such that* $\textsc{ic}[v_1, v_2, \ldots, v_k]$.

(2.1e) *For every induced hamiltonian subgraph H and every $e_1 \in E(H)$, there exist $e_2, \ldots, e_k \in V(H)$ such that* $\textsc{ic}[e_1, e_2, \ldots, e_k]$.

Proof. **(2.0v) ⇒ (2.1v)**: Suppose G satisfies (2.0v) = (1.3), H is an induced hamiltonian subgraph of G, and $v_1 \in V(H)$. Let C be a minimum length cycle (and so an induced cycle) of H that contains v_1. Since Lemma 1 implies that $|C| = k$, suppose $C = v_1, v_2, \ldots, v_k, v_1$. Therefore, $\textsc{ic}[v_1, v_2, \ldots, v_k]$ holds in H.

(2.1v) ⇒ (2.1e): Suppose G satisfies (2.1v)—and so G satisfies (2.0v) = (1.3)—and H is an induced hamiltonian subgraph of G and $e_1 \in E(H)$. Let C be a minimum length cycle (and so an induced cycle) of H that contains e_1. Since Lemma 1 implies that $|C| = k$, suppose $C = e_1, e_2, \ldots, e_k, e_1$. Therefore, $\textsc{ic}[e_1, e_2, \ldots, e_k]$ holds in H.

(2.1*e*) ⇒ (2.0*e*): This implication is immediate.

(2.0*e*) ⇒ (2.0*v*): Suppose G satisfies (2.0e) and H is any induced hamiltonian subgraph of G.

Suppose for the moment that $k = 3$. Let C be any induced cycle in H and let H' be the hamiltonian subgraph induced by $V(C)$. By (2.0e), there are pairwise adjacent edges $e_1, e_2, e_3 \in E(H') = E(C)$, and C being an induced cycle implies that C is a triangle with $E(C) = \{e_1, e_2, e_3\}$. Suppose $V(C) = \{v_1, v_2, v_3\}$. Therefore, $\textsc{ic}[v_1, v_2, v_3]$ holds in H.

Now suppose $k > 3$. By (2.0e), there exist edges $e_1, \ldots, e_k \in E(H)$ such that $e_1, \ldots, e_k, e_1$ is an induced cycle C. Suppose $V(C) = \{v_1, \ldots, v_k\}$. Therefore, $\textsc{ic}[v_1, \ldots, v_k]$ holds in H. □

Lemma 3. *The following are equivalent for every graph*:

(3.0v) *For every induced hamiltonian subgraph H, there exist* $v_1, v_2, v_3 \in V(H)$ *such that* $\textsc{ic}[v_1, v_2, v_3]$.

(3.0e) *For every induced hamiltonian subgraph H, there exist* $e_1, e_2, e_3 \in E(H)$ *such that* $\textsc{ic}[e_1, e_2, e_3]$.

(3.1v) *For every induced hamiltonian subgraph H and every* $v_1 \in V(H)$, *there exist* $v_2, v_3 \in V(H)$ *such that* $\textsc{ic}[v_1, v_2, v_3]$.

(3.1e) *For every induced hamiltonian subgraph H and every* $e_1 \in E(H)$, *there exist* $e_2, e_3 \in E(H)$ *such that* $\textsc{ic}[e_1, e_2, e_3]$.

(3.2v) *For every induced hamiltonian subgraph H and every two adjacent vertices* $v_1, v_2 \in V(H)$, *there exists* $v_3 \in V(H)$ *such that* $\textsc{ic}[v_1, v_2, v_3]$.

(3.2e) *For every induced hamiltonian subgraph H and every two adjacent edges* $e_1, e_2 \in E(H)$, *there exists* $e_3 \in E(H)$ *such that* $\textsc{ic}[e_1, e_2, e_3]$.

Proof. **(3.0*v*) ⇔ (3.0*e*) ⇔ (3.1*v*) ⇔ (3.1*e*)** is the $k = 3$ case of Lemma 2.

(3.1*e*) ⇔ (3.2*v*): This equivalence is straightforward: For the ⇒ direction, let $e_1 = v_1v_2$, and then take v_3 to be the shared endpoint of e_2 and e_3. For the ⇐ direction, let v_1 and v_2 to be the endpoints of e_1, and then take $e_2 = v_1v_3$ and $e_3 = v_2v_3$.

(3.1*e*) ⇔ (3.2*e*): The ⇐ direction is immediate. For the ⇒ direction, suppose G satisfies (3.1e), H is an induced hamiltonian subgraph of G, and e_1, e_2 are adjacent edges of H. Since H is 2-connected, H has a minimum length cycle C that contains e_1, e_2 as adjacent edges, and so every chord of C is incident with both e_1 and e_2. Let H' be the hamiltonian subgraph induced by $V(C)$. By (3.1e), there exist $f_2, f_3 \in E(H') \subseteq E(H)$ such that $\textsc{ic}[e_1, f_2, f_3]$; thus either e_1, f_2, f_3 are all incident with a common vertex or e_1, f_2, f_3 form a triangle. If e_1, f_2, f_3 are all incident with a common vertex v, then v is also incident with e_2 (because every chord of C is incident with both e_1, e_2) and so either $\textsc{ic}[e_1, e_2, f_2]$ (if $f_2 \neq e_2$) or $\textsc{ic}[e_1, e_2, f_3]$ (if $f_3 \neq e_2$) holds in H. If e_1, f_2, f_3 form a triangle, then one of f_2, f_3 must be e_2 (because every chord of C is incident with both e_1, e_2), and so again either $\text{IC}[e_1, e_2, f_2]$ (if $f_3 = e_2$) or $\text{IC}[e_1, e_2, f_3]$ (if $f_2 = e_2$) holds in H. □

The 'domino' graph—formed by inserting a chord into C_6 joining two vertices that are directly opposite across C_6—shows that v_1, v_2, v_3 and e_1, e_2, e_3 in Lemma 3 cannot be replaced with v_1, v_2, v_3, v_4 and e_1, e_2, e_3, e_4—every two adjacent vertices of the domino are in a 4-cycle, but not every two adjacent edges are.

Lemma 4. *If every induced cycle is a k-cycle, then every cycle C has $|C| \equiv 2$ (modulo $k-2$).*

Proof. Suppose every induced cycle is a k-cycle and [arguing by contradiction] some cycle C has minimum length $|C| \not\equiv 2$ (congruences in this proof are modulo $k-2$). Every induced cycle being a k-cycle implies $|C| \geq k$; thus $|C| \not\equiv 2$ further implies $|C| > k$. Therefore, C has a chord e and there are two cycles C_1 and C_2 such that $e \in E(C_i) \subset E(C) \cup \{e\}$ for each i and $|C| = |C_1| + |C_2| - 2$. Since each $|C_i| < |C|$, the minimality of $|C|$ implies that both $|C_i| \equiv 2$, and so that $|C_1| + |C_2| - 2 \equiv 2$ [contradicting $|C| \not\equiv 2$]. □

2. Graphs that induce only k-cycles

A graph is *chordal* if every cycle of length four or more has a chord. See [1, 5] for other characterizations of this highly-studied graph property. Theorem 5 will give two new, very simple characterizations.

Theorem 5. *Each of the following is equivalent to every induced cycle of a graph G being a 3-cycle*:

(5) *G is chordal.*

(5v) *In every induced hamiltonian subgraph of G, every two adjacent vertices are adjacent to a third vertex.*

(5e) *For every induced hamiltonian subgraph of G, every two adjacent edges are adjacent to a third edge.*

Proof. The definition of choral graphs shows that condition (5) is equivalent to every induced cycle being a triangle. Lemma 1 then implies that (5) $\Leftrightarrow$ (3.0v). Also, of course, (5v) = (3.2v) and (5e) = (3.2e). Theorem 5 then follows from Lemma 3. □

(Theorem 5 could, of course, also include restatements of conditions (3.0e), (3.1v) and (3.1e), but (5v) and (5e) immediately imply the others.)

A graph is *chordal bipartite* if it is bipartite and every cycle of length six or more has a chord, see [1, 5].

Theorem 6. *Each of the following is equivalent to every induced cycle of a graph G being a 4-cycle*:

(6) *G is chordal bipartite.*

(6v) *In every induced hamiltonian subgraph of G, every vertex is contained in a 4-cycle.*

(6e) *In every induced hamiltonian subgraph of G, every edge is contained in a 4-cycle.*

Proof. The definition of choral bipartite graphs shows that condition (6) is equivalent to every induced cycle being a 4-cycle. Lemma 1 then implies that (6) $\Leftrightarrow$ (2.0v). Also, (6v) $\Leftrightarrow$ (2.1v) and (6e) $\Leftrightarrow$ (2.1e) (since triangles are induced hamiltonian subgraphs, each of (6v) and (6e) implies there are no triangles; thus, every 4-cycle is induced). Theorem 6 then follows from the $k = 4$ case of Lemma 2. □

As in [3], define a graph to be a C_k*-tree* recursively as follows: Every k-cycle is a C_k-tree, and every graph obtained by identifying an edge of a C_k-tree with an edge of a new copy of C_k is a C_k-tree. (The C_k-trees are called *trees of k-cycles* in [2], and C_3-trees are traditionally called 2-*trees*; see [1, 2].)

Theorem 7. *If $k \geq 5$ is odd, then each of the following is equivalent to every induced cycle of a graph G being a k-cycle*:

(7) *Every maximal* 2*-connected subgraph of G is a C_k-tree.*

(7v) *In every induced hamiltonian subgraph of G, every vertex is contained in a k-cycle.*

(7e) *In every induced hamiltonian subgraph of G, every edge of is contained in a k-cycle.*

Proof. Suppose $k \geq 5$ is odd. Define a C_k*-ear* of a graph G to be G itself if $G \cong C_k$ and, if $G \ncong C_k$, to be an induced k-cycle C of G with $xy \in E(C)$ such that $G - \{x\}$ and $G - \{y\}$ are connected, but $G - \{x, y\}$ is not connected. (If $G \ncong C_k$, then a C_k-ear will contain exactly $k - 2$ degree-2 vertices of G.) A simple recursive argument shows that being a C_k-tree is equivalent to being 2-connected with every induced 2-connected subgraph having a C_k-ear.

Lemma 1 implies that every induced cycle being a k-cycle is equivalent to (2.0v). Also, (7v) $\Leftrightarrow$ (2.1v) and (7e) $\Leftrightarrow$ (2.1e) (since induced cycles are induced hamiltonian subgraphs, each of (7v) and (7e) implies there are no h-cycles with $h < k$; thus, every k-cycle is induced). Therefore, by Lemma 2, (2.0v) is equivalent to (7v) and to (7e). Every C_k-tree satisfies (2.0v), and so (7) $\Rightarrow$ (2.0v). Finally, (7) $\Leftarrow$ (2.0v) will follow from showing the following: *Every* 2*-connected graph that has only induced k-cycles has a C_k-ear.*

Suppose G is 2-connected and has only induced k-cycles; thus, $n = |V(G)| \geq k \geq 5$. Argue by induction on $n \geq k$ that G has a C_k-ear. The $n = k$ case is immediate (since then $G \cong C_k$). Suppose $n > k$ and every induced 2-connected subgraph G' of G with $|V(G')| < n$ has a C_k-ear. Since G is 2-connected has only induced k-cycles, G contains an induced k-cycle $C = v_1, v_2, \ldots, v_k, v_1$.

Suppose for the moment that there is a v_1-to-v_j path π in G with $4 \leq j \leq k - 1$ and $V(\pi) \cap V(C) = \{v_1, v_j\}$. Let C' be the cycle with edge set $E(\pi) \cup \{v_1v_2, \ldots, v_{j-1}v_j\}$ and let C'' be the cycle with edge set $E(\pi) \cup \{v_jv_{j+1}, \ldots, v_kv_1\}$. Since k is odd, it follows that

$1 \le |C'| - |C''| \le k-4$, and so $|C'| \not\equiv |C''| \pmod{k-2}$ with both $|C'|, |C''| < k$. Thus, the at least one of $|C'|$ and $|C''|$ is not congruent to 2 modulo $k-2$ [contradicting Lemma 4].

Therefore, for each $j \in \{4, \ldots, k-1\}$, there is no v_1-to-v_j path π in G with $V(\pi) \cap V(C) = \{v_1, v_j\}$. Therefore, $G - \{v_2, v_k\}$ is not connected. Let G_1 be the component of $G - \{v_2, v_k\}$ that contains v_1 and let G_1^+ be the subgraph of G induced by $V(G_1) \cup V(C)$. Let G_3 be the union of the components of $G - \{v_2, v_k\}$ that do not contain v_1 (so G_3 contains v_3) and let G_3^+ be the subgraph of G induced by $V(G_3) \cup V(C)$. Each vertex of each G_i will have the same degree in G_i^+ as in G. Each G_i^+ is 2-connected. Also, since $n > k$, at least one $|V(G_i^+)| < n$; by inductive hypothesis, that G_i^+ will contain a C_k-ear that is a C_k-ear of G. □

Notice that $k = 3$ cannot be allowed in Theorem 7, since every induced cycle of the chordal graph K_4 is a 3-cycle, but K_4 is not a C_3-tree.

For $d \ge 2$ and k even, define Θ_k^d to be the graph that consists of two degree-d vertices x and y connected by paths $\pi_1, \ldots, \pi_d$ where each π_i has length $k/2$, each internal vertex of each π_i has degree two, and $V(\pi_i) \cap V(\pi_j) = \{x, y\}$ when $i \ne j$. Thus, each of the $\binom{d}{2}$ cycles in Θ_k^d is a k-cycle, and $\Theta_k^2 \cong C_k$.

Generalizing C_k-trees, define a graph to be a Θ_k*-tree* recursively as follows: Every Θ_k^d is a Θ_k-tree whenever $d \ge 2$, and any graph obtained by identifying an edge of a Θ_k-tree with an edge of a new copy of any Θ_k^d (for any $d \ge 2$) is a Θ_k-tree. (The Θ_4^d-trees also appear in [4], where they are called $2*$*-trees.*)

Theorem 8. *If $k \ge 6$ is even, then each of the following is equivalent to every induced cycle of a graph G being a k-cycle*:

(8) *Every maximal 2-connected subgraph of G is a Θ_k-tree.*

($8v$) *In every induced hamiltonian subgraph of G, every vertex is contained in a k-cycle.*

($8e$) *In every induced hamiltonian subgraph of G, every edge is contained in a k-cycle.*

Proof. Suppose $k \ge 6$ is even. Define a Θ_k*-ear* of a graph G to be G itself if $G \cong \Theta_k^d$ and to be an induced subgraph $H \cong \Theta_k^d$ with $xy \in E(H)$ such that $G - \{x\}$ and $G - \{y\}$ are connected, but $G - \{x, y\}$ is not connected. A simple recursive argument shows that being a Θ_k-tree is equivalent to being 2-connected with every induced 2-connected subgraph having a Θ_k-ear.

Lemma 1 implies that every induced cycle being a k-cycle is equivalent to ($2.0v$). Also, ($8v$) $\Leftrightarrow$ ($2.1v$) and ($8e$) $\Leftrightarrow$ ($2.1e$) (since induced cycles are induced hamiltonian subgraphs, each of ($8v$) and ($8e$) implies there are no h-cycles with $h < k$; thus, every k-cycle is induced). Therefore, by Lemma 2, ($2.0v$) is equivalent to ($8v$) and to ($8e$). Every Θ_k-tree satisfies ($2.0v$), and so (8) $\Rightarrow$ ($2.0v$). Finally, (8) $\Leftarrow$ ($2.0v$) will follow from showing the following: *Every 2-connected graph that has only induced k-cycles has a Θ_k-ear.*

Suppose G is 2-connected and has only induced k-cycles; thus, $n = |V(G)| \ge k \ge 6$. Argue by induction on $n \ge k$ that G has a Θ_k-ear. The $n = k$ case is immediate (since then $G \cong C_k = \Theta_k^2$). Suppose $n > k$ and every induced 2-connected subgraph G' of G with $|V(G')| < n$ has a Θ_k-ear. Since G is 2-connected and has only induced k-cycles, G contains an induced k-cycle $C = v_1, v_2, \ldots, v_k, v_1$.

Suppose there is a path π between distinct, nonadjacent vertices of C. By rotating the subscripts of the vertices of C, assume π is a v_1-to-v_j path in G with $4 \leq j \leq k-1$ and $V(\pi) \cap V(C) = \{v_1, v_j\}$. If $j \neq \frac{k+2}{2}$ (in other words, if v_j is not directly opposite from v_1 across C), then define cycles C' and C'' exactly as in the proof of Theorem 7, noting that $j \neq \frac{k+2}{2}$ implies $|C'| \neq |C''|$, and so $1 \leq |C'| - |C''| \leq k-4$ [leading to a contradiction of Lemma 4]. Thus, assume $j = \frac{k+2}{2}$ (and so $|\pi| = \frac{k}{2}$, since G has only induced k-cycles).

Suppose for the moment that there is a second path π' between vertices $v_{i'}$ and $v_{j'}$ that are directly opposite across C, where $\{v_{i'}, v_{j'}\} \cap \{v_1, v_j\} = \emptyset$ (and so $|\pi'| = \frac{k}{2}$ and the paths π and π' are internally vertex disjoint, since G has only induced k-cycles.) Without loss of generality, suppose $v_1, v_{i'}, v_j, v_{j'}$ come in that order around C and let $\pi_1, \pi_{i'}, \pi_j$, $\pi_{j'}$ be, respectively, the v_1-to-$v_{i'}$, $v_{i'}$-to-v_j, v_j-to-$v_{j'}$, $v_{j'}$-to-v_1 paths along C. Since v_1 and v_j are directly opposite across C, as are $v_{i'}$ and $v_{j'}$, it follows that each of $|\pi_1| + |\pi_{i'}|$, $|\pi_{i'}| + |\pi_j|$, $|\pi_j| + |\pi_{j'}|$ and $|\pi_{j'}| + |\pi_1|$ equals $\frac{k}{2}$; thus, $|\pi_1| = |\pi_k|$ and $|\pi_{i'}| = |\pi_{j'}|$. If C' is the cycle formed by $E(\pi) \cup E(\pi') \cup E(\pi_1) \cup E(\pi_j)$ and C'' is the cycle formed by $E(\pi) \cup E(\pi') \cup E(\pi_{i'}) \cup E(\pi_{j'})$, then at least one of $|C'|$ and $|C''|$ (or both if $|\pi_1| = |\pi_{i'}|$) is not congruent to 2 modulo $k-2$ [contradicting Lemma 4].

Therefore, the only paths between distinct, nonadjacent vertices of C can be assumed to be $\pi_1, \ldots, \pi_\delta$, each between v_1 and v_j, where these paths are pairwise internally vertex disjoint (since G has only induced k-cycles). Thus $E(C) \cup E(\pi_1) \cup \cdots E(\pi_\delta)$ forms a subgraph $H \cong \Theta_k^{\delta+2}$, and $G - \{v_1, v_j\}$ is not connected. Let G_2 be the component of $G - \{v_1, v_j\}$ that contains v_2 and let G_2^+ be the subgraph of G induced by $V(G_2) \cup V(H)$. Let G_k be the union of the components of $G - \{v_1, v_j\}$ that do not contain v_2 (so G_k contains v_k) and let G_k^+ be the subgraph of G induced by $V(G_k) \cup V(H)$. Each G_i^+ is 2-connected. Also, if $G \neq \Theta_k^{\delta+2}$, at least one $|V(G_i^+)| < n$; by inductive hypothesis, that G_i^+ will contain a Θ_k-ear that is a Θ_k-ear of G. □

Notice that $k = 4$ cannot be allowed in Theorem 8, since every induced cycle of the chordal bipartite graph $K_{3,3}$ is a 4-cycle, but $K_{3,3}$ is not a Θ_4-tree.

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