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On Properly Colored Hamiltonian Cycles in Cubes of Distance-Colored Grids

Derya Dogan
Ege University
Izmir, Turkey.
Kyle Kolasinski and Ping Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA.
e-mail: *ping.zhang@wmich.edu*

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Abstract

For a connected graph G and a positive integer k, the kth power G^k of G is the graph with $V(G^k) = V(G)$ where $uv \in E(G^k)$ if the distance $d_G(u, v)$ between u and v is at most k. The edge coloring of G^k defined by assigning each edge uv of G^k the color $d_G(u, v)$ produces an edge-colored graph G^k called a distance-colored graph. A distance-colored graph G^k is Hamiltonian-colored if G^k contains a properly colored Hamiltonian cycle. For a grid $G = P_n \,\square\, P_m$ with $n, m \geq 2$, it is known that G^3 is Hamiltonian-colored and that G^2 is Hamiltonian-colored if and only if $nm \equiv 0 \pmod 4$. It is shown here that (i) G^3 contains a properly colored Hamiltonian cycle whose edges are colored only 1 or 3 if and only if nm is even unless $n = m = 2$ or $\{n, m\}$ is $\{2, 3\}$ or $\{2, 7\}$ and (ii) G^3 contains a properly colored Hamiltonian cycle whose edges are colored only 2 or 3 if and only if $nm \equiv 0 \pmod 4$ unless $n = m = 2$.

Keywords: powers of a graph, distance-colored graph, Hamiltonian-colored graph.

2010 Mathematics Subject Classification: 05C12, 05C15, 05C38, 05C45.

1. Introduction

For a connected graph G and a positive integer k, the *kth power* G^k of G is the graph with $V(G^k) = V(G)$ and where uv is an edge of G^k if the distance $d_G(u, v)$ between u and v (the length of a shortest $u - v$ path in G) is at most k. The graphs G^2 and G^3 are called the *square* and *cube*, respectively, of G. The *k-step graph* $G^{[k]}$ of G is the graph whose vertex set is $V(G)$ and where uv is an edge of $G^{[k]}$ if $d_G(u, v) = k$. Thus $G^{[1]} = G$. A graph H is a *step graph* of G if $H = G^{[k]}$ for some positive integer k.

In 1960 Sekanina [6] proved that the cube of every connected graph G of order 3 or more is Hamiltonian-connected, that is, every two vertices of G^3 are connected by a Hamiltonian path. As a consequence of this, the cube of every connected graph of order 3 or more is Hamiltonian. The square of a connected graph need not be Hamiltonian however. For example, the square of the graph G of Figure 1 is not Hamiltonian. In 1974 Fleischner [4] proved that the square of every 2-connected graph is Hamiltonian.

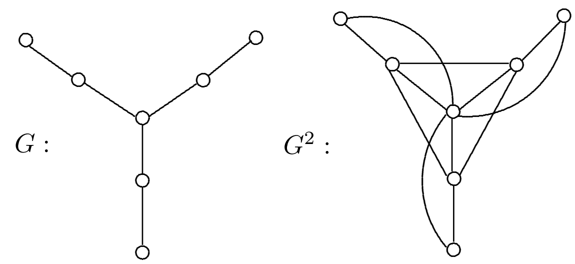

Figure 1: A connected graph whose square is not Hamiltonian

As a consequence of Sekanina's result, it follows that for every connected graph G of order $n \geq 3$, there exists a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ of G^3 such that $v_i v_{i+1}$ belongs to one of the graphs $G^{[1]}$, $G^{[2]}$, $G^{[3]}$ for $i = 1, 2, \ldots, n$.

In [1] for a connected graph G and a positive integer k, the graph G^k is called *distance-colored* if an edge uv of G^k is colored i $(1 \leq i \leq k)$ where $uv \in E(G^{[i]})$. The distance-colored graph G^k is *Hamiltonian-colored* (see [1, 5]) if G^k contains a properly colored Hamiltonian cycle. That is, G^k if Hamiltonian-colored if G^k contains a Hamiltonian cycle C in which every two consecutive edges in C belong to distinct step graphs of G.

In [2] as a solution to a road network problem, it was determined for a grid G (the Cartesian product $P_n \square P_m$ of paths of orders $n \geq 2$ and $m \geq 2$) conditions under which G^3 and G^2 are Hamiltonian-colored. In particular, it was shown in [2] that G^3 is Hamiltonian-colored and that G^2 is Hamiltonian-colored if and only if $nm \equiv 0 \pmod 4$. Equivalently, the latter result shows which grids $G = P_n \square P_m$ have the property that G^3 contains a properly colored Hamiltonian cycle whose edges belong to one of the two step graphs $G^{[1]}$ and $G^{[2]}$. In this paper, we complete the solution of the problem of determining which grids $G = P_n \square P_m$ have the property that G^3 contains properly colored Hamiltonian cycles whose edges belong to two of three specified step graphs $G^{[1]}$, $G^{[2]}$ and $G^{[3]}$. In general, we follow the book [3] for graph theory terminology and notation.

2. Properly Colored Hamiltonian Cycles in Cubes of Grids With Prescribed Conditions

We begin by determining those grids $G = P_n \square P_m$ for which G^3 contains a properly colored Hamiltonian cycle each edge of which belongs to $G = G^{[1]}$ or $G^{[3]}$.

Theorem 2.1. *For integers $n, m \geq 2$, the distance-colored graph $(P_n \square P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 1 and 3 if and only if nm is even unless $n = m = 2$ or $\{n, m\}$ is $\{2, 3\}$ or $\{2, 7\}$.*

Proof. First, observe that if nm is odd, then no Hamiltonian cycle in $(P_n \square P_m)^3$ can be properly colored with only two colors. If $n = m = 2$, then the diameter of $P_n \square P_m$ is 2 and no edge in $(P_n \square P_m)^3$ is colored 3. If $\{n, m\} = \{2, 3\}$, then there are only two edges colored 3 in $(P_n \square P_m)^3$ and so no properly colored Hamiltonian cycle using the colors 1 and 3 exists in $(P_n \square P_m)^3$.

Next, we show the distance-colored graph $(P_2 \square P_7)^3$ does not contains a properly colored Hamiltonian cycle using only the colors 1 and 3. Assume, to the contrary, that such a properly colored Hamiltonian cycle C exists. Let the vertices of $P_2 \square P_7$ be labeled as shown in Figure 2. Necessarily, each vertex of C is incident with an edge colored 1 and an edge colored 3. We consider two cases, according to the edge of C colored 3 that is incident with the vertex x_1.

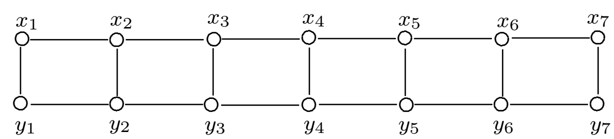

Figure 2: The graph $P_2 \square P_7$.

Case 1**.** The edge x_1x_4 is on C.

(See Figure 3, where we always represent an edge colored 1 by a solid edge and an edge colored 3 by a dotted edge.) Since the only possible edges on C that are colored 3 and incident with y_2 are y_2x_4 and y_2y_5, the cycle C must contain y_2y_5. Since the only possible edges on C that are colored 3 and incident with x_7 are x_4x_7 and y_5x_7, a contradiction is produced.

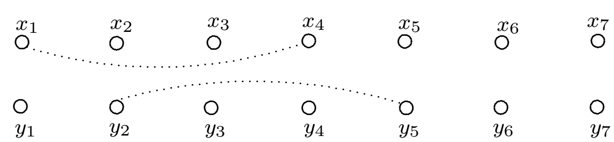

Figure 3: The situation where C contains the edge x_1x_4.

Since Case 1 cannot occur, none of the edges x_1x_4, y_1y_4, x_4x_7 and y_4y_7 can belong to C.

Case 2**.** All of the edges x_1y_3, y_1x_3, x_5y_7 and y_5x_7 belong to C.

(See Figure 4.) Since the only possible edges on C that are colored 3 and incident with x_2 are x_2x_5 and x_2y_4, the edge x_2y_4 must belong to C. Similarly, the only possible edges on C that are colored 3 and incident with x_6 are x_3x_6 and y_4x_6. Therefore, the edge y_4x_6 must belong to C, which produces a contradiction.

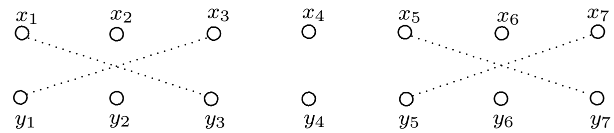

Figure 4: The situation where C contains the edges and x_5y_7, x_1y_3, y_1y_3, y_5y_7.

For the converse, assume that $n \geq 2$ and $m \geq 2$ are integers such that nm is even and such that neither $n = m = 2$ nor $\{n, m\} = \{2, 3\}$ or $\{2, 7\}$ occurs. We show that the distance-colored graph $(P_n \,\square\, P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 1 and 3. Since nm is even, we may assume that n is even. The graph $P_n \,\square\, P_m$ consists of n paths P_m which we denote by $P_{m,i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,m})$ for $1 \leq i \leq n$ such that $v_{i,t}$ is adjacent to $v_{j,t}$ $(1 \leq t \leq m)$ when $|i-j| = 1$. With this labeling, the graph $P_2 \,\square\, P_5$ is shown in Figure 5.

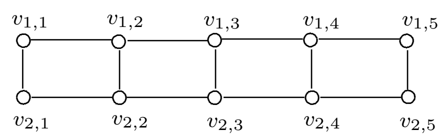

Figure 5: The graph $P_2 \,\square\, P_5$.

We first show by induction on $m \geq 4$ with $m \neq 7$ that the distance-colored graph $(P_2 \,\square\, P_m)^3$ has a properly colored Hamiltonian cycle using only the colors 1 and 3 and containing the edges $v_{1,m}v_{2,m}$, $v_{1,m-2}v_{1,m-1}$ and $v_{2,m-2}v_{2,m-1}$. First, observe that this is true for $m \in \{4, 5, 6, 11\}$ as shown in Figure 6. (Again, the solid edges are colored 1 and dotted edges are colored 3.)

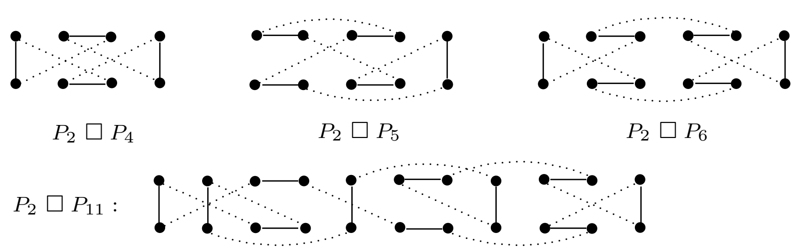

Figure 6: Properly colored Hamiltonian cycles in $(P_2 \,\square\, P_m)^3$ for $m \in \{4, 5, 6, 11\}$ containing $v_{1,m}v_{2,m}$, $v_{1,m-2}v_{1,m-1}$ and $v_{2,m-2}v_{2,m-1}$.

For $m \geq 4$ with $m \neq 7$, assume that $(P_2 \,\square\, P_m)^3$ has a properly colored Hamiltonian cycle C' using the colors 1 and 3 and containing the edges $v_{1,m}v_{2,m}$, $v_{1,m-2}v_{1,m-1}$ and $v_{2,m-2}v_{2,m-1}$. For the properly colored Hamiltonian cycle C'' in $(P_2 \,\square\, P_4)^3$ shown in Figure 7, we delete the edge $v_{1,m}v_{2,m}$ from C' and the edge $v_{1,m+1}v_{2,m+1}$ from C'' and add the edges $v_{1,m}v_{1,m+1}$ and $v_{2,m}v_{2,m+1}$ to produce a properly colored Hamiltonian cycle C

in $(P_2 \square P_{m+4})^3$ containing the edges $v_{1,m+4}v_{2,m+4}$, $v_{1,m+2}v_{1,m+3}$ and $v_{2,m+2}v_{2,m+3}$.

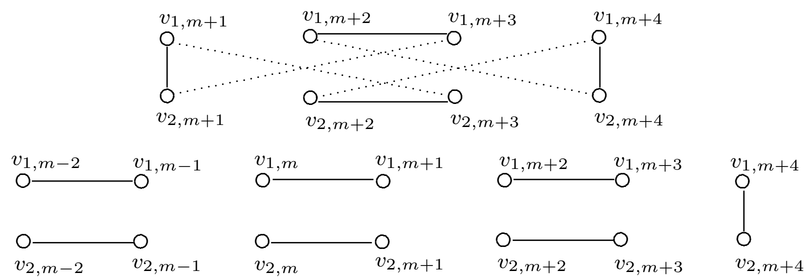

Figure 7: Constructing a properly colored Hamiltonian cycle in $(P_2 \square P_{m+4})^3$.

Therefore, $(P_2 \square P_m)^3$ contains a properly colored Hamiltonian cycle using the colors 1 and 3 and containing the edges $v_{1,m}v_{2,m}$, $v_{1,m-2}v_{1,m-1}$ and $v_{2,m-2}v_{2,m-1}$ for every integer $m \geq 4$ with $m \neq 7$.

Next, we show by induction on even integers $n \geq 2$ that except for $n = m = 2$ and $\{n, m\} = \{2, 3\}, \{2, 7\}$ the distance-colored graph $(P_n \square P_m)^3$ has a properly colored Hamiltonian cycle using the colors 1 and 3 and containing the edge $v_{n,m-2}v_{n,m-1}$. We have seen that this is true for $n = 2$. Assume first that this is true for an even integer $n \geq 2$ and an integer $m \geq 2$ where $m \notin \{2, 3, 7\}$. Then there exists a properly colored Hamiltonian cycle C' in $(P_n \square P_m)^3$ using the colors 1 and 3 and containing the edge $v_{n,m-2}v_{n,m-1}$. Consider the graph $P_2 \square P_m$ whose vertices are labeled as in Figure 8

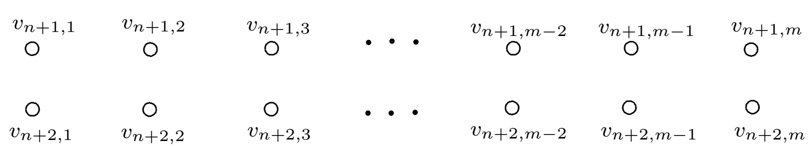

Figure 8: The vertices of a graph $P_2 \square P_m$.

We have seen that when $m \notin \{2, 3, 7\}$, there is a properly colored Hamiltonian cycle C'' in $(P_2 \square P_m)^3$ using only the colors 1 and 3 and containing the edges $v_{n+1,m-2}v_{n+1,m-1}$ and $v_{n+2,m-2}v_{n+2,m-1}$. We now delete the edge $v_{n,m-2}v_{n,m-1}$ of C' and the edge $v_{n+1,m-2}v_{n+1,m-1}$ of C'' and add the two edges $v_{n,m-2}v_{n+1,m-2}$ and $v_{n,m-1}v_{n+1,m-1}$, producing a properly colored Hamiltonian cycle C in $(P_{n+2} \square P_m)^3$ using only the colors 1 and 3 and containing the edge $v_{n+2,m-2}v_{n+2,m-1}$. Consequently, for every even integer $n \geq 2$ and every integer $m \geq 2$ with $m \notin \{2, 3, 7\}$, there is a properly colored Hamiltonian cycle using only the colors 1 and 3 in $(P_n \square P_m)^3$.

To complete the proof, it remains to show that $(P_n \square P_3)^3$ and $(P_n \square P_7)^3$ contain properly colored Hamiltonian cycles using only the colors 1 and 3 for each even integer $n \geq 4$. We verify by induction on even integers $n \geq 4$ that such a cycle exists containing the edge $v_{n,m-2}v_{n,m-1}$ for $m = 3$ and $m = 7$. That this is true for $n = 4$ and $n = 6$ is shown in Figure 9 (where the edges $v_{n,m-2}v_{n,m-1}$ are drawn in bold).

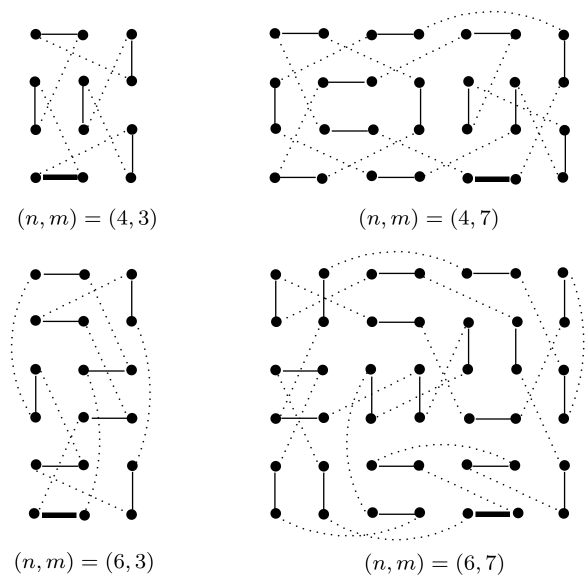

Figure 9: Properly colored Hamiltonian cycles in $(P_n \square P_m)^3$ using only the colors 1 and 3 and containing the edge $v_{n,m-2}v_{n,m-1}$ for four pairs (n, m).

Assume for an even integer $k \geq 6$ that $(P_j \square P_3)^3$ and $(P_j \square P_7)^3$ contain properly colored Hamiltonian cycles using only the colors 1 and 3 and containing the edge $v_{j,m-2}v_{j,m-1}$ for every even integer j with $4 \leq j \leq k$ and $m \in \{3, 7\}$. We show that $(P_{k+2} \square P_3)^3$ and $(P_{k+2} \square P_7)^3$ contain properly colored Hamiltonian cycles using only the colors 1 and 3 and containing the edge $v_{k+2,m-2}v_{k+2,m-1}$ for $m \in \{3, 7\}$. From the induction hypothesis, it follows that $(P_{k-2} \square P_3)^3$ and $(P_{k-2} \square P_7)^3$ contain properly colored Hamiltonian cycles C'_1 and C'_2, respectively, using only the colors 1 and 3 and containing the edge $v_{k-2,m-2}v_{k-2,m-1}$ where $m \in \{3, 7\}$.

Each of the graphs $(P_4 \square P_3)^3$ and $(P_4 \square P_7)^3$ consists of four paths P_m ($m \in \{3, 7\}$) which we denote by $P_{m,i} = (v_{k+i,1}, v_{k+i,2}, \ldots, v_{k+i,m})$ for $-1 \leq i \leq 2$ such that $v_{i,t}$ is adjacent to $v_{j,t}$ ($1 \leq t \leq m$) when $|i - j| = 1$. Then $(P_4 \square P_3)^3$ and $(P_4 \square P_7)^3$ contain properly colored Hamiltonian cycles C''_1 and C''_2, respectively, using only the colors 1 and 3 and containing the edge $v_{k-1,m-2}v_{k-1,m-1}$ and $v_{k+2,m-2}v_{k+2,m-1}$ for $m \in \{3, 7\}$. Deleting the edge $v_{k-2,m-2}v_{k-2,m-1}$ of C'_i and the edge $v_{k-1,m-2}v_{k-1,m-1}$ of C''_i for $i = 1, 2$ and adding the edges $v_{k-2,m-2}v_{k-1,m-2}$ and $v_{k-2,m-1}v_{k-1,m-1}$ result in properly colored Hamiltonian cycles C_1 and C_2 in $(P_{k+2} \square P_3)^3$ and $(P_{k+2} \square P_7)^3$, respectively, using only the colors 1 and 3 and containing the edge $v_{k+2,m-2}v_{k+2,m-1}$ for $m \in \{3, 7\}$, completing the proof. □

We now determine all those grids $G = P_n \square P_m$ possessing the property that G^3 contains a properly colored Hamiltonian cycle all of whose edges belong to $G^{[2]}$ and $G^{[3]}$.

Theorem 2.2. *For integers $n, m \geq 2$, the distance-colored graph $(P_n \square P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 2 and 3 if and only if $nm \equiv 0 \pmod 4$ unless $n = m = 2$.*

Proof. Suppose first that $(P_n \square P_m)^3$ contains a properly colored Hamiltonian cycle C using only the colors 2 and 3. Since the diameter of $P_2 \square P_2$ is 2, it is impossible that $n = m = 2$. Let $C = (v_1, v_2, \ldots, v_{nm-1}, v_{nm}, v_1)$. Since $P_n \square P_m$ is bipartite, it has two partite sets A and B. We may assume that $v_1 \in A$ and that v_1v_2 on C is colored 2. Thus the distance between v_1 and v_2 in $P_n \square P_m$ is 2 and $v_2 \in A$ as well. Since v_2v_3 is colored 3 and v_3v_4 is colored 2, both v_3 and v_4 belong to B. Because v_4v_5 is colored 3, it follows that $v_5 \in A$. This implies that the nm vertices of $P_n \square P_m$ are encountered cyclically on C in groups of 4, the first pair of which belongs to A and the second pair of which belongs to B. Thus $nm \equiv 0 \pmod 4$.

For the converse, let $n, m \geq 2$ be integers with $nm \equiv 0 \pmod 4$ such that $(n, m) \neq (2, 2)$. We show that the distance-colored graph $(P_n \square P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 2 and 3. Since nm is even, we may assume that n is even. The graph $P_n \square P_m$ consists of n paths of order m, which we denote by $P_{m,i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,m})$ for $1 \leq i \leq n$ such that $v_{i,t}$ is adjacent to $v_{j,t}$ $(1 \leq t \leq m)$ when $|i - j| = 1$.

We first show by induction on the even integers $m \geq 4$ that the distance-colored graph $(P_2 \square P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the edges $v_{1,1}v_{2,2}$ and $v_{1,m-1}v_{2,m}$ colored 2. That this is true for $m = 4$ and $m = 6$ is shown in Figure 10 (where the solid edges are colored 2 and dotted edges are colored 3).

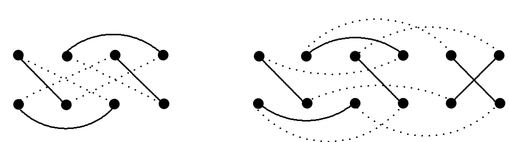

Figure 10: Properly colored Hamiltonian cycles in $(P_2 \square P_m)^3$ for $m = 4, 6$ using only the colors 2 and 3.

Assume for an even integer $m \geq 6$ that for every even integer t with $4 \leq t \leq m$, the distance-colored graph $(P_2 \square P_t)^3$ contains a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the edges $v_{1,1}v_{2,2}$ and $v_{1,t-1}v_{2,t}$. We claim that the distance-colored graph $(P_2 \square P_{m+2})^3$ contains a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the edges $v_{1,1}v_{2,2}$ and $v_{1,m+1}v_{2,m+2}$. By the induction hypothesis, the distance-colored graph $(P_2 \square P_{m-2})^3$ contains a properly colored Hamiltonian cycle C' using only the colors 2 and 3 and containing the edges $v_{1,1}v_{2,2}$ and $v_{1,m-3}v_{2,m-2}$ (see Figure 11). The graph $(P_2 \square P_4)^3$ consists of two paths P_4 which we denote by $(v_{1,m-1}, v_{1,m}, v_{1,m+1}, v_{1,m+2})$ and $(v_{2,m-1}, v_{2,m}, v_{2,m+1}, v_{2,m+2})$, respectively, as shown in Figure 11 where the vertices of $(P_2 \square P_4)^3$ are drawn as solid vertices. Let C''

be the cycle in $(P_2 \,\Box\, P_4)^3$ also shown in Figure 11. We now construct a properly colored Hamiltonian cycle C from C' and C'' by removing the edge $v_{1,m-3}v_{2,m-2}$ from C' and the edge $v_{1,m-1}v_{2,m}$ from C'' and adding the edges $v_{1,m-3}v_{2,m-1}$ and $v_{2,m-2}v_{2,m}$. The cycle C uses only colors 2 and 3 and contains the edges $v_{1,1}v_{2,2}$ and $v_{1,m+1}v_{2,m+2}$. This verifies the claim.

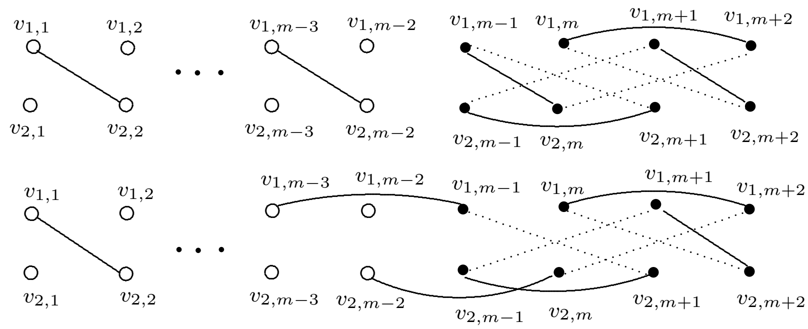

Figure 11: Constructing a properly colored Hamiltonian cycle in $(P_2 \,\Box\, P_{m+2})^3$ using only the colors 2 and 3 and containing the $v_{1,1}v_{2,2}$ and $v_{1,m+1}v_{2,m+2}$.

Next, we show by induction on the even integers $n \geq 2$ that for every even integer $m \geq 2$, except for $m = 2$ when $n = 2$, the distance-colored graph $(P_n \,\Box\, P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the edge $v_{n,1}v_{n,3}$ colored 2. We have already seen that this is true for $n = 2$. Assume that this is true for an even integer $n \geq 2$ and every even integer $m \geq 2$ where $(n, m) \neq (2, 2)$. Then for each even integer $m \geq 2$, there is a properly colored Hamiltonian cycle C' in $(P_n \,\Box\, P_m)^3$ using only the colors 2 and 3 and containing the edge $v_{n,1}v_{n,3}$. For this integer m, let $P_2 \,\Box\, P_m$ consist of two paths P_m which we denote by $(v_{n+1,1}, v_{n+1,2}, \ldots, v_{n+1,m})$ and $(v_{n+2,1}, v_{n+2,2}, \ldots, v_{n+2,m})$, respectively, and $v_{n+1,i}$ is adjacent to $v_{n+2,i}$ for $1 \leq i \leq m$. We have seen that there is a properly colored Hamiltonian cycle C'' in $(P_2 \,\Box\, P_m)^3$ using only the colors 2 and 3 and containing the edges $v_{n+1,2}v_{n+1,4}$ and $v_{n+2,1}v_{n+2,3}$. The cycle in $(P_{n+2} \,\Box\, P_m)^3$ obtained by deleting the edges $v_{n,1}v_{n,3}$ and $v_{n+1,2}v_{n+1,4}$ from C' and C'', respectively, and adding the edges $v_{n,1}v_{n+1,2}$ and $v_{n,3}v_{n+1,4}$ produces a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the edge $v_{n+2,1}v_{n+2,3}$. It therefore follows that for every two even integers $n \geq 2$ and $m \geq 2$ with $(n, m) \neq (2, 2)$, the distance-colored graph $(P_n \,\Box\, P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 2 and 3.

To complete the proof, it remains to show that the distance-colored graph $(P_n \,\Box\, P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 2 and 3 when $n \equiv 0 \pmod 4$ and $m \geq 3$ is odd or equivalently, when $n \geq 3$ is odd and $m \equiv 0 \pmod 4$. We verify this latter formulation of the statement.

First, we show by induction on the integers $m \geq 4$ with $m \equiv 0 \pmod 4$ that the distance-colored graph $(P_3 \square P_m)^3$ has a properly colored Hamiltonian cycle using only the colors 2 and 3. We use the same notation as before for the vertex set of $P_3 \square P_m$. In fact, we show that $(P_3 \square P_m)^3$ has a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the two edges $v_{1,1}v_{2,2}$ and $v_{1,m-1}v_{2,m}$, both colored 2. That this is true for $m = 4$ is shown in Figure 12.

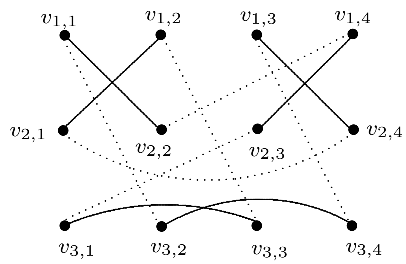

Figure 12: A properly colored Hamiltonian cycle in $(P_3 \square P_4)^3$ using only the colors 2 and 3 and containing the edges $v_{1,1}v_{2,2}$ and $v_{1,3}v_{2,4}$.

Assume, for an integer $m \geq 4$ with $m \equiv 0 \pmod 4$ that the distance-colored graph $(P_3 \square P_m)^3$ has a properly colored Hamiltonian cycle C' using only the colors 2 and 3 and containing the edges $v_{1,1}v_{2,2}$ and $v_{1,m-1}v_{2,m}$. Let $P_3 \square P_4$ consist of three paths P_4 which we denote by $(v_{i,m+1}, v_{i,m+2}, v_{i,m+3}, v_{i,m+4})$ for $i = 1, 2, 3$. We have seen that the distance-colored graph $(P_3 \square P_4)^3$ has a properly colored Hamiltonian cycle C'' using only the colors 2 and 3 and containing the edges $v_{1,m+1}v_{2,m+2}$ and $v_{1,m+3}v_{2,m+4}$. Then the cycle C in $(P_3 \square P_{m+4})^3$ constructed by deleting the edges $v_{1,m-1}v_{2,m}$ and $v_{1,m+1}v_{2,m+2}$ from C' and C'', respectively, and adding the edges $v_{1,m-1}v_{1,m+1}$ and $v_{2,m}v_{2,m+2}$ (both colored 2) is a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the edges $v_{1,1}v_{2,2}$ and $v_{1,m+3}v_{2,m+4}$. This verifies the statement. Furthermore, observe that the cycle C constructed also contains the edge $v_{3,1}v_{3,3}$, which is colored 2.

We now show by induction on the odd integers $n \geq 3$ that for every integer $m \geq 4$ with $m \equiv 0 \pmod 4$ there exists a properly colored Hamiltonian cycle in the distance-colored graph $(P_n \square P_m)^3$ that uses only the colors 2 and 3 and contains the edge $v_{n,1}v_{n,3}$. We have already seen that this is true when $n = 3$. Assume for an odd integer $n \geq 3$ and for every integer $m \geq 4$ with $m \equiv 0 \pmod 4$ there exists a properly colored Hamiltonian cycle C' in the distance-colored graph $(P_n \square P_m)^3$ that uses only the colors 2 and 3 and contains the edge $v_{n,1}v_{n,3}$.

For an integer $m \geq 4$ with $m \equiv 0 \pmod 4$, let $P_2 \square P_m$ consist of two paths P_m which we denote by $P_{m,i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,m})$ for $i = n+1, n+2$. We have seen that $(P_2 \square P_m)^3$ has a properly colored Hamiltonian cycle C'' using only the colors 2 and 3 and containing the edges $v_{n+1,2}v_{n+1,4}$ and $v_{n+2,1}v_{n+2,3}$. By deleting the edges $v_{n,1}v_{n,3}$ and $v_{n+1,2}v_{n+1,4}$ from C' and C'', respectively, and adding the edges $v_{n,1}v_{n+1,2}$ and $v_{n,3}v_{n+1,4}$ (both colored 2), we obtain a properly colored Hamiltonian cycle C in the distance-colored graph $(P_{n+2} \square P_m)^3$ using only the colors 2 and 3 and containing the edge $v_{n+2,1}v_{n+2,3}$. This completes the proof. $\square$

In [2] all pairs $n, m \geq 2$ of integers were determined for which the distance-colored graphs $(P_n \square P_m)^3$ and $(P_n \square P_m)^2$ contain a properly colored Hamiltonian cycle. In the second case, this is equivalent to determining all pairs $n, m \geq 2$ of integers for which the distance-colored graph $(P_n \square P_m)^3$ has a properly colored Hamiltonian cycle the colors of whose edges follow the permutation (1 2). In this paper we have also answered the question for the permutations (1 3) and (2 3). This might suggest another question: For which pairs $n, m \geq 2$ of integers, does the distance-colored graph $(P_n \square P_m)^3$ contain a properly colored Hamiltonian cycle the colors of whose edges follow the permutation (1 2 3)? Of course, this is only possible if $nm \equiv 0 \pmod 3$. Suppose that such a cycle $C = (v_1, v_2, \ldots, v_{nm-1}, v_{nm}, v_1)$ exists. Since $P_n \square P_m$ is bipartite, it has two partite sets A and B. If nm is even, then $|A| = |B|$; while if nm is odd, then one of $|A|$ and $|B|$ exceeds the other by 1, say $|B| = |A| + 1$. We may assume that $v_1 \in A$ and that v_1v_2 is colored 1, which implies that $v_2 \in B$. Thus v_2v_3 is colored 2, v_3v_4 is colored 3 and v_4v_5 is colored 1. Therefore, $v_3 \in B$, $v_4 \in A$ and $v_5 \in B$. This implies that two-thirds of the vertices of $P_n \square P_m$ belong to B and one-third to A, that is, $|B| = 2|A|$. This is impossible if $|A| = |B|$. If $|B| = |A| + 1$, then $|B| = 2$, $|A| = 1$ and $(P_2 \square P_1)^3$ is not Hamiltonian. Hence there are no integers $n, m \geq 2$ for which the distance-colored graph $(P_n \square P_m)^3$ has a properly colored Hamiltonian cycle the colors of whose edges follow the permutation (1 2 3).

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References

[1] G. Chartrand, R. Jones, K. Kolasinski and P. Zhang, On the Hamiltonicity of distance-colored graphs, *Congr. Numer.*, **202** (2010) 129-136.

[2] G. Chartrand, K. Kolasinski and P. Zhang, The colored bridges problem, *Geographical Analysis*, **43** (2011) 370-382.

[3] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & Digraphs*, Fifth Edition, Chapman & Hall/CRC, Boca Raton, FL (2011).

[4] H. Fleischner, The square of every two-connected graph is Hamiltonian, *J. Combin. Theory Ser. B*, **16** (1974) 29-34.

[5] R. Jones, K. Kolasinski and P. Zhang, On Hamiltonian-colored graphs, *Util. Math.*, (To appear).

[6] M. Sekanina, On an ordering of the set of vertices of a connected graph, *Publ. Fac. Sci. Univ. Brno.*, **412** (1960) 137-142.