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Achieving All Radio Numbers

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Abstract

For a connected graph G, a radio labeling is a function $c : V(G) \to \mathbb{Z}^+$ such that for every pair of vertices (u, v) in $V(G)$, the radio condition is satisfied:

$$d(u, v) + |\, c(u) - c(v)\, | \geq diam(G) + 1.$$

The span of a radio labeling c is the largest integer in the image of c. The radio number of a graph G is the smallest integer M such that $span(c) = M$ for some radio labeling c. It is known that a graph of n vertices has a radio number of at least n and at most $\frac{(n-1)^2}{2} + r$, where r is determined by the parity of n. This paper defines and examines three-parameter graphs known as Sok graphs. We show that for all but two integers between the known minimum and maximum, there exists a Sok graph whose radio number is that integer. The results of this work entirely settle the question of what the possible radio numbers are for graphs of order n.

Keywords: Multi-level distance labeling, Radio labelling, Radio number.

2010 Mathematics Subject Classification: 05C78.

1. Introduction

Radio labeling, a type of multilevel distance labeling is first explored by Chartrand, Erwin, Zhang, and Harary in [1]. The motivation for radio labeling comes from the restrictions that must be considered when assigning frequencies to radio channels. Neighbouring channels require large frequency differences, while channels at large distances from one another may have channel frequencies that differ only slightly. This situation is modelled using connected graphs $G = (V(G), E(G))$. We write $d(u, v)$ for the *distance* between vertices u and v, and use $diam(G)$ to indicate the *diameter* of G(the maximum distance over all pairs of verticies).

For a connected graph G, a *radio labeling* is a function $c : V(G) \to \mathbb{Z}^+$ such that for every pair of vertices (u, v) in $V(G)$, the *radio condition* is satisfied:

$$d(u, v) + |c(u) - c(v)| \geq diam(G) + 1.$$

The *span* of a radio labeling c is the largest integer in the image of c. The *radio number* of a graph G is the smallest integer M such that $span(c) = M$ for some radio labeling c. Note that the diameter of the graph is always at least as large as the distance between two vertices. Thus the radio condition requires that $|c(u) - c(v)| \geq 1$ for all pairs (u, v). This implies that each label must be distinct, thus the radio number of a graph of order n is always at least n. This minimum can be realized with a complete graph of order n. The diameter of this graph is 1, so as long as the labels all differ by 1, the radio condition is satisfied.

In [1], the authors provide an upper bound for the radio number of paths. Liu and Zhu determine that this bound is in fact the radio number in [4]. The radio number of a path of order n is $\frac{(n-1)^2}{2} + \frac{3}{2}$ when n is even or $\frac{(n-1)^2}{2} + 3$ when n is odd [4]. Further, in [5], Canales, Tomova and Wyels determine that the path has the highest radio number of any graph on n vertices. In recent years, explorations have been done with the goal of determining which intermediate values are achievable, i.e. what the possible radio numbers for a graph of order n are. One important result of these explorations is that for graphs of odd order, the radio number $\frac{(n-1)^2}{2} + 2$ is not achievable [5]. In this paper we will prove that every other intermediate integer is achievable for graphs of both even and odd order.

This paper focuses on a type of graph we have called a Sok graph, which consists of a complete or semi-complete graph between two path graphs. We have divided the body of this paper into three sections. The first determines the radio numbers of Sok graphs that do not have edges removed from the central complete subgraph. The second modifies the initial graphs by removing edges, thus allowing the modified Sok graphs to achieve a multitude of radio numbers. We also determine the relatively small number of intermediate integers these graphs do not achieve. Finally, in Section 4, we define one more specific family of Sok graphs with radio numbers that are the integers not achieved by the graphs analyzed in Section 3. The final conclusion of this paper is that all intermediate integers except $\frac{(n-1)^2}{2} + 2$ are achievable.

2. Exploring the Basic Sok Graphs

In this section we define two basic graph types we call Sok graphs and determine their radio numbers. We will later modify these graphs to show that all integers under consideration are achievable radio numbers. Sok graphs are characterized by both their order and their diameter. Graphs with diameters of different parity must be treated separately. First we introduce a lower bound technique known as distance maximization that will be used throughout the paper.

The technique was commonly used in several articles on radio labeling (cf. [4] [3]); the proof follows directly from the definition of radio labeling (cf. [2]).

Lemma 1. *For any graph G of order n and diameter α, we have*

$$rn(G) \geq (n-1)(\alpha+1)+1-max_p\left(\sum_{i=1}^{n-1} d(x_i, x_{i+1})\right), \tag{2.1}$$

where the maximum is taken over all bijections p from the vertices of the graph to the set $\{x_1, x_2, ..., x_n\}$.

To simplify notation, for a labeling c with corresponding bijection $p : V(G) \rightarrow \{x_1, x_2, ..., x_n\}$, we say that $v_i = x_j$ if $p(v_i) = x_j$.

2.1. Sok Graphs with Even Diameter

We begin by determining the radio numbers of Sok graphs with even diameters. To do this, we first define the graphs in question and find then equate upper and lower bounds for the radio numbers of these graphs.

Definition 2. *Let an even-Sok graph $S_{n,\alpha}$, where $n > \alpha$, be a graph of order n and even diameter $\alpha = 2p$ with vertex sets $V_1 = \{v_1, v_2, v_3...v_p, v_{p+2}...v_{2p+1}\}$ and $V_2 = \{w_1, w_2....w_{n-2p}\}$ along with edge sets $E_1 = \{v_iv_{i+1} \mid 1 \leq i \leq 2p, i \neq p, p+1\}$, $E_2 = \{w_iw_j \mid 1 \leq i < j \leq n-2p\}$, $E_3 = \{v_pw_i \mid 1 \leq i \leq n-2p\}$, $E_4 = \{w_iv_{p+2} \mid 1 \leq i \leq n-2p\}$. We refer to elements of V_1 as* path verticies *and elements of V_2 as* web verticies*.*

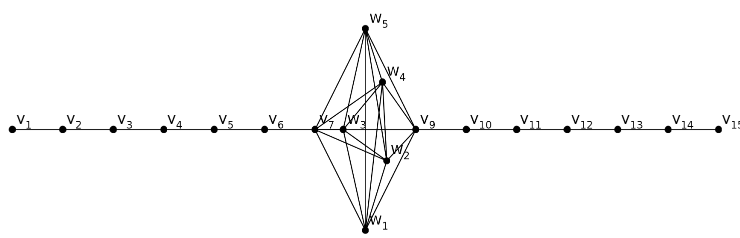

Figure 2.1: $S_{19,14}$

Theorem 3. *For positive integers n and α, with $n > \alpha$ and α even, we have*

$$rn(S_{n,\alpha}) \geq \frac{-\alpha^2}{2} + (n-1)\alpha + 2. \tag{2.2}$$

Proof. To find the maximum sum of distances used in our lower bound formula, we first define a function $\sigma : V(G) \rightarrow \{1, 2, ..., 2p+1\}$ where $\sigma(v_i) = i$ for all v_i in V_1, and $\sigma(w_j) = p+1$ for all w_j in V_2.

We may now compute the maximum distance sum by noting

1. $d(v_i, v_j) = |\,\sigma(v_i) - \sigma(v_j)\,|$ for $v_i, v_j \in V_1$
2. $d(v_i, w_j) = |\,\sigma(v_i) - \sigma(w_j)\,|$ for $v_i \in V_1$, and $w_j \in V_2$
3. $d(w_i, w_j) = |\,\sigma(w_i) - \sigma(w_j)\,| + 1$ for $w_i, w_j \in V_2$

We leave the third quantity unsimplified so that all three cases have a similar absolute value term. Using these quantities we obtain

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) = \sum_{i=1}^{n-1} |\,\sigma(x_i) - \sigma(x_{i+1})\,| + q,$$

where q is the additional distance contributed from the extra 1s in distance case 3. In order to determine an upper bound for the maximum sum of distances, we split the sum into two parts and maximize each part individually.

Part 1. Maximizing the sums of absolute values.

The quantity $\sum_{i=1}^{n-1} |\,\sigma(x_i) - \sigma(x_{i+1})\,|$ contains a total of $2n-2$ terms of the form $\sigma(x_i)$. Each vertex in G will contribute two σ terms except the vertices x_1 and x_n, which will each contribute one term to the sum. Because of the absolute values, half of the σ terms must be added and half must be subtracted. Thus to maximize distances, we let all values of size at least $p+2$ be added and all values of size p or less be subtracted. The remaining values, all of size $p+1$, are then split evenly between addition and subtraction. In order to achieve the maximum we must have $\sigma(x_1) = p+1$ and $\sigma(x_n) = p+1$, because allowing any other values to occur only once would decrease our distance sum. Thus our distribution is

Addition. $2p+1, 2p+1, 2p, 2p, ..., p+3, p+3, p+2, p+2, (p+1)(n-\alpha-1)$

Subtraction. $1, 1, 2, 2....p, p, (p+1)(n-\alpha-1)$

Taking the difference between the added and subtracted values we obtain a maximum of $2p(p+1)$ for Part 1.

Part 2. Maximizing q

We have some number of 1s to add to our sum due to the increase of 1 in distance case 3. The number of extra 1s is maximized if and only if the web vertices are labeled consecutively. In this case there will be $n-\alpha-1$ pairs of the form (x_i, x_{i+1}) where both x_i and x_{i+1} are elements of V_2. Each of these pairs contributes 1 to q. Thus the maximum for Part 2 is $n-\alpha-1$.

If both parts of the sum are maximized, we have a maximum sum of distances of

$$2p^2 + 2p + n - \alpha - 1 = \frac{\alpha^2}{2} + n - 1.$$

Substituting this into Lemma 1 we obtain

$$\begin{aligned} rn(S_{n,\alpha}) &\geq (n-1)(\alpha+1)+1-\frac{\alpha^2}{2}-n+1 \\ &= \frac{-\alpha^2}{2}+n\alpha-\alpha+n-1+1-n+1 \\ &= \frac{-\alpha^2}{2}+(n-1)\alpha+1. \end{aligned}$$

We will now argue that this lower bound can be increased by 1. Note that to achieve the current lower bound, there must exist a labeling order that maximizes both Parts 1 and 2.

Suppose that there exists a labeling c that maximizes both Part 1 and Part 2. The maximization of Part 1 implies that $\sigma(x_1) = \sigma(x_n) = p+1$. The maximization of Part 2 implies that all of the vertices in V_2 are labeled consecutively. Since $|V_2| < n$, it is impossible that both x_1 and x_n are in V_2. However $\sigma(x_i) = p+1$ implies $x_i \in V_2$. This implies that it is not that case that $\sigma(x_1) = \sigma(x_n) = p+1$, which is a contradiction. Thus no labeling can maximize both parts of the distance sum. Thus we can conclude

$$rn(S_{n,\alpha}) \geq \frac{-\alpha^2}{2}+(n-1)\alpha+2.$$

□

We now have the desired lower bound. In order to show that this is the radio number of an even-Sok graph, we will provide a matching upper bound.

Theorem 4. *For positive integers n and α, with $n > \alpha$ and α even, we have*

$$rn(S_{n,\alpha}) = \frac{-\alpha^2}{2}+(n-1)\alpha+2. \tag{2.3}$$

Proof. We will provide a radio labeling whose span is equal to the lower bound given in Theorem 3. The table below defines a labeling order and provides the distance between consecutively labeled vertices. The fourth column is calculated by determining the smallest label difference for which the radio condition holds for consecutively labeled vertices.

To ensure that this is a radio labeling, we must show that the radio condition is satisfied for any pair of vertices. First note that because the sum of the third and fourth columns in every row is $\alpha+1$, the radio condition is satisfied for consecutively labeled vertices. By adding the entries in the fourth column of the rows corresponding to x_i and x_{i+1} we see that the label difference between vertices x_i and x_{i+2} is at least α. Since the distance between these two vertices is at least 1, the sum of the distance and label difference is at least $\alpha+1$, which means the radio condition is satisfied for all pairs (x_i, x_{i+2}) where $1 \leq i \leq n-2$. Similarly, if the labeling indices differ by more than two then the label

x_i	Vertex Name	$d(x_i, x_{i+1})$	$c(x_{i+1}) - c(x_i)$
x_1	$w_{n-\alpha}$	$\frac{\alpha}{2}$	$\frac{\alpha}{2}+1$
x_2	v_1	$\frac{\alpha}{2}+1$	$\frac{\alpha}{2}$
x_3	v_{p+2}	$\frac{\alpha}{2}$	$\frac{\alpha}{2}+1$
x_4	v_2	$\frac{\alpha}{2}+1$	$\frac{\alpha}{2}$
x_5	v_{p+3}	$\frac{\alpha}{2}$	$\frac{\alpha}{2}+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2p}	v_p	$\frac{\alpha}{2}+1$	$\frac{\alpha}{2}$
x_{2p+1}	v_{2p+1}	$\frac{\alpha}{2}$	$\frac{\alpha}{2}+1$
x_{2p+2}	w_1	1	α
x_{2p+3}	w_2	1	α
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-1}	$w_{n-\alpha-2}$	1	α
x_n	$w_{n-\alpha-1}$		

Table 1: Labeling Scheme for Even-Sok Graphs

difference is greater than $\alpha + 1$ and the radio conditions is satisfied for all pairs (x_i, x_j) where $1 \leq i \leq n-3$ and $i \leq j+3$.

Thus the table above provides a radio labeling of $S_{n,\alpha}$. To determine the span of this labeling we sum the fourth column and add 1 for the value $c(x_1) = 1$. Vertices x_1 through x_{2p} form $\frac{\alpha}{2}$ pairs each of whose sum is $\alpha+1$. Vertex x_{2p+1} contributes an additional $\frac{\alpha}{2}+1$ Finally there are $n-\alpha-2$ vertices from x_{2p+2} to x_{n-1}, each of which contribute α to the sum. From this we obtain

$$\begin{aligned} span(c) &= (\alpha+1)\left(\frac{\alpha}{2}\right) + \left(\frac{\alpha}{2}+1\right) + (n-\alpha-2)(\alpha) + 1 \\ &= \frac{-\alpha^2}{2} + (n-1)\alpha + 2. \end{aligned}$$

As this matches our lower bound, we conclude

$$rn(S_{n,\alpha}) = \frac{-\alpha^2}{2} + (n-1)\alpha + 2.$$

□

2.2. Sok Graphs of Odd Diameter

The analogous graphs with odd diameter are very similar, but the analysis is slightly simpler because we do not need to provide a labeling. The primary difference between the graphs types is that V_1 contains an odd number of vertices and hence the path on one side of the web has one more vertex than the path on the other side.

Definition 5. *Let an odd-Sok graph* $S_{n,\alpha}$*, where* $n > \alpha$ *and* $\alpha \geq 3$*, be a graph of order* n *and odd diameter* $\alpha = 2p-1$ *with vertex sets* $V_1 = \{v_1, v_2 ... v_{p-1}, v_{p+1} ... v_{2p}\}$ *and* $V_2 = \{w_1, w_2, w_3, ..., w_{n-\alpha}\}$ *along with edge sets* $E_1 = \{v_i v_{i+1}; 0 \leq i \leq 2p-1,\ i \neq p-1, p\}$*,* $E_2 = \{w_i w_j; 1 \leq i \leq n-\alpha\}$*,* $E_3 = \{v_{p-1} w_i; 1 \leq i < j \leq n-\alpha\}$*,* $E_4 = \{w_i v_{p+1}; 1 \leq i \leq n-\alpha\}$*. We refer to elements of* V_1 *as path vertices and elements of* V_2 *as web vertices.*

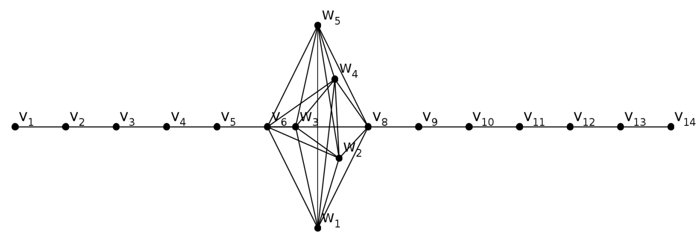

Figure 2.2: $S_{18,\,13}$

Theorem 6. *For positive integers* n *and* α*, with* $n > \alpha \geq 3$ *and* α *odd, we have*

$$rn(S_{n,\alpha}) = \frac{-\alpha^2}{2} + (n-1)\alpha + \frac{3}{2}. \tag{2.4}$$

Proof. We will again use distance maximization to provide a lower bound for these graphs. We define $\tau : V(G) \to \{1, 2, ...2p\}$ where $\tau(v_i) = i$ for all v_i in V_1, and $\tau(w_j) = p$ for all w_j in V_2.

Just as in Section 2.1, for any pair of vertices in $S_{n,\alpha}$ we have the following distances:

1. $d(v_i, v_j) = |\,\tau(v_i) - \tau(v_j)\,|$ for $v_i, v_j \in V_1$
2. $d(v_i, w_j) = |\,\tau(v_i) - \tau(w_j)\,|$ for $v_i \in V_1$, and $w_j \in V_2$
3. $d(w_i, w_j) = |\,\tau(w_i) - \tau(w_j)\,| + 1$ for $w_i, w_j \in V_2$

We will again separate the distance sum into two parts and determine the maximum for each part

Part 1. Maximizing the sums of absolute values.

As in Section 2.1, this sum contains a total of $2n-2$ terms of the form $\tau(x_i)$, half of which must be added and half of which must be subtracted. Thus to maximize distances we let all values of size at least $p+1$ be added and all values of size $p-1$ or less, along with two values of size p, be subtracted. The remaining values, all of size p, are then split evenly between values to be added and those to be subtracted. In order to achieve the maximum, we must have $\tau(x_1) = p$ and $\tau(x_n) = p$, because allowing any other values to occur only once would decrease our distance sum. Thus our distribution is

Addition. $2p, 2p, 2p-1, 2p-1, ..., p+3, p+3, p+2, p+2, p+1, p+1, p(n-\alpha-2)$,

Subtraction. $1, 1, 2, 2....p-1, p(n-\alpha)$

Thus the maximum distance for Part 1 is $2p^2$.

Part 2. Maximizing the sum of additional one values.

We have some number of 1s to add to our sum due to the increase of 1 in distance case (3). This maximum value is again $n-\alpha-1$.

Since we are determining a lower bound, and the sum of distances is subtracted, we may assume that both quantities can be maximized simultaneously.

This gives us a maximum sum of distances of $2p^2+n-\alpha-1 = \frac{(\alpha+1)^2}{2}+n-\alpha-1$. Applying this to Lemma 1 we obtain

$$\begin{aligned} rn(S_{n,\alpha}) &\geq (n-1)(\alpha+1)+1-\frac{(\alpha+1)^2}{2}-n+\alpha+1 \\ &= \frac{-\alpha^2}{2}+n\alpha-\alpha+n-1+1-\alpha-\frac{1}{2}-n+\alpha+1 \\ &= \frac{-\alpha^2}{2}+(n-1)\alpha+\frac{1}{2}. \end{aligned}$$

Again we must argue that the lower bound can be increased by one. The argument is identical to the argument in Section 2.1 except $p+1$ is replaced with p. The maximization of Part 1 implies that $\tau(x_1) = \tau(x_n) = p$. The maximization of Part 2 implies that it is not that case that $\tau(x_1) = \tau(x_n) = p$. Thus no labeling can maximize both parts of the distance sum, and we can increase our lower bound by one. We conclude

$$rn(S_{n,\alpha}) \geq \frac{-\alpha^2}{2}+(n-1)\alpha+\frac{3}{2}.$$

It has been shown that the maximum radio number for a graph of order n and odd diameter α is precisely $\frac{-\alpha^2}{2}+(n-1)\alpha+\frac{3}{2}$ [5]. Thus we may conclude:

$$rn(S_{n,\alpha}) = \frac{-\alpha^2}{2}+(n-1)\alpha+\frac{3}{2}.$$

□

3. Achieving the intermediate integers

We will now modify the Sok graphs examined in Section 2 to fill the number line of achievable radio numbers. We will discuss the case where the diameter is even in detail and outline the changes that are required to apply this idea to graphs of odd diameter.

3.1. Radio Numbers of M-Even Sok Graphs

The new family of graphs we define are subgraphs of even-Sok graphs with some number of edges removed between web vertices. The purpose of removing these edges is to decrease the radio numbers of the graphs in question.

Definition 7. *Let the* m-even-Sok graph, $S_{n,\alpha,m}$, *for some* $1 \le m \le n - \alpha - 2$, *be the even-Sok graph* $S_{n,\alpha}$ *with edges* $\{w_i w_{i+1} \,|\, 1 \le i \le m\}$ *removed from* E_2.

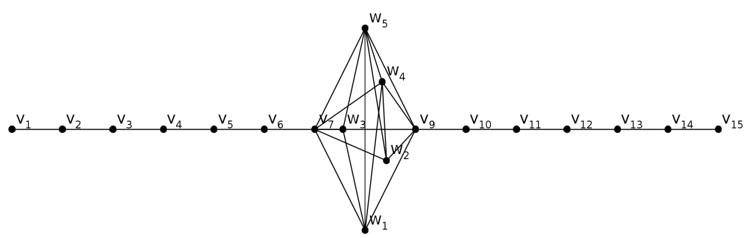

Figure 3.1: $S_{19,14,3}$

Note that the order and diameter of $S_{n,\alpha,m}$ are n and α, respectively.

Theorem 8. *For positive integers* n, α *and* m, *with* $n - \alpha - 2 \ge m$, *and* α *even, we have*

$$rn(S_{n,\alpha,m}) = \frac{-\alpha^2}{2} + (n-1)\alpha + 2 - m. \tag{3.5}$$

Proof. We will use distance maximization to determine a lower bound for these graphs. The σ function is identical to that of Section 2.1 and the distance between vertices is very similar, except now we have:

1. $d(v_i, v_j) = |\,\sigma(v_i) - \sigma(v_j)\,|$ for $v_i, v_j \in V_1$
2. $d(v_i, w_j) = |\,\sigma(v_i) - \sigma(w_j)\,|$ for $v_i \in V_1,\ w_j \in V_2$
3. $d(w_i, w_{i+1}) = |\,\sigma(w_i) - \sigma(w_{i+1})\,| + 2$ for $w_i, w_{i+1} \in V_2$; $1 \le i \le m$
4. $d(w_i, w_j) = |\,\sigma(w_i) - \sigma(w_j)\,| + 1$ for $w_i, w_j \in V_2$, $i > m$ or $i \ne j \pm 1$

As in Section 2.1, we have

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) = \sum_{i=1}^{n-1} |\,\sigma(x_i) - \sigma(x_{i+1})\,| + q,$$

except here q is the additional sum from distance cases 3 and 4. We again split the distance sum into two parts and maximize each part separately.

Part 1. The maximum of absolute values is identical to that of Section 2.1 and yields a value of $2p(p+1)$.

Part 2. Maximizing q.

As in Section 2.1, if the vertices in V_2 are labeled consecutively, there are a total of $n-\alpha-1$ pairs of the form (x_i, x_{i+1}) where both x_i and x_{i+1} are elements of V_2. However, in this case up to m pairs will contribute 2 to q, while the remaining pairs will contribute 1. Thus the maximum sum is $2m + 1(n-\alpha-1-m) = n-\alpha-1+m$. Applying this to Lemma 1 we obtain

$$rn(S_{n,\alpha,m}) \geq \frac{-\alpha^2}{2} + (n-1)\alpha + 1 - m.$$

Since all of the vertices in V_2 must again be labeled consecutively in order to maximize Part 2, an argument identical to the one used in the last paragraph of the proof of Theorem 3 allows us to conclude that no labeling order maximizes both parts of the sum simultaneously, thus allowing us to conclude:

$$rn(S_{n,\alpha,m}) \geq \frac{-\alpha^2}{2} + (n-1)\alpha + 2 - m.$$

We provide a labeling to show that this bound is tight. The labeling order here is identical to that of Section 2.1.

x_i	Vertex Name	$d(x_i, x_{i+1})$	$c(x_{i+1}, x_i)$
x_1	$w_{n-\alpha}$	$\frac{\alpha}{2}$	$\frac{\alpha}{2}+1$
x_2	v_1	$\frac{\alpha}{2}+1$	$\frac{\alpha}{2}$
x_3	v_{p+2}	$\frac{\alpha}{2}$	$\frac{\alpha}{2}+1$
x_4	v_2	$\frac{\alpha}{2}+1$	$\frac{\alpha}{2}$
x_5	v_{p+3}	$\frac{\alpha}{2}$	$\frac{\alpha}{2}+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2p}	v_p	$\frac{\alpha}{2}+1$	$\frac{\alpha}{2}$
x_{2p+1}	v_{2p+1}	$\frac{\alpha}{2}$	$\frac{\alpha}{2}+1$
x_{2p+2}	w_1	2	$\alpha-1$
x_{2p+3}	w_2	2	$\alpha-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2p+m+1}	w_m	2	$\alpha-1$
x_{2p+m+2}	w_{m+1}	1	α
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-1}	$w_{n-\alpha-2}$	1	α
x_n	$w_{n-\alpha-1}$		

Table 2: Labeling Scheme for M-Even-Sok Graph

The argument that this is actually a radio labeling is identical to that of Section 2.1. Summing and adding one, we obtain a span of $\frac{-\alpha^2}{2}+(n-1)\alpha+2-m$, which allows us to conclude

$$rn(S_{n,\alpha,m}) = \frac{-\alpha^2}{2} + (n-1)\alpha + 2 - m.$$

□

3.2. Radio Numbers of M-Odd Sok Graphs

We now define the analogous Sok graphs with odd diameter. The changes between this section and Section 2.2 are almost identical to the changes between Sections 2.1 and 3.1 . We will run through the arguments quickly so the reader can note the changes.

Definition 9. *Let the m-odd-Sok graph $S_{n,\alpha,m}$, for some $1 \le m \le n-\alpha-2$, be an odd-Sok graph $S_{n,\alpha}$ with edges $\{w_i w_{i+1} \,|\, 1 \le i \le m\}$ removed from E_2.*

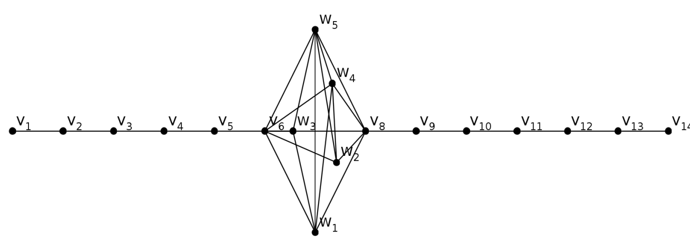

Figure 3.2: $S_{18,13,3}$

Distance maximization will give us a Part 1 maximum of $2p^2$, just as it did for odd-Sok graphs in Section 2.2. Here the Part 2 maximum is the same as in the even case, $n-\alpha-1+m$. In total we get a maximum sum of distances of $\frac{(\alpha+1)^2}{2}+n-\alpha-1+m$.

Again applying Lemma 2 we obtain our lower bound:

$$rn(S_{n,\alpha,m}) \ge \frac{-\alpha^2}{2} + (n-1)\alpha + \frac{1}{2} - m.$$

We again have a contradiction in our assumption that both parts of the distance sum are maximized simultaneously. Thus we conclude

$$rn(S_{n,\alpha,m}) \ge \frac{-\alpha^2}{2} + (n-1)\alpha + \frac{3}{2} - m.$$

We provide a slightly modified labeling order to prove that this is in fact the radio number. See Table 3 for the labeling.

x_i	Vertex Name	$d(x_i, x_{i+1})$	$c(x_{i+1}, x_i)$
x_1	v_{p+1}	$\frac{\alpha+1}{2}$	$\frac{\alpha+1}{2}$
x_2	v_1	$\frac{\alpha+3}{2}$	$\frac{\alpha-1}{2}$
x_3	v_{p+2}	$\frac{\alpha+1}{2}$	$\frac{\alpha+1}{2}$
x_4	v_2	$\frac{\alpha+3}{2}$	$\frac{\alpha-1}{2}$
x_5	v_{p+3}	$\frac{\alpha+1}{2}$	$\frac{\alpha+1}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2p}	v_{2p}	$\frac{\alpha+1}{2}$	$\frac{\alpha+1}{2}$
x_{2p+2}	w_1	2	$\alpha - 1$
x_{2p+3}	w_2	2	$\alpha - 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2p+m+1}	w_m	2	$\alpha - 1$
x_{2p+m+2}	w_{m+1}	1	α
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-1}	$w_{n-\alpha-1}$	1	α
x_n	$w_{n-\alpha-2}$		

Table 3: Labeling Scheme for M-Odd-Sok Graph

The argument that this is in fact a radio labeling is the same as that in Section 2.1. Summing and adding 1 we obtain

$$\begin{aligned} span(c) &= p(\alpha) + (\alpha-1)m + \alpha(n-\alpha-m-1) + 1 \\ &= \frac{-\alpha^2}{2} + (n-1)\alpha + \frac{3}{2} - m. \end{aligned}$$

Thus our upper bound matches the lower bound as desired. We conclude

$$rn(S_{n,\alpha,m}) = \frac{-\alpha^2}{2} + (n-1)\alpha + \frac{3}{2} - m.$$

3.3. Consequences to the Number Line

Here we will examine to what extent the radio numbers of Sok graphs fill the number line of possibly achievable radio numbers. For graphs with an even diameter α, we know that the maximum radio number of a Sok graph is

$$\frac{-\alpha^2}{2} + (n-1)\alpha + 2.$$

Sok graphs of this diameter fill every radio number until m reaches its maximum of $n-\alpha-2$.

Thus the minimum radio number for a Sok graph with even diameter α is

$$\frac{-\alpha^2}{2} + n\alpha - n + 4.$$

To see whether any integers are not achieved as radio numbers, we determine the maximum radio number of a Sok graph with odd diameter $\alpha - 1$. The maximum radio number of such a graph is

$$-\frac{(\alpha-1)^2}{2} + (n-1)(\alpha-1) + \frac{3}{2} = \frac{-\alpha^2}{2} + n\alpha - n + 2.$$

Thus the only integer that is not a radio number of a Sok graph when decreasing the diameter of a graph from an even number to an odd number is $\frac{-\alpha^2}{2} + n\alpha - n + 3$.

Similar analysis applies for decreasing the diameter from an odd number to an even number. For a graph with odd diameter we have a radio number of

$$\frac{-\alpha^2}{2} + (n-1)\alpha + \frac{3}{2} - m.$$

The minimum radio number of a Sok graph with this diameter is obtained when m is $n - \alpha - 1$. Thus our minimum is

$$\frac{-\alpha^2}{2} + n\alpha - n + \frac{5}{2}.$$

The maximum for a graph of even diameter $\alpha - 1$ is

$$-\frac{(\alpha-1)^2}{2} + (n-1)(\alpha-1) + 2 = \frac{-\alpha^2}{2} + n\alpha - n + \frac{5}{2}.$$

Thus when α is odd, every integer between the minimum for diameter $\alpha - 1$ and maximum for diameter α is the radio number of a Sok graph. The rest of the paper will focus on designing graphs with radio numbers of the form $\frac{-\alpha^2}{2} + n\alpha - n + 3$ when α is even.

4. The Final Set of Achievable Integers

In the case in which the diameter of the graph is even, the labeling provided only decreases with increasing m for $m \leq n - \alpha - 2$. Removing the edge between $w_{n-\alpha-1}$ and $w_{n-\alpha}$ would make the distance maximizing lower bound two lower than the radio number and thus the radio number would remain the same as the radio number before the edge was removed. We will not prove that fact but will instead consider one more graph family to achieve the integers left by this restriction.

Definition 10. *For $\alpha \geq 4$, Let an* empty-Sok graph *$E_{n,\alpha}$ be an even-Sok graph $S_{n,\alpha}$ with $\{w_i w_{i+1} \,|\, 1 \leq i \leq n - \alpha - 1\}$ removed from E_2 and the edge $\{v_p w_{n-\alpha}\}$ removed.*

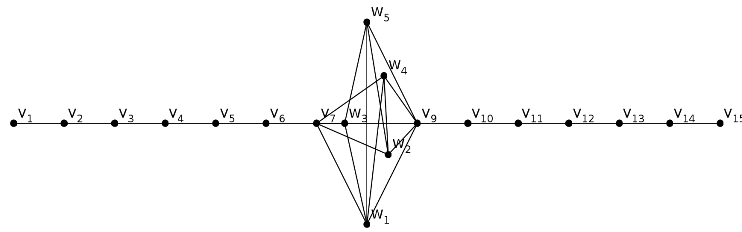

Figure 4.1: $E_{19,14}$

Theorem 11. *For positive integers n and α, with α even and $\alpha \geq 4$, we have*

$$rn(E_{n,\alpha}) = \frac{-\alpha^2}{2} + n\alpha - n + 1. \tag{4.6}$$

Proof. To determine the distance maximizing lower bound we will again begin by determining the distance between any pair of vertices. We use the σ function from Sections 2.1 and 3.1. In this case we have the following distances:

1. $d(v_i, v_j) = |\,\sigma(v_i) - \sigma(v_j)\,|$ for $v_i, v_j \in V_1$
2. $d(v_i, w_j) = |\,\sigma(v_i) - \sigma(w_j)\,|$ for $v_i \in V_1,\ w_j \in V_2 \setminus \{w_{n-\alpha}\}$
3. $d(w_i, w_{i+1}) = |\,\sigma(w_i) - \sigma(w_{i+1})\,| + 2$ for $w_i, w_{i+1} \in V_2$
4. $d(w_i, w_j) = |\,\sigma(w_i) - \sigma(w_j)\,| + 1$ for $w_i, w_j \in V_2,\ i \neq j \pm 1$
5. $d(v_i, w_{n-\alpha}) = |\,\sigma(v_i) - \sigma(w_{n-\alpha})\,| + 1$ for $v_i \in \{v_1, v_2..., v_p\}$
6. $d(v_i, w_{n-\alpha}) = |\,\sigma(v_i) - \sigma(w_{n-\alpha})\,|$ for $v_i \in \{v_{p+2}, v_{p+3}..., v_{2p+1}\}$.

Again we split the maximizing of distances into two parts. The maximization of absolute values is identical to Sections 1.1 and 2.1 and obtains a value of $2p(p+1)$. Here, Part 2 is again maximized by having $n-\alpha-1$ pairs (x_i, x_{i+1}) with two vertices in V_2 along with the pair $(v_i, w_{n-\alpha})$ where v_i in $\{v_1, v_2, ..., v_p\}$. The pair of vertices in V_2 each contribute 2 and the additional pair contributes 1. This gives us a maximum for part 2 of $2(n-\alpha-1)+1$. Thus our upper bound for the maximum sum of distances is $2p^2 + 2p + 2n - 2\alpha - 1 = \frac{\alpha^2}{2} + 2n - \alpha - 1$. Placing this in the lower bound formula we obtain

$$\begin{aligned} rn(E_{n,\alpha}) &\geq (n-1)(\alpha+1) + 1 - \frac{\alpha^2}{2} - 2n + \alpha + 1 \\ &= \frac{-\alpha^2}{2} + n\alpha - \alpha + n - 1 + 1 - 2n + \alpha + 1 \\ &= \frac{-\alpha^2}{2} + n\alpha - n + 1. \end{aligned}$$

We need to show that this lower bound must be increased by 2. First note that in order for a span within one of the lower bound to be achieved, one of the two distance sums must be maximized. Thus we will consider two cases for the two separate sums of distances.

Case 1. Suppose the distance sum in part 1 is maximized

This tells us that both x_1 and x_n contribute $p+1$ and are hence elements of V_2. Thus there are at least two pairs of consecutively labeled vertices that contain one element of V_1 and one element of V_2. Thus there are at most $n-\alpha-2$ pairs with two vertices from V_2. Thus pairs of vertices in V_2 contribute a maximum of $2(n-\alpha-2)$. The only other pair that can contribute to the sum in Part 2 is a pair with $w_{n-\alpha}$ and a vertex in V_1 which would contribute an additional one to the sum. Thus the maximum of the distance sum in Part 2, given that the distance sum in Part 1 is maximized, is $2n-2\alpha-3$. This is two lower than our original maximum for Part 2. Thus any labeling that maximizes the sum of distances in Part 1 has a span at least two greater than our original lower bound.

Case 2. Suppose the sum in Part 2 is maximized.

Then as for all Sok graphs, the vertices of V_2 must be labeled consecutively, thus it is not that case that $\sigma(x_1) = \sigma(x_n) = p+1$. If neither $\sigma(x_1)\,\text{or}\,\sigma(x_n)$ equals $p+1$ then the sum or absolute values is at least two less than the maximum and the lower bound increases by 2 as desired.

Suppose without loss of generality that $\sigma(x_1) = p+1$. Thus $\{x_1, x_2, ..., x_{n-\alpha}\}$ are all in V_2 and $\{x_{n-\alpha+1}, x_{n-\alpha+2}, ..., x_n\}$ are all in V_1. Since Part 2 is maximized, we know $w_{n-\alpha}$ is labeled consecutively with some vertex in V_1, thus $x_{n-\alpha} = w_{n-\alpha}$. Suppose that this vertex has a label $c(x_{n-\alpha}) = k$. We now apply Lemma 1 to vertices $x_{n-\alpha}$ through x_n. We have only the absolute value terms to consider, except for the first pair, which may include an additional 1. Here we have $2p-2$ values of the form $\sigma(x_i)$. Every value from 1 to $2p+1$ occurs twice except $\sigma(x_{n-\sigma}) = p+1$ and $\sigma(x_n)$. In this case we maximize distances with

Addition. $2p+1\,, 2p+1\,, 2p\,, 2p....p+3, p+3, p+2, p+1$

Subtraction. $1, 1, 2, 2...., p-1, p-1, p, p$

Summing these terms and adding 1 we obtain a maximum distance of

$$\sum_{i=n-\alpha}^{n-1} d(x_i, x_{i+1}) = 2p(p+1) - 1 + 1 = \frac{\alpha^2}{2} + \alpha.$$

Using this distance with a modified form of Lemma 1 we have:

$$c(x_n) - c(x_{n-\alpha}) \geq (\alpha+1-1)(\alpha+1) - \sum_{i=n-\alpha}^{n-1} d(x_i, x_{i+1})$$

$$span(c) - k \geq (\alpha)(\alpha+1) - (\frac{\alpha^2}{2} + \alpha)$$

$$span(c) \geq k + \frac{\alpha^2}{2}.$$

We would like to increase this lower bound by 1. Now suppose a labeling order c maximizes $\sum_{i=n-\alpha}^{n-1} d(x_i,x_{i+1})$. We will show that a jump is required. We have

$$span(c) = k + \frac{\alpha^2}{2} + \sum_{i=n-\alpha}^{n-1} j_i,$$

where j_i is the nonnegative integer such that

$$d(x_i, x_{i+1}) + c(x_{i+1}) - c(x_i) = diam(G) + 1 + j_i.$$

Suppose to the contrary that $\sum_{i=n-\alpha}^{n-1} j_i = 0$. Let x_l be an element of $\{v_1, v_{2p+1}\}$ such that $l \neq n - \alpha + 1$. Assume without loss of generality that $x_l = v_1$. Since c is distance maximizing, v_1 is not labeled last, thus x_{l-1} and x_{l+1} both exist. Also since 1 is subtracted twice in a distance maximizing labeling, we know that x_{l-1} and x_{l+1} must both have indices that are added at least once in a distance maximizing labeling. Thus both x_{l-1} and x_{l+1} are in $\{v_{p+2}, v_{p+3}...v_{2p+1}\}$. Suppose $x_{l-1} = v_q$ and $x_{l+1} = v_r$ for $p + 2 \leq q, r \leq 2p + 1$. Assume without loss of generality that $r > k$. Thus we have

$$\begin{aligned} c(x_l) - c(x_{l-1}) &= 2p + 1 - d(v_1, v_q) = 2p - q + 2 \\ c(x_{l+1}) - c(x_l) &= 2p + 1 - d(v_1, v_r) = 2p - r + 2 \end{aligned}$$

Since c is a radio labeling we have

$$\begin{aligned} c(x_{l+1}) - c(x_{l-1}) &\geq 2p + 1 - d(v_l, v_n) = 2p - l + n + 1 \\ 4p - r - q + 4 &\geq 2p - r + q + 1 \\ 2p + 3 &\geq 2q \end{aligned}$$

This is a contradiction because $q \geq p + 2$. Thus for a c that maximizes the remaining distances, $\sum_{i=n-\alpha}^{n-1} j_i \neq 0$. This allows us to conclude

$$span(c) \geq k + \frac{\alpha^2}{2} + 1.$$

We can now apply Lemma 1 to the vertices labeled first. From this we have

$$k \geq (n - \alpha - 1)(\alpha + 1) + 1 - \sum_{i=1}^{n-\alpha-1} d(x_i, x_{i+1}).$$

Since all of these vertices are in V_2, the maximum of each pair is 2. Since there are a total of $n-\alpha-1$ pairs, the maximum of this sum of distances is $2(n-\alpha-1)$. Substituting this into our lower bound formula we have:

$$\begin{aligned} span(c) &\geq (n-\alpha-1)(\alpha+1)+1+\frac{\alpha^2}{2}+1-2(n-\alpha-1) \\ &= n\alpha+n-\alpha^2-\alpha-\alpha-1+1+\frac{\alpha^2}{2}+1-2n+2\alpha+2 \\ &= \frac{-\alpha^2}{2}+n\alpha-n+3. \end{aligned}$$

Thus we conclude that if Part 2 of the distance sum is maximized, the span of the labeling must be at least two higher than the original lower bound.

This exhausts all cases and allows us to conclude

$$rn(E_{n,\alpha}) \geq \frac{-\alpha^2}{2}+n\alpha-n+3.$$

Finally we will provide a labeling to show that this bound is in fact the radio number.

x_i	Vertex Name	$d(x_i, x_{i+1})$	$c(x_{i+1}, x_i)$
x_1	w_1	$\frac{\alpha}{2}$	$\frac{\alpha}{2}+1$
x_2	v_{2p+1}	$\frac{\alpha}{2}+1$	$\frac{\alpha}{2}$
x_3	v_p	$\frac{\alpha}{2}$	$\frac{\alpha}{2}+1$
x_4	v_{2p}	$\frac{\alpha}{2}+1$	$\frac{\alpha}{2}$
x_5	v_p	$\frac{\alpha}{2}$	$\frac{\alpha}{2}+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2p-1}	v_{p+2}	$\frac{\alpha}{2}+1$	$\frac{\alpha}{2}$
x_{2p}	v_1	$\frac{\alpha}{2}+1$	$\frac{\alpha}{2}$
x_{2p+1}	$w_{n-\alpha}$	2	$\alpha-1$
x_{2p+2}	$w_{n-\alpha-1}$	2	$\alpha-1$
x_{2p+3}	$w_{n-\alpha-2}$	2	$\alpha-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{n-1}	w_3	2	$\alpha-1$
x_n	w_2		

Table 4: Labeling Scheme for Empty-Sok Graph

The same argument used in Section 2.1 shows that this is a radio labeling. Summing and adding one we obtain

$$\begin{aligned} Span(c) &= (\alpha+1)(\frac{\alpha}{2}) + \frac{\alpha}{2} + (\alpha-1)(n-\alpha-2) + 1 \\ &= \frac{-\alpha^2}{2} + n\alpha - n + 3, \end{aligned}$$

which allows us to conclude

$$rn(E_{n,\alpha}) = \frac{-\alpha^2}{2} + n\alpha - n + 3.$$

□

Thus we have shown that the radio numbers of empty-Sok graphs match the desired integers.

5. Conclusion

Consider the radio numbers of $S_{n,\alpha,m}$ and $E_{n,\alpha}$ as established in Sections 2 and 3. Recall that α is the diameter of the graph, and m is the number of edges removed from the web. We argue that by letting m and α vary we will fill the number line for graphs of both even or odd order. The proofs and labellings provided for $S_{n,\alpha,m}$ are valid for $2 \leq \alpha \leq n-2$. This restriction ensures there are at least 2 vertices in the web and also that removing edges from the web does not change the diameter of the graph. In addition, the radio numbers of $E_{n,\alpha}$, for even α greater than 4, fill all of the gaps outlined in Section 2.3. Thus the Sok graphs fill all radio numbers from the minimum of an even-Sok graph with $\alpha = 2$ to the maximum for a Sok graph with $\alpha = n-2$. This minimum is $n+2$ for graphs of both even and odd order. We achieve the integer $n+1$ by noting that it is the radio number of a star graph on n verticies [6]. For graphs of even order, the maximum is $\frac{(n-1)^2}{2} + \frac{1}{2}$, one less than the radio number of the path. For graphs of odd order, the maximum is $\frac{(n-1)^2}{2} + 1$, two less than the radio number of the path. This is to be expected because of the known unachievable radio number for graphs of odd order. Thus using Sok graphs, and known facts about radio numbers, we have completely answered the question of achievable radio numbers for graphs of order n.

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