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Quorum Colorings of Graphs

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Abstract

Let $G = (V, E)$ be a graph. A partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of the vertex set V of G into k *color classes* V_i, with $1 \leq i \leq k$, is called a *quorum coloring* if for every vertex $v \in V$, at least half of the vertices in the closed neighborhood $N[v]$ of v have the same color as v. In this paper we introduce the study of quorum colorings of graphs and show that they are closely related to the concept of defensive alliances in graphs. Moreover, we determine the maximum quorum coloring of a hypercube.

Keywords: quorum coloring, defensive alliance, hypercube.

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1. Introduction to Colorings

Let $G = (V, E)$ be a simple graph of *order* $n = |V|$. For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$ of vertices adjacent to v, and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighborhood* of S is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. For any subset $S \subseteq V$, the *subgraph induced by* S is the graph $G[S]$ with vertex set S that inherits all edges of G between vertices in S.

A set S is called *independent* if no two vertices in S are adjacent. The *vertex independence number*, denoted $\beta_0(G)$, equals the maximum cardinality of an independent set in G.

A graph G is called *k-regular* if every vertex $v \in V$ has degree k, that is $|N(v)| = k$. A *cubic graph* is a 3-regular graph.

Let K_n denote the complete graph of order n, and let $\overline{K_n}$ denote the complement of K_n, that is, the graph consisting of n isolated vertices. By the *join* of two graphs G and H we mean the graph $G + H$ consisting of the disjoint union of G and H together with all edges between the vertices in G and the vertices in H. If G consists of a single vertex v, we just write $v + H$. By the *corona* $G \circ H$ of two graphs G and H we mean the graph obtained from a copy of G by replacing v by $v + H$, for every vertex $v \in V(G)$.

A $\mathcal{P}$*-coloring* of a graph G is a partition $\pi = \{V_1, V_2, \ldots, V_k\}$ having the property that every class V_i of π is a set of vertices having some specified property $\mathcal{P}$. The number $k = |\pi|$ is the *order* of the coloring. The minimum order of a $\mathcal{P}$-coloring of a graph G is called the $\mathcal{P}$*-chromatic number* and is denoted $\chi_{\mathcal{P}}(G)$.

We say that a $\mathcal{P}$-coloring $\pi = \{V_1, V_2, \ldots, V_k\}$ is *minimal* if the union of any two color classes $V_i \cup V_j$ is a set that does not have property $\mathcal{P}$. The maximum order of a minimal $\mathcal{P}$-coloring of a graph G is called the $\mathcal{P}$*-achromatic number* and is denoted $\psi_{\mathcal{P}}(G)$.

We say that a $\mathcal{P}$-coloring $\pi = \{V_1, V_2, \ldots, V_k\}$ is *proper* if every color class V_i is an independent set. A $\mathcal{P}$-coloring of order k is called *complete* if for every $1 \leq i < j \leq k$, there is a vertex $u \in V_i$ and a vertex $v \in V_j$ such that u is adjacent to v. The well studied *chromatic number* $\chi(G)$ equals the minimum order of a proper coloring of G, while the *achromatic number* $\psi(G)$ equals the maximum order of a complete proper coloring.

In this paper we are concerned with the concept of a *quorum*, which is normally understood to mean a simple majority, that is, at least half. With this in mind, consider any $\mathcal{P}$-coloring $\pi = \{V_1, V_2, \ldots, V_k\}$ of the vertices of a graph G into k *color classes*. For each vertex $v \in V$, if $v \in V_i$ we say that v is colored with color i. Define $c[v]$ to equal the number of vertices in $N[v]$ having the same color as v, including v itself. We say that a $\mathcal{P}$-coloring π is a *quorum coloring* if, for every vertex $v \in V$, we have $c[v] \geq |N[v]|/2$, that is, at least half (a quorum) of the vertices in $N[v]$ have the same color as v. The maximum order of a quorum coloring of G is called the *quorum coloring number* of G and is denoted $\psi_q(G)$, and a quorum coloring of order $\psi_q(G)$ is called a ψ_q*-coloring*. Notice that, if every vertex $v \in V$ is colored the same, say color 1, then the coloring is automatically a quorum coloring. Thus, every graph has a quorum coloring of order 1. Therefore, we seek to color the vertices of V with a maximum number of colors that results in a quorum coloring.

A *satisfactory partition* is a concept closely related to quorum coloring, but it is of slightly different form. It is due originally to Gerber and Kobler [3] (see also [4, 5]). Given a set $S \subseteq V$ of a graph $G = (V, E)$, we say that a vertex $v \in S$ is *satisfied* if it has at least as many neighbors in S as it does in $V - S$. The set S is called *cohesive* if every vertex $v \in S$ is satisfied. A bipartition $\pi = \{V_1, V_2\}$ of V is called a *satisfactory partition* if every vertex $v \in V_1$ is satisfied with respect to V_1 and every vertex $w \in V_2$ is satisfied with respect to V_2; that is, both V_1 and V_2 are cohesive sets. This concept was generalized to partitions $\pi = \{V_1, V_2, \ldots, V_k\}$ of arbitrary order by Shafique and Dutton [13].

Thus, the difference between a quorum coloring and a satisfactory partition is that with quorum colorings at least half of the vertices in the closed neighborhood $N[v]$ have the same color as v, while in a satisfactory partition, at least half of the vertices in the open neighborhood $N(v)$ have the same color as v. Since it is known that certain graphs do not have a satisfactory partition, for example, complete graphs, the primary focus of research on satisfactory partitions is (i) the complexity question: given an arbitrary graph G, does G have a satisfactory partition? and (ii) finding classes of graphs that do or do not have satisfactory partitions. On the other hand, all graphs have quorum colorings, and the primary focus of quorum colorings is on the maximum order of a quorum coloring of a given graph.

2. A general setting for quorum colorings

There are several reasons why the study of quorum colorings can be considered to be "natural". First, they offer a dual view of the well-studied chromatic number. With the chromatic number there is no limit on the number of different colors that can appear in any closed neighborhood, but the objective is to minimize the total number of colors used. With the quorum colorings there is a limit on the number of different colors that can appear in any closed neighborhood, but the objective is to maximize the total number of colors used.

Quorum colorings have a variety of real-world applications. For example, they are a natural model of *quorum sensing* in bacteria, in which if sufficiently many molecules of a given type are in the neighborhood of a community of bacteria, they will all exhibit a common behavior, for example, emitting luminescence, or launching a virulent attack [8].

Similarly, the notion of a quorum is widespread in social and political groups, in which at least some number k of individuals must be present (usually, a simple majority) in order for certain group actions to be considered "official".

The idea of a majority or a quorum in a neighborhood is also found in the concept of a signed dominating set. A function $f : V \to \{-1, +1\}$ is called a *signed dominating function* on a graph $G = (V, E)$ if for every vertex $v \in V$, we have $f(N[v]) \geq 1$, where $f(N[v]) = \Sigma_{u \in N[v]} f(u)$. Thus, if we define $V_1 = f^{-1}(+1)$, and $V_2 = f^{-1}(-1)$, then a signed dominating function can be viewed as a bi-partition $\pi = \{V_1, V_2\}$ of V, such that a majority of the vertices in every closed neighborhood $N[v]$ are in the set V_1, that is, are assigned the value $+1$. Signed domination in graphs was introduced by Dunbar, Hedetniemi, Henning and Slater in [1].

Quorum colorings also apply to the study of wireless sensor networks, in which a threshold is associated with each sensor node, such that when the inputs from the neighborhood of a sensor exceed the threshold (or quorum), a signal is sent by the sensor to a gateway sensor node [12].

Similarly, a set $S \subseteq V$ is called a *defensive alliance* if for every vertex $v \in S$, we have $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. The *defensive alliance number* denoted $a(G)$ equals the minimum order of a defensive alliance in G. In a defensive alliance, at least a simple majority (a quorum) of vertices in the closed neighborhood $N[v]$ of every vertex $v \in S$ are in S. The study of alliances in graphs was introduced by Hedetniemi, Hedetniemi and Kristiansen in 2004 (cf. [7]). In 2007 Haynes and Lachniet [6] introduced the *alliance partition number* $\psi_a(G)$ of a graph, as follows. A partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that each block V_i is a defensive alliance is called an *alliance partition*. The *alliance partition number* $\psi_a(G)$ is the maximum order of an alliance partition of G. It can be seen from the definitions that a quorum coloring is the same thing as an alliance partition, that is each class V_i in a quorum coloring is a defensive alliance in G. Thus, for any graph G, we have $\psi_a(G) = \psi_q(G)$.

Although the study of quorum colorings can be viewed from the perspective of alliances in graphs, most of the motivation for our study of quorum colorings comes from coloring theory. As such, we will adopt the coloring notation and terminology.

3. Basic properties of quorum colorings

We begin with the most fundamental observations about quorum colorings.

Proposition 1. *If $G \cup H$ denotes the disjoint union of two graphs G and H, then*

$$\psi_q(G \cup H) = \psi_q(G) + \psi_q(H).$$

Proposition 2. *Let G be a graph of order n. Then $\psi_q(G) \leq n$, and $\psi_q(G) = n$ if and only if $\Delta(G) \leq 1$, that is G consists of isolated vertices and disjoint copies of the complete graph K_2 of order 2.*

From these two propositions we can see that the addition of an isolated vertex to a graph G increases its quorum coloring number by one. But even if a connected graph G of order n has no isolated vertices, $\psi_q(G)$ can be arbitrarily close to n.

Proposition 3. *Let $n \geq 1$ and $G = K_n \circ \overline{K_n}$ of order $n^2 + n$. Then $\psi_q(G) = n^2 + 1$.*

Proposition 4. *Let $G = (V, E)$ be a graph without isolated vertices, and let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a quorum coloring of G. Then, for every color class V_i, if $|V_i| = 1$ then the only vertex in V_i is a leaf in G; otherwise $|V_i| \geq 2$.*

This gives the following upper bound for graphs with minimum degree $\delta(G) \geq 2$.

Corollary 5. *Let G be a graph of order n without isolated vertices or leaves. Then $\psi_q(G) \leq \lfloor n/2 \rfloor$.*

This bound is achieved, for example, by a cycle C_n of even order, in which the vertices are colored consecutively with pairs of adjacent colors $1, 1, 2, 2, 3, 3, 4, 4, \ldots$; in this case each vertex v is in a color class of size two, and at least half of the vertices in $N[v]$ have the same color as v.

This bound can be refined in terms of the defensive alliance number $a(G)$, as follows. In [2], Fricke, Lawson, Haynes, Hedetniemi and Hedetniemi showed the following.

Theorem 6. *Let G be a graph of order n. Then $a(G) \leq \lceil n/2 \rceil$.*

The alliance number provides us with a simple upper bound for the quorum coloring number.

Proposition 7. *Let G be a graph of order n with defensive alliance number $a(G)$ and without isolated vertices. Then $\psi_q(G) \leq n/a(G) \leq \lfloor n/2 \rfloor$.*

Proof. Let $\psi_q(G) = k$, let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a ψ_q-coloring and for $1 \leq i \leq k$, let $|V_i| = n_i$. It follows that each color class must be a defensive alliance, and therefore have at least $n_i \geq a(G)$ vertices. Therefore, $n = n_1 + n_2 + \ldots + n_k \geq ka(G)$. Therefore, $\psi_q(G) \leq n/a(G)$. □

In [7], Kristiansen, Hedetniemi and Hedetniemi pointed out that for 4-regular and 5-regular graphs, $a(G) = girth(G)$, where the *girth* of a graph equals the smallest order of a cycle in G.

Corollary 8. *Let G be a 4-regular or 5-regular graph of order n. Then $\psi_q(G) \leq n/girth(G)$.*

Another basic property of a ψ_q-coloring of a graph G is that each color class must induce a connected subgraph.

Proposition 9. *Let G be a graph, and let $\pi = \{V_1, V_2, \ldots, V_k\}$ be any ψ_q-coloring of G. Then, for every i, $1 \leq i \leq k$, the induced subgraph $G[V_i]$ is connected.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a ψ_q-coloring of a graph G, assume that for some color class V_i, the subgraph $G[V_i]$ is not connected, and arbitrarily let V_{i_1} be a connected subgraph of $G[V_i]$. Then it can be seen that $\pi = \{V_1, V_2, \ldots, V_{i_1}, V_i - V_{i_1}, \ldots, V_k\}$ is a quorum coloring of order greater than $\psi_q(G)$, a contradiction. □

Another refinement of the upper bound of Corollary 5 is given by the following result. A *matching* in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ having the property that no two edges in M have a vertex in common. We say that the edges in a matching form an *independent* set of edges. The *matching number* $\beta_1(G)$ equals the maximum cardinality of a matching in G.

Proposition 10. *Let G be a graph of order n with minimum degree $\delta(G) \geq 2$. Then*

$$\psi_q(G) \leq \beta_1(G) \leq \lfloor n/2 \rfloor.$$

Proof. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a ψ_q-coloring of G. Since $\delta(G) \geq 2$, each color class V_i has size at least two and induces a connected subgraph, and therefore contains at least one pair of adjacent vertices. This implies that G contains at least k independent edges, and therefore has a matching of order at least k. Therefore, $\beta_1(G) \geq k = \psi_q(G)$. □

It is worth noting that the $\beta_1(G)$ upper bound does not apply to graphs with $\delta(G) = 1$. For the path P_4 of order 4, $\psi_q(P_4) = 3$ but $\beta_1(P_4) = 2$.

The following result shows that the upper bound in Proposition 10 is sharp for cubic graphs.

Theorem 11. *Let G be a cubic graph. Then $\psi_q(G) = \beta_1(G)$.*

Proof. From Proposition 10 we know that $\psi_q(G) \leq \beta_1(G)$. Let $\beta_1(G) = k$ and let $M = \{x_1y_1, x_2y_2, \ldots, x_ky_k\}$ be a matching of maximum cardinality. Furthermore, let $V(M) = \{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$ and let $V - V(M) = \{u_1, u_2, \ldots, u_r\}$. It follows that $V - V(M)$ is an independent set of vertices.

Consider then the partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of $V(M)$, where $V_i = \{x_i, y_i\}$ for $1 \leq i \leq k$. We can then augment the partition π to a partition of all of V as follows: for each vertex $u_i \in V - V(M)$, for $1 \leq i \leq r$, arbitrarily select an adjacent vertex, say x_j, and add u_i to the class V_j of π. Since G is a cubic graph, each vertex x_i and y_i in V_i is adjacent to at most two vertices not in V_i, and thus at least half of the vertices in their closed neighborhoods have color i. Similarly, each vertex, say u_i belongs to a class containing an adjacent vertex colored the same as u_i, and u_i can have at most two neighbors not colored the same as its color. Therefore, π is a quorum coloring of order k. This implies that $\psi_q(G) \geq \beta_1(G)$. □

Theorem 11 can be extended to 4-regular graphs having no triangles.

Theorem 12. *If G is a triangle-free, 4-regular graph, then $\psi_q(G) \leq \beta_1(G)/2$.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a ψ_q-coloring of a triangle-free, 4-regular graph G. Consider each induced subgraph $G[V_i]$. Each vertex in $G[V_i]$ will have to have degree at least two. This means, since G is triangle-free, that $|V_i| \geq 4$ and that each $G[V_i]$ contains at least two independent edges. This in turn means that G has a matching containing at least $2k$ edges, that is, $\beta_1(G) \geq 2k = 2\psi_q(G)$. □

For paths and cycles, quorum colorings are slightly different. For example, for the path P_6, the following is a ψ_q-coloring: $\{1\}, \{2, 3\}, \{4, 5\}, \{6\}$.

Proposition 13. *Let P_n be the path of order n. Then $\psi_q(P_n) = \lfloor (n + 2)/2 \rfloor$.*

Proposition 14. *Let C_n be the cycle of order n. Then $\psi_q(C_n) = \lfloor n/2 \rfloor$.*

It is easy to see that the requirement, that each vertex be in a quorum color class, prevents the vertices in a complete graph of odd order from being colored with more than one color.

Proposition 15. *For the complete graph K_n of odd order, $\psi_q(K_n) = 1$, while for any complete graph K_n of even order, $\psi_q(K_n) = 2$.*

Proposition 16. $\psi_q(K_{1,n}) = \lceil (n+2)/2 \rceil$, *for the star $K_{1,n}$.*

Proposition 17. *For the complete bipartite graph $K_{m,n}$, where $2 \leq m \leq n$, $\psi_q(K_{3,3}) = 3$, but otherwise $\psi_q(K_{m,n}) = 2$.*

4. Graphs with $\psi_q(G) = 1$

The simple question, which graphs G have $\psi_q(G) = 1$?, seems to be very difficult to answer. In fact we raise it as an interesting algorithmic complexity question, as follows.

QUORUM-ONE
INSTANCE: Graph $G = (V, E)$.
QUESTION: Is $\psi_q(G) > 1$?

We know of only a few classes of graphs for which $\psi_q(G) = 1$. We have already seen the first class, namely complete graphs K_{2n+1} of odd order. From Theorem 6, we can conclude the following.

Proposition 18. *If G is a graph of odd order n for which $a(G) = \lceil n/2 \rceil$, then $\psi_q(G) = 1$.*

Proof. It follows from the definitions that if $\pi = \{V_1, V_2, \ldots, V_n\}$ is a quorum coloring of order $k = \psi_q(G)$, then each color class must be a defensive alliance. But if $a(G) = \lceil n/2 \rceil$ and n is odd, then each color class must have strictly more than half the total number of vertices. Only one color class can have this property, and therefore $\psi_q(G) = 1$. □

We can also observe the following.

Proposition 19. *If G is a graph having order $n \geq 2$ and either minimum degree $\delta(G) = 0$ or $\delta(G) = 1$, then $\psi_q(G) \geq 2$.*

A *bridge* in a connected graph is an edge whose removal disconnects the graph.

Proposition 20. *If G is a connected graph having a bridge, then $\psi_q(G) \geq 2$.*

A second class of graphs for which $\psi_q(G) = 1$ can be constructed using the join operation.

Proposition 21. *For any graph $G = K_r + \overline{K_s}$, where $r + s$ is odd, $\psi_q(G) = 1$.*

Proof. For a graph $G = K_r + \overline{K_s}$, where $r+s$ is odd, we can show that $a(G) \geq \lceil n/2 \rceil$. From this, it follows from Proposition 18 that $\psi_q(G) = 1$. Let $V(K_r) = X$, and $V(\overline{K_s}) = Y$, and S be a defensive alliance of minimum order $a(G)$, and let $u \in S$.

Case 1. $u \in X$.

In this case, $|N[u]| = r + s$, since u is adjacent to every vertex $v \in V - \{u\}$, and $r + s$ is odd. But since u is in the defensive alliance S, we know that at least half of the vertices in $N[u]$ are in S. Thus, $|S| \geq \lceil (r+s)/2 \rceil$.

Case 2. $u \in Y$.

Since $u \in S$, $deg(u) = r$, and since at least half the vertices in $N[u]$ must be in S, there must exist a vertex $v \in X$ such that v is adjacent to u and $v \in S$. But then, as in Case 1 we know that $|N[v]| = r + s$, and this implies again that $|S| \geq \lceil (r+s)/2 \rceil$. □

It remains for us an open problem to find any other class of graphs for which $\psi_q(G) = 1$.

5. Quorum coloring numbers of hypercubes

A *hypercube* or *n-cube* Q_n of dimension n is defined recursively by $Q_1 = K_2$ and

$$Q_n = Q_{n-1} \square K_2,$$

for $n > 1$, where the *Cartesian product* $G_1 \square G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph having vertex set $V_1 \times V_2$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$.

Equivalently, the n-cube Q_n can be defined to be the graph having 2^n vertices, consisting of all n-tuples of 0's and 1's, such that two vertices are adjacent if and only if their corresponding n-tuples differ in precisely one coordinate.

We need some basic facts of hypercubes. Let u and v be two adjacent vertices in Q_n. Then, by definition, they differ in exactly one coordinate. We say that we can get from u to v by *flipping* that coordinate. Thus, we can make any walk in Q_n by consecutively flipping coordinates. Let w and v be two vertices that differ in exactly i coordinates. Then, to get from w to v, we have to flip at least the coordinates in which they differ. But flipping these coordinates consecutively only once gets us already from w to v. Hence their distance is i. This is the so called *Hamming distance* on the n-tuples of 0' and 1's, which is precisely the graph distance in Q_n. Moreover, if we flip only one coordinate of w in which w and v differ, then we get one step closer to v. So there are exactly i neighbors of w one step closer to v. Since Q_n is bipartite and n-regular, all other $n - i$ neighbors of w are one step further away from v. These observations give us Equations (1) and (2) below.

In this section we determine the quorum coloring number of the n-cubes Q_n. To achieve this, we first prove a more general result on subgraphs of Q_n with minimum degree at least

k. We need some notation. Let d be the distance function of Q_n. For any subgraph G of Q_n, the neighborhood of a vertex x in G is denoted by $N_G(x)$. Note that $N(x)$ is the neighborhood of x in Q_n. Fix a vertex v of Q_n. The *i-th level* $L_i(v)$ of v in Q_n is defined by

$$L_i(v) = \{ \ w \mid w \text{ in } Q_n \text{ such that } d(v, w) = i \ \}.$$

For any vertex w in $L_i(v)$ we have the following simple equations

$$|N(w) \cap L_{i-1}(v)| = i, \tag{1}$$

$$|N(w) \cap L_{i+1}(v)| = n - i. \tag{2}$$

In Figure 1 we depict two consecutive levels in Q_n as well as in G with respect to a 'fixed' vertex v as an aid for reading the proof of the next theorem. Let $\delta(G)$ denote the minimum degree in G.

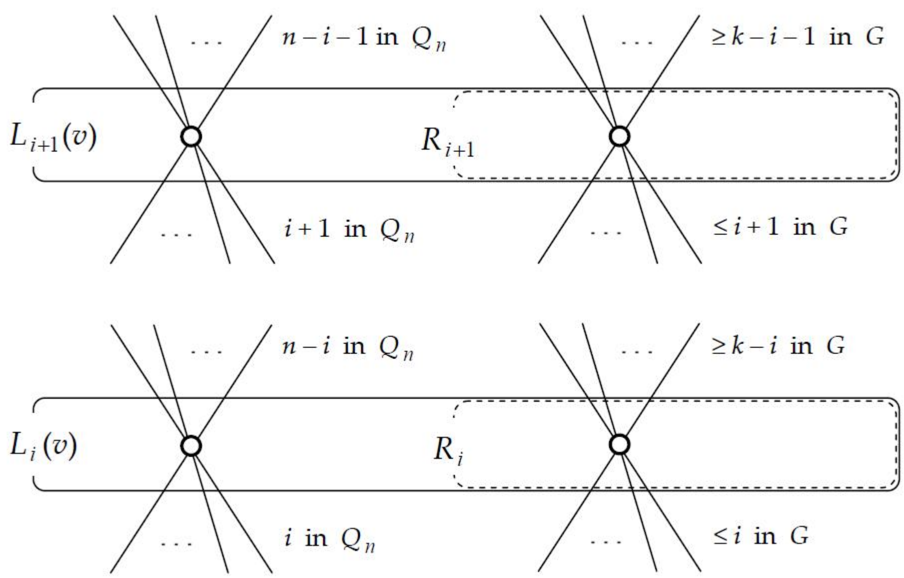

Figure 1: Consecutive levels in Q_n and G

Theorem 22. *Let $G = (V, E)$ be a subgraph of Q_n with the property that $\delta(G) \geq k$, for some k with $0 \leq k \leq n$. Then $|V| \geq 2^k$.*

Proof. Let $G = (V, E)$ be a subgraph of Q_n with minimum degree at least k. If $k = 0$, then, clearly, we have $|V| \geq 1 = 2^0$, and we are done. So let $k > 0$. Take any vertex v of

G. We write

$$R_i = L_i(v) \cap V,$$

for $i = 0, 1, \ldots n$. Then $R_0 = \{v\}$, and $R_1 = N_G(v)$, so we have $|R_0| = 1$ and $|R_1| \geq k$. Take any i with $0 < i < k$. Assume that $R_i \neq \emptyset$, and let x be a vertex in R_i. Then, by Equation (1), we have

$$|N_G(x) \cap R_{i-1}| \leq i, \tag{3}$$

and, by the fact that $|N_G(x)| \geq k$, we have

$$|N_G(x) \cap R_{i+1}| \geq k - i. \tag{4}$$

Since $i < k$ and $R_i \neq \emptyset$, it follows from (4) that R_{i+1} is also nonempty. So, since $|R_1| \geq k > 0$, we deduce that R_i is nonempty, for $i = 0, 1, \ldots, k$. Note that R_i may be empty for $i > k$.

Now consider the subgraph $H = (R_i \cup R_{i+1}, F)$ of G induced by $R_i \cup R_{i+1}$ with $0 \leq i < k$. By Equation (4) the vertices in R_i have degree at least $k - i$ in H, and, by Equation (3), the vertices in R_{i+1} have degree at most $i + 1$ in H. We count the number of edges in H twice, first by summing the degrees of the vertices in R_i, second by summing the degrees in R_{i+1}. Then we get the following lower bound and upper bound for this number:

$$|R_i|(k - i) \leq |F| \leq |R_{i+1}|(i + 1). \tag{5}$$

Equation (5) gives us

$$|R_{i+1}| \geq |R_i| \frac{k - i}{i + 1}. \tag{6}$$

Since $|R_1| \geq k$, it now follows by induction that $|R_i| \geq \binom{k}{i}$. Indeed, inserting this in Equation (6), we get

$$|R_{i+1}| \geq \binom{k}{i} \frac{k - i}{i + 1} = \binom{k}{i + 1}. \tag{7}$$

Since $V = \bigcup_{i=0}^{n} R_i$, Equation (7) gives us the desired bound: $|V| \geq 2^k$. □

Before proving our next result, we cite some results from the literature. A $(0, 2)$-*graph* is a connected graph in which any two distinct vertices have exactly two common neighbors or none at all. These were introduced in [9], where it was proved that they are regular. In [10] the following theorem was proved. This theorem also follows from results in [9, 11].

Theorem A. *Let G be a bipartite $(0, 2)$-graph of degree k. If G is of order 2^k then G is isomorphic to Q_k.*

Now we prove the result from which we can deduce the quorum coloring number of Q_n and the structure of any ψ_q-coloring of Q_n.

Theorem 23. *Let k be a number with $0 \leq k \leq n$, and let $G = (V, E)$ be a subgraph of minimum order of Q_n such that the minimum degree in G is at least k. Then G is isomorphic to Q_k.*

Proof. First note that any Q_k in Q_n is a subgraph of minimum degree at least k. So we have $|V| \leq 2^k$. Hence, by Theorem 22, we have $|V| = 2^k$. To get this equality, we must have equalities all over in every step of the proof of Theorem 22. Also we must have $R_i = \emptyset$, for $i > k$. On the other hand, for any vertex x in R_i with $1 \leq i \leq k$, there exist exactly i neighbors in R_{i-1}, so there is a path from x towards v. Hence G is connected, and, moreover, any vertex in R_2 has exactly two neighbors in R_1, so exactly two common neighbors with v. Clearly any other vertex has no common neighbor with v. Since v was chosen arbitrarily in G in the proof of Theorem 22, it follows that any two vertices have two common neighbors or none at all, whence G is a $(0, 2)$-graph. So G is regular, and, as there is a vertex of degree k, the graph is k-regular. Finally, G being a subgraph of Q_n, it must be bipartite. So G satisfies all conditions in Theorem A, and G is a Q_k. □

Take any ψ_q-coloring of Q_n. Due to Proposition 9, any color class induces a connected subgraph of minimum degree at least $\lfloor n/2 \rfloor$. The subgraph of this kind of minimum order is $Q_{\lfloor n/2 \rfloor}$. Since we can partition the vertex set of Q_n into $2^{\lceil n/2 \rceil}$ copies of $Q_{\lfloor n/2 \rfloor}$, any ψ_q-coloring of Q_n is of this type, and $\psi_q(Q_n) = 2^{\lceil n/2 \rceil}$. Hence we have the following theorem.

Theorem 24. *$\psi_q(Q_n) = 2^{\lceil n/2 \rceil}$, and any ψ_q-coloring of Q_n consists of $2^{\lceil n/2 \rceil}$ disjoint copies of $Q_{\lfloor n/2 \rfloor}$.*

6. Open Problems

The following problems are suggested from this preliminary study of quorum colorings.

1. Can you characterize the class of graphs for which $\psi_q(G) = 1$ or the class of graphs for which $\psi_q(G) > 1$? In fact, can you find any infinite family of graphs other than those of the form K_{2n+1} or $K_r + \overline{K_s}$ for $r + s$ odd and $r \geq 2$, for which $\psi_q(G) = 1$?

2. Is $\psi_q(G) = 1$ if and only if $a(G) = \lceil n/2 \rceil$ and n is odd?

3. If $\psi_q(G) = 1$, is $diam(G) \leq 2$?

4. Is $\lfloor diam(G)/2 \rfloor \leq \psi_q(G)$? It is easy to prove the following.

   Proposition 25. *For any tree T,*

   $$\lfloor diam(T)/2 \rfloor \leq \psi_q(T).$$

5. It is easy to see that for any graph G, $\psi_q(G) \leq \psi_q(G \circ K_1)$. But is a more refined result possible? For example, when is this inequality strict?

6. What is the complexity of the following decision problem:

   QUORUM-ONE
   INSTANCE: Graph $G = (V, E)$.
   QUESTION: Is $\psi_q(G) > 1$?

7. What is the complexity of the following decision problem:

   QUORUM-K
   INSTANCE: Graph $G = (V, E)$, positive integer $K \leq |V|$.
   QUESTION: Does G have a quorum coloring of order at least K?

8. It is easy to see that for 1-regular graphs G of order n, $\psi_q(G) = n$. It is also easy to determine the value of $\psi_q(G)$ for any 2-regular graph G. In addition, since $\psi_q(G) = \beta_1(G)$ for 3-regular graphs G, it is easy to determine, in polynomial time, the value of $\psi_q(G)$ for 3-regular graphs. This leads us to the following decision problem:

   4-REGULAR QUORUM
   INSTANCE: A 4-regular graph $G = (V, E)$, positive integer $K \leq |V|$.
   QUESTION: Does G have a quorum coloring of order at least K?

9. Can you design a linear algorithm for computing the value of $\psi_q(T)$ for any tree T?

10. What are good Gaddum-Nordhaus bounds for $\psi_q(G) + \psi_q(\overline{G})$ and $\psi_q(G) \times \psi_q(\overline{G})$? We believe that the following is true:

    Conjecture 26. *For any graph G of order $n \geq 4$,*

    $$4 \leq \psi_q(G) + \psi_q(\overline{G}) \leq n + 2.$$

11. A *sub-quorum coloring* is an onto partial function $p : V \to \{1, 2, \ldots, k\}$ having the property that for every vertex $v \in V$, if $p(v)$ is defined, then at least half of the vertices in $N[v]$, that have been colored, have the same color as v. The *sub-quorum coloring number* $\psi_{sq}(G)$ equals the maximum value k in a sub-quorum coloring of G. Notice that by definition, $\psi_q(G) \leq \psi_{sq}(G)$. What can you say about the sub-quorum coloring number of a graph? For example, it is easy to see that $\beta_0(G) \leq \psi_{sq}(G)$.

12. A *partial quorum coloring* is a partition $\pi = \{V_1, V_2, \ldots, V_k\}$ having the property that each color class V_i contains at least one quorum vertex. The *partial quorum coloring number* $\psi_{pq}(G)$ equals the maximum value k in a partial quorum coloring of G. Notice that by definition, $\psi_q(G) \leq \psi_{pq}(G)$. What can you say about the partial quorum coloring number of a graph?

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