

CONSTRUCTING UNION-FREE PAIRS OF k -ELEMENT SUBSETS

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Abstract

It is proved that one can choose $\lfloor \frac{1}{2} \binom{n}{k} \rfloor$ disjoint pairs of k -element subsets of an n -element set in such a way that the unions of the pairs are all different, supposing that $n > n(k)$.

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1. Introduction

The following notations will be used: $[n] = \{1, 2, \dots, n\}$ for an n -element set, and $\binom{[n]}{k}$ for the family of its all k -element subsets.

The main aim of the present paper is to prove the following theorem.

Theorem 1. *If $1 \leq k, n$ are integers and n is large enough, that is $n \geq n(k)$, then one can find*

$$\left\lfloor \frac{1}{2} \binom{n}{k} \right\rfloor$$

unordered pairs $\{A_i, B_i\}$ so that all these sets are distinct elements of $\binom{[n]}{k}$, $A_i \cap B_i = \emptyset$ holds for every pair and

$$A_i \cup B_i \neq A_j \cup B_j \text{ holds for all } i \neq j. \quad (1)$$

Our method however proves the following stronger, perhaps less natural statement.

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Theorem 2. *If $n \geq n(k)$, then the members of $\binom{[n]}{k}$ can be listed in the following way:*

$$A_1, A_2, \dots, A_N, A_{N+1} = A_1$$

where $N = \binom{n}{k}$, $A_i \cap A_{i+1} = \emptyset$ ($1 \leq i \leq N$), and

$$A_i \cup A_{i+1} \neq A_j \cup A_{j+1} \text{ holds for all } i \neq j. \quad (2)$$

Theorem 2 trivially implies Theorem 1: take the pairs $\{A_{2\ell-1}, A_{2\ell}\}$ ($\ell = 1, 2, \dots$).

The proof of Theorem 2 will be based on a Hamiltonian type theorem that is a generalization of the following theorem of Dirac [2].

Theorem 3. *If G is a simple graph on $N \geq 3$ vertices and every degree in G is at least $\frac{N}{2}$, then G has a Hamiltonian cycle.*

We will use a similar theorem for the graph

$$G_1 = \left(\binom{[n]}{k}, E \right)$$

where two vertices are adjacent (their pair is in E) if the k -element sets are disjoint. The simple application of Dirac's theorem for this graph G_1 would lead to a cyclic listing of all k -element sets in such a way that the neighboring ones are disjoint. However we have to forbid simultaneous occurrences of certain pairs along the cycle. This will be formulated in a more general context.

The forbidden pairs of edges form a new graph with vertex set E . Let $G_2 = (E, F)$ be the new graph where F is the set of forbidden pairs of non-adjacent edges (that is edges without common vertices). We need to find a Hamiltonian cycle in G_1 which contains no two edges whose pair is in F .

Theorem 4. *Let $G_1 = (V, E)$ and $G_2 = (E, F)$ be graphs where F contains pairs of edges non-adjacent in G_1 . Let further the minimum degree in G_1 be δ and the maximum degree in G_2 be Δ . Suppose that*

$$\text{a path of three edges in } G_1 \text{ cannot contain an element of } F \quad (i)$$

and that the inequality

$$\delta \geq N \left(1 - \frac{1}{2(\Delta + 1)} \right) \quad (ii)$$

holds. Then there is a Hamiltonian cycle in G_1 containing no pair of edges $\in F$.

Historical comments and a list of similar results will be given in Section 4.

2. Proof of Theorem 4

We will use an indirect way. Suppose that the desired Hamiltonian cycle does not exist. Delete elements of F one by one. When G_2 becomes empty then a Hamiltonian cycle is to be found without any restriction. Therefore (ii) and Dirac's theorem implies its existence. On the way from G_2 to the empty graph there is a place of jump: there is no Hamiltonian cycle if G_2 is replaced by $G_2^* = (E, F^* \cup \{f\})$ but there is one if it is replaced by $G_2^{**} = (E, F^*)$. The Hamiltonian cycle in the latter case must contain two edges which are the elements of f . Deleting one of these edges from the Hamiltonian cycle a Hamiltonian path is obtained that contains no pair of edges from F^* .

Let the vertices in this Hamiltonian path be ordered in the following way: v_1, v_2, \dots, v_N . Let P denote the set of edges along this path: $P = \{\{v_i, v_{i+1}\} : 1 \leq i < N\}$. Then no element of F^* is a subset of P . Define $A^1 \subset P$ as the set of edges along the path having a pair in F^* which is incident to v_1 :

$$A^1 = \{\{v_i, v_{i+1}\} : 1 < i < N, \{\{v_1, v_\ell\}, \{v_i, v_{i+1}\}\} \in F^* \text{ holds for some } \ell\}.$$

Let L denote the set of endpoints of edges in G_1 with the starting point v_1 :

$$L = \{\ell : \{v_1, v_\ell\} \in E\}.$$

Suppose that $\{v_i, v_{i+1}\} \in A^1$ and either $v_i \in L$ or $v_{i+1} \in L$ holds. Then $\{\{v_1, v_\ell\}, \{v_i, v_{i+1}\}\} \in F^*$ holds for some $\ell \neq i, i+1$. Hence $\{v_1, v_\ell\}, \{v_i, v_{i+1}\}$ and either $\{v_1, v_i\}$ or $\{v_1, v_{i+1}\}$ form a path of length 3 in E . Two of them form a pair in F^* by the definition of A^1 . This contradicts (i) therefore $\{v_i, v_{i+1}\}$ cannot be in A^1 . The conclusion is that the edges $\{v_i, v_{i+1}\} (1 < i)$ in the path cannot be in A^1 if one of their endpoints is in L . The number of such edges along the path is at least $|L| \geq \delta$. Hence we have $|A^1| \leq N - 2 - \delta$. Here $A^1 \subset E$ is a set of vertices of G_2 . The degrees in G_2 are $\leq \Delta$ the total number of vertices in G_2 adjacent to at least one element of A^1 is $\leq \Delta(N - 2 - \delta)$. There are at least δ edges in E of the form $\{v_1, v_\ell\}$, we have at least $\delta - \Delta(N - 2 - \delta)$ of them which do not form an edge in G_2 with an edge in P . Here $\delta - \Delta(N - 2 - \delta) \geq \frac{N}{2}$ holds by (ii). If B^1 denotes the set of edges $\{v_1, v_\ell\}$ in G_1 which are not vertices in G_2 adjacent to vertices $\in P$ then we obtain the conclusion of this paragraph

$$|B^1| \geq \frac{N}{2}. \quad (3)$$

This can be repeated with the other end of the path. Let B^N denote the set of edges $\{v_N, v_\ell\}$ in G_1 which are not vertices in G_2 adjacent to vertices $\in P$. We have

$$|B^N| \geq \frac{N}{2}. \quad (4)$$

The edge $\{v_i, v_{i+1}\}$ is called *start-pinned* if $\{v_N, v_i\} \in B^N$. On the other hand, it is *end-pinned* if $\{v_1, v_{i+1}\} \in B^1$. By (3) and (4) there is an edge $\{v_r, v_{r+1}\}$ which is both start-

and end-pinned. Hence the edges $\{v_1, v_{r+1}\}$ and $\{v_N, v_r\}$ can be added to P without violating the conditions, that is $P \cup \{\{v_1, v_{r+1}\}, \{v_N, v_r\}\}$ contains no element of F^* . (The relation $\{\{v_1, v_{r+1}\}, \{v_N, v_r\}\} \notin F^*$ is a direct consequence of (i).) The sequence of vertices $v_1, v_{r+1}, v_{r+2}, \dots, v_N, v_r, v_{r-1}, \dots, v_2, v_1$ determines a Hamiltonian cycle satisfying the conditions. This is a contradiction, the statement is proved. \square

3. Proof of Theorem 2

Use Theorem 4 for the following graphs G_1 and G_2 .

$$G_1 = \left(\binom{[n]}{k}, E \right)$$

where $\{A, B\} \in E$ if and only if $A \cap B = \emptyset$, F consists of the pairs $\{\{A, B\}, \{C, D\}\}$ ($\{A, B\} \neq \{C, D\}$) satisfying

$$A, B, C, D \in \binom{[n]}{k}, A \cap B = C \cap D = \emptyset, A \cup B = C \cup D. \quad (5)$$

Check the conditions of Theorem 4 for these graphs. (i) is obvious: if (5) holds then A and C cannot be disjoint.

The number of vertices of G_1 is $N = \binom{n}{k}$. Both graphs are regular. $\delta = \binom{n-k}{k}$. Observe that G_2 is a vertex disjoint union of clicks of size $\frac{1}{2} \binom{2k}{k}$. Therefore $\Delta = \frac{1}{2} \binom{2k}{k} - 1$. (ii) has the following form:

$$\binom{n-k}{k} \geq \binom{n}{k} \left(1 - \frac{1}{\binom{2k}{k}} \right). \quad (6)$$

For fixed k and large n the two binomial coefficients are asymptotically equal. The coefficient on the right hand side is less than 1. Therefore (6) holds for large n . \square

More detailed analysis (see below) of (6) gives that $n(k) = k^2 \binom{2k}{k} + k$ is a possible threshold. The following inequality is an equivalent form of (6).

$$\frac{(n-k)(n-k-1) \dots (n-2k+1)}{n(n-1) \dots (n-k+1)} \geq \left(1 - \frac{1}{\binom{2k}{k}} \right). \quad (7)$$

Apply the inequality

$$\frac{n-k-i}{n-i} \geq \frac{n-2k}{n-k} \quad (0 \leq i < k)$$

on the left hand side of (7) then use the Bernoulli-inequality ($n \geq 2k$ can be supposed):

$$\frac{(n-k)(n-k-1) \dots (n-2k+1)}{n(n-1) \dots (n-k+1)} \geq$$

$$\left(\frac{n-2k}{n-k}\right)^k \geq \left(1 - \frac{k}{n-k}\right)^k \geq 1 - \frac{k^2}{n-k}.$$

Here the right hand side gives the right hand side of (7) for $n(k) = k^2 \binom{2k}{k} + k$.

4. Historical comments, similar results

The method of using a Hamiltonian type theorem for constructing a family of disjoint pairs satisfying certain properties goes back to [1]. The following theorem was proved there.

Theorem 5. *If $1 \leq k, n$ are integers and n is large enough, that is $n \geq n(k)$, then one can find $\lfloor \frac{1}{2} \binom{n}{k} \rfloor$ unordered pairs $\{A_i, B_i\}$ of disjoint k -element subsets ($A_i \cap B_i = \emptyset, |A_i| = |B_i| = k$) of $[n]$ such that*

$$\min\{|A_i \cap A_j|, |B_i \cap B_j|\} \leq \frac{k}{2}$$

which implies

$$\min\{|A_i \cap B_j|, |B_i \cap A_j|\} \leq \frac{k}{2}$$

by the symmetry.

The proof of this theorem is based on the following Hamiltonian type statement.

Theorem 6. [1] *Let $G_0 = (V, E_0)$ and $G_1 = (V, E_1)$ be simple graphs on the same vertex set $|V| = N$, such that $E_0 \cap E_1 = \emptyset$. Let r be the minimum degree of G_0 and let s be the maximum degree of G_1 . Suppose, that*

$$2r - 8s^2 - s - 1 > N$$

holds, then there is a Hamiltonian cycle in G_0 such that if (a, b) and (c, d) are two vertex-disjoint edges of the cycle, then they do not form an alternating cycle with two edges of G_1 .

It was discovered in [3] that Theorem 5 can be made stronger in the following way.

Theorem 7. *If $1 \leq k, n$ are integers and n is large enough, that is $n \geq n(k)$, then one can find $\lfloor \frac{1}{2} \binom{n}{k} \rfloor$ unordered pairs $\{A_i, B_i\}$ of disjoint k -element subsets ($A_i \cap B_i = \emptyset, |A_i| = |B_i| = k$) of $[n]$ such that*

$$|A_i \cap A_j| + |B_i \cap B_j| \leq k$$

which implies

$$|A_i \cap B_j| + |B_i \cap A_j| \leq k$$

by the symmetry.

Its proof was based on a theorem similar to Theorem 6, but it is much more complicated.

Observe that Theorems 5 and 7 require the disjoint pairs $\{A_i, B_i\}$ to be “close” to each other. On the other hand our present Theorem 1 does not allow them to be too “close”.

Both Theorems 4 and 6 can be interpreted in the following way. Given a graph and a 4-graph (4-uniform hypergraph) on the same vertex set. Give conditions ensuring the existence of a Hamiltonian cycle in the graph avoiding the given 4-edges as the union of two edges of the cycle. These interpretations lead to some Hamiltonian problems and theorems involving hypergraphs. The first result of this type was [5]. The survey paper [4] contains similar questions and answers. There are some important newer results in the following papers: [7], [8], [9], [6].

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