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HANDICAP DISTANCE ANTIMAGIC GRAPHS AND INCOMPLETE TOURNAMENTS

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Abstract

Let $G = (V, E)$ be a graph of order n. A bijection $f : V \to \{1, 2, \ldots, n\}$ is called a *distance magic labeling* of G if there exists a positive integer μ such that $\sum_{u \in N(v)} f(u) = \mu$ for all $v \in V$, where $N(v)$ is the open neighborhood of v. The constant μ is called the *magic constant* of the labeling f. Any graph which admits a distance magic labeling is called a *distance magic graph.* The bijection $f : V \to \{1, 2, \ldots, n\}$ is called a *d-distance antimagic labeling* of G if for $V = \{v_1, v_2, \ldots, v_n\}$ the sums $\sum_{u \in N(v_i)} f(u)$ form an arithmetic progression with difference d.

We introduce a generalization of the well-known notion of *magic rectangles* called *magic rectangle sets* and use it to find a class of graphs with properties derived from the distance magic graphs. Then we use the graphs to construct a special kind of incomplete round robin tournaments, called *handicap tournaments.*

Keywords: Distance magic labeling, magic constant, handicap incomplete tournament.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terms we refer to [2].

Distance magic labelings of graphs belong to a wide family od magic type labelings that can be vaguely described as follows. Graph elements (that is, either vertices, or edges, or both) are labeled by consecutive integers $1, 2, \ldots, t$ where $t = n$ or m or $n+m$, respectively, is such a way that the sum of labels of the elements incident and/or adjacent to every vertex or every edge is equal to a given constant μ.

Magic square is a well known structure. It is an $a \times a$ array filled with numbers $1, 2, \ldots, a^2$, each appearing once, such that the sum of each row, column, and the main and main backward diagonal is equal to $a(a^2+1)/2$. For a nice survey of results related to magic squares, we refer the reader to [3].

Motivated by properties of magic squares, Vilfred [13] in his doctoral thesis introduced the concept of sigma labelings. The same concept was introduced by Miller et al. [10] under the name 1-vertex magic vertex labeling. Sugeng et al. [12] introduced the term distance magic labeling for this concept. In this paper we use the term distance magic labeling as it seems to be the prevailing term recently.

Definition 1.1. [13] *A distance magic labeling of a graph G of order n is a bijection $f: V \to \{1,2,\dots,n\}$ with the property that there is a positive integer μ such that*

$$\sum_{y \in N(x)} f(y) = \mu \quad \textit{for every} \quad x \in V.$$

The constant μ is called the magic constant of the labeling f. The sum $\sum\limits_{y\in N(x)} f(y)$ is called the weight of vertex x and is denoted by $w(x)$. A graphs that admits a distance magic labeling is called a distance magic graph.

A survey of results on distance magic graphs can be found in [1].

In this paper we will introduce two closely related concepts, namely the distance antimagic and handicap distance antimagic labelings and show their relationship to certain types of incomplete round robin tournaments. To construct these labelings, we use a new generalization of magic rectangles, which are in turn a generalization of magic squares.

2. Fair and equalized incomplete tournaments

Author's interest in this topic was sparked by a real life situation, when he was asked to schedule a soccer tournament. The organizers wanted to schedule an one-divisional tournament, but there was not enough time to play the complete round robin. After some discussion, the organizers and the scheduler agreed on the following format. They decided to schedule a *fair incomplete round robin tournament* with the following properties:

1. Every team plays the same number of opponents.
2. The difficulty of the tournament for each team mimics the difficulty of the complete round robin tournament.

Condition 2 can be justified as follows. If we know the strength of each team based on team standings in the previous year, the teams can be ranked from 1 to n. Based on their rankings, we can define the *strength* of the i-th ranked team (or just team i for short) in a tournament with n teams as $s_n(i) = n+1-i$. The *total strength of opponents* of team i in a complete round robin tournament is then defined as $\overline{S}_{n,n-1}(i) = n(n+1)/2 - s_n(i) =$

$(n+1)(n-2)/2+i$. We observe that the total strengths form an increasing arithmetic progression with difference one. Therefore, we want the total strengths of opponents for respective teams in our incomplete tournament to form such a progression as well. In general, we want to find a tournament of n teams with each team playing g games in which the total strength of opponents of the i-th ranked team is $\overline{S}_{n,g}(i) = (n+1)(n-2)/2+i-c$ for some integer c.

Obviously, this is equivalent to finding the set of games that are left out of the complete tournament with the property that the total strength of opponents in the $n-g-1$ left out games, $S_{n,n-g-1}(i)$, is equal to some constant c for every team i.

A *fair incomplete tournament* of n teams with g rounds, $\mathrm{FIT}(n,g)$, is a tournament in which every team plays g other teams and the total strength of the opponents that team i plays is $\overline{S}_{n,g}(i) = (n+1)(n-2)/2+i-c$ for every i and some fixed constant c. The total strength of the opponents that each team misses is then equal to c. Hence, we can view the games that are *not* played as a complement of $\mathrm{FIT}(n,g)$, which is itself an incomplete tournament. In an *equalized incomplete tournament* of n teams with r rounds, $\mathrm{EIT}(n,r)$, every team plays exactly r other teams and the total strength of the opponents that team i plays is $S_{n,r}(i) = c$ for every i. Notice that $\mathrm{EIT}(n,n-g-1)$ is the complement of $\mathrm{FIT}(n,g)$. Therefore, a $\mathrm{FIT}(n,g)$ exists if and only if an $\mathrm{EIT}(n,n-g-1)$ exists.

One can notice that finding an $\mathrm{EIT}(n,r)$ is equivalent to finding a distance magic labeling of *any* r-regular graph on n vertices. We also observe that the complementary $\mathrm{FIT}(n,n-r-1)$ is a *distance antimagic graph.*

Definition 2.1. *A distance d-antimagic labeling of a graph $G(V,E)$ with n vertices is a bijection $\bar{f} : V \to \{1,2,\ldots,n\}$ with the property that there exists an ordering of the vertices of G such that the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an arithmetic progression with difference d. When $d=1$, then $\bar{f}$ is called just distance antimagic labeling. A graph G is a distance d-antimagic graph if it allows a distance d-antimagic labeling, and distance antimagic graph when $d=1$.*

Notice that the order of vertices whose weights form the arithmetic progression is in general not specified, and need not be related to their respective labels. We will define later a special case of such antimagic labeling where the sequences $(w(x_1), w(x_2), \ldots, w(x_n))$ and $(f(x_1), f(x_2), \ldots, f(x_n))$ will be closely related.

The weight $w(x)$ of a vertex x in a $\mathrm{FIT}(n,k)$ or $\mathrm{EIT}(n,r)$ is equal to $\overline{S}_{n,k}(x)$ or $S_{n,r}(x)$, respectively.

In the language of distance magic graphs, our observation can be stated as follows.

Observation 2.2. *If a graph G is distance magic, then its complement $\overline{G}$ is distance antimagic.*

It was observed independently by several sets of authors that if G is an r-regular distance magic graph, then r is even.

Observation 2.3. [9, 10, 11, 13] *For odd r, there is no distance magic r-regular graph.*

The remaining feasible values of r for r-regular distance magic graphs with an even number of vertices were found in [5].

Theorem 2.4. [5] *For even n, an r-regular distance magic graph with n vertices exists if and only if $2 \leq r \leq n-2, r \equiv 0 \pmod 2$ and either $n \equiv 0 \pmod 4$ or $r \equiv 0 \pmod 4$.*

For graphs with an odd number of vertices, the existence question of regular distance magic graphs was partially answered in [4].

Theorem 2.5. [4] *Let n, q be odd integers and s an integer, $q \geq 3$, $s \geq 1$. Let $r = 2^s q$, $q \mid n$ and $n \geq r + q$. Then, an r-regular distance magic graph of order n exists.*

When the maximum odd divisor of r does not divide n, somewhat weaker result can be proved.

Theorem 2.6. [4] *Let n, q be odd integers and s an integer, $q \geq 3$, $s \geq 1$. Let $r = 2^s q$, $q \nmid n$ and $n \geq (7r+4)/2$. Then, an r-regular distance magic graph of order n exists.*

The proofs are based on an application of magic rectangles, which are a natural generalization of magic squares.

Definition 2.7. *A magic rectangle $MR(a, b)$ is an $a \times b$ array with entries from the set $\{1, 2, \ldots, ab\}$, each appearing once, with all its row sums equal to a constant ρ and with all its column sums equal to a constant σ.*

The sum of all entries in the array is $ab(ab+1)/2$; it follows that

$$\sigma = \sum_{i=1}^{a} m_{ij} = a(ab+1)/2 \text{ for all } j \quad \text{and} \tag{1}$$

$$\rho = \sum_{j=1}^{b} m_{ij} = b(ab+1)/2 \text{ for all } i. \tag{2}$$

Hence a and b must either both be even or both odd. It was proved in [7, 8] that such an array exists whenever a and b have the same parity, except for the impossible cases when exactly one of a and b is 1, or when $a = b = 2$. We state the result formally here.

Theorem 2.8. [7, 8] *A magic rectangle $MR(a, b)$ exists if and only if $a, b > 1, ab > 4$, and $a \equiv b \pmod 2$.*

To prove our main result, we will need the following generalization of magic rectangles.

Definition 2.9. *A magic rectangle set* $\mathcal{M} = MRS(a, b; c)$ *is a collection of* c *arrays* $(a \times b)$ *whose entries are elements of* $\{1, 2, \ldots, abc\}$*, each appearing once, with all row sums in every rectangle equal to a constant* ρ *and all column sums in every rectangle equal to a constant* σ*.*

Observe that this generalization is less restrictive than the notion of n-dimensional magic rectangle, which was introduced by Hagedorn in [6].

Definition 2.10. *An* n*-dimensional magic rectangle* $\mathcal{D} = n\text{-}MR(a_1, a_2, \ldots, a_n)$ *is an* $a_1 \times a_2 \times \cdots \times a_n$ *array with entries* $d_{i_1,i_2,\ldots,i_n}$ *which are elements of* $\{1, 2, \ldots, a_1a_2 \ldots a_n\}$*, each appearing once, such that all sums in the* k*-th direction are equal to a constant* σ_k*. That is, for every* k*,* $1 \le k \le n$*, we have*

$$\sum_{j=1}^{a_k} d_{b_1,b_2,\ldots,b_{k-1},j,b_{k+1},\ldots,b_n} = \sigma_k$$

for every selection of indices $b_1, b_2, \ldots, b_{k-1}, b_{k+1}, \ldots, b_n$*, and* $\sigma_k = a_k(a_1a_2 \ldots a_n + 1)/2$*.*

The following existence results were also proved in [6].

Theorem 2.11. [6] *If an* n*-dimensional magic rectangle* $\mathcal{D} = n\text{-}MR(a_1, a_2, \ldots, a_n)$ *exists, then* $a_1 \equiv a_2 \equiv \cdots \equiv a_n \pmod 2$*.*

Theorem 2.12. [6] *An* n*-dimensional magic rectangle* $\mathcal{D} = n\text{-}MR(a_1, a_2, \ldots, a_n)$ *with* $a_1 \le a_2 \le \cdots \le a_n$ *and all* a_i *even exists if and only if* $4 \le a_2 \le \cdots \le a_n$*.*

3. Handicap tournaments

Although the fair incomplete tournaments mimic the structure of the complete round-robin tournaments, they in fact favor the highest ranked team, because the total strength of its opponents, $\overline{S}_{n,k}(1)$, is the lowest. Even in an equalized tournament the highest ranked team has the best chance of winning, because all teams face opponents with the same total strength and it is apparent that if the strongest and the weakest team both play a set of opponents with the same strength, the strongest team has much higher probability of finishing with more wins than the weakest team. If we want to give all teams roughly the same chance of winning, we need to schedule a tournament with handicaps.

A *handicap incomplete tournament* of n teams with r rounds, $\mathrm{HIT}(n, r)$, is a tournament in which every team plays r other teams and the total strength of the opponents that team i plays is $\vec{S}_{n,r}(i) = t - i$ for every i and some fixed constant t. This means that the strongest team plays strongest opponents, and the lowest ranked team plays weakest opponents. In terms of distance magic graphs this restriction corresponds to finding a distance antimagic graph with the additional property that the sequence $w(1), w(2), \ldots, w(n)$ (where team i is again the i-th ranked team) is an increasing arithmetic progression with difference one. We call this special case *handicap distance antimagic graphs.*

Definition 3.1. *A handicap distance d-antimagic labeling of a graph $G(V,E)$ with n vertices is a bijection $\vec{f} : V \to \{1, 2, \ldots, n\}$ with the property that $\vec{f}(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an increasing arithmetic progression with difference d. A graph G is a handicap distance d-antimagic graph if it allows a handicap distance d-antimagic labeling, and handicap distance antimagic graph when $d = 1$.*

Notice that this is an inverse ordering compared with the ordering of labeled vertices in a complete distance antimagic graph, or any distance magic graph which is a complement of a regular distance magic graph. There we have $w(1) > w(2) > \ldots > w(n)$, while in a graph with a handicap distance antimagic labeling we have $w(1) < w(2) < \ldots < w(n)$.

In the next section, we prove our main result, using the notion of magic rectangle sets. By $c(K_a \square K_b)$ we denote the graph consisting of c mutually disjoint copies of the Cartesian product of complete graphs K_a and K_b.

Theorem 3.2. *Let a and b be even positive integers such that $2 \leq a \leq b$, $4 < ab$, and c be any positive integer. Let $n = abc$ and $G = c(K_a \square K_b)$. Then the complement of G is a handicap distance antimagic graph with n vertices.*

4. Existence of magic rectangle sets

To prove non-existence of a class of magic rectangle sets, we start with a simple observation.

Observation 4.1. *If a magic rectangle set $MRS(a, b; c)$ exists, then both $MR(a, bc)$ and $MR(ac, b)$ exist.*

To construct $MR(a, bc)$ (or $MR(ac, b)$), we simply take all $a \times b$ rectangles and "glue" them together into one $a \times bc$ (or $ac \times b$) rectangle.

The non-existence result then can be proved as follows.

Theorem 4.2. *If $a \equiv 1 \pmod 2$ and either $b \equiv 0 \pmod 2$ or $c \equiv 0 \pmod 2$, then a magic rectangle set $MRS(a, b; c)$ does not exist.*

Proof. Suppose that $a \equiv 1 \pmod 2$ and an $MRS(a, b; c)$ exists. Then by Observation 4.1 a magic rectangle $MR(a, bc)$ also exists. But $a \equiv 1 \pmod 2$ and $bc \equiv 0 \pmod 2$ and such $MR(a, bc)$ cannot exist by Theorem 2.8. This contradiction proves our claim. □

The above result also follows from Theorem 2.11.

Now we show that for a and b both even, a magic rectangle set $MRS(a, b; c)$ can be constructed for any c. Notice that for c even the proof follows from Theorem 2.12 as every of the c layers of n-$MR(a, b, c)$ is one of the $a \times b$ rectangles.

We start with an elementary step.

Lemma 4.3. *If $b \equiv 0 \pmod 2$ and $b \geq 4$, then a magic rectangle set $MRS(2, b; c)$ exists for every c.*

Proof. Denote by $x^s_{i,j}$ the entry in the i-th row and j-th column of the s-th rectangle. Set $x^1_{1,j} = j$ and $x^1_{2,j} = 2bc+1-j$ for $j = 1, 3, \ldots, b-1$, and $x^1_{2,j} = j$ and $x^1_{1,j} = 2bc+1-j$ for $j = 2, 4, \ldots, b$. The remaining rectangles (if $c > 1$) are defined recursively as $x^s_{i,j} = x^{s-1}_{i,j} + b$ when $x^{s-1}_{i,j} < bc$ and $x^s_{i,j} = x^{s-1}_{i,j} - b$ when $x^{s-1}_{i,j} > bc$ for $s = 2, 3, \ldots, c$. Apparently, every column adds up to $2bc + 1$ and every row in each rectangle has the sum equal to $b(2bc + 1)/2$. □

We can now modify Observation 4.1 to see that the following theorem holds. To construct $MRS(a, b; c)$ with $a \equiv b \equiv 0 \pmod 2$, we first use Lemma 4.3 to build a set $MRS(2, b; ca')$ where $a = 2a'$ and then glue together subsets of a' rectangles to obtain the set $MRS(a, b; c)$. The following then obviously holds.

Theorem 4.4. *If $a \equiv b \equiv 0 \pmod 2$, $a \geq 2$ and $b \geq 4$, then a magic rectangle set $MRS(a, b; c)$ exists for every c.*

5. Construction of handicap tournaments from magic rectangle sets: Proof of Theorem 3.2

Let $G = c(K_a \Box K_b)$ with $V(G) = \{v^k_{i,j} \mid 1 \leq i \leq a, 1 \leq j \leq b, 1 \leq k \leq c\}$ and $E(G) = \{v^k_{i,j} v^k_{i,l} \mid 1 \leq i \leq a,\ 1 \leq j < l \leq b,\ 1 \leq k \leq c\} \cup \{v^k_{i,j} v^k_{l,j} \mid 1 \leq i < l \leq a,\ 1 \leq j \leq b,\ 1 \leq k \leq c\}$ and $\mathcal{M} = \{R^k \mid 1 \leq k \leq c\}$ be a magic rectangle set $MRS(a, b; c)$ with row sums ρ and column sums σ. The labeling $\bar{f}(v^k_{i,j}) = r^k_{i,j}$ is obviously a distance 2-antimagic labeling, for when $\bar{f}(v^k_{i,j}) = r^k_{i,j} = p$, then $w_G(v^k_{i,j}) = \rho + \sigma - 2p$. Hence, the following observation holds.

Observation 5.1. *The graph $G = c(K_a \Box K_b)$ admits a distance 2-antimagic labeling f such that $f(x) = p$ implies $w_G(x) = (a + b)(abc + 1)/2 - 2p$ for every $x \in V(G)$ whenever there exists a magic rectangle set $MRS(a, b; c)$.*

The proof of Theorem 3.2 then follows easily. We show that $\overline{G}$, the complement of G, has a handicap antimagic labeling $\vec{f}$. We define $\vec{f}(v^k_{i,j}) = \bar{f}(v^k_{i,j})$. For $v^k_{i,j}$ with $\vec{f}(v^k_{i,j}) = p$ we have

$$w_G(v^k_{i,j}) + w_{\bar{G}}(v^k_{i,j}) = n(n + 1)/2 - p$$

and because

$$w_G(v^k_{i,j}) = \rho + \sigma - 2p,$$

we have

$$w_{\bar{G}}(v^k_{i,j}) = n(n + 1)/2 - \rho - \sigma + p.$$

The values of p are $1, 2, \ldots, n$ and $\overline{G}$ has a handicap antimagic labeling. This completes the proof of Theorem 3.2.

6. Conclusion

From a practical point of view, an r-regular handicap incomplete tournament $\mathrm{HIT}(n, r)$ should have r close to $n-1$, or at least greater than $(n-1)/2$. Therefore, it should arise from a magic rectangle $MR(a, b)$ with $2 < a \leq b$ or from a magic rectangle set $MRS(a, b; c)$ with $c \geq 2$.

It is nevertheless obvious that our construction does not cover the whole spectrum of orders of handicap tournaments. For instance, when n is a prime, then $\mathrm{HIT}(n, r)$ cannot be constructed from magic rectangles. Therefore, we pose the following problem.

Problem 6.1. *For what pairs* (n, r) *there exists a handicap tournament* $\mathrm{HIT}(n, r)$, *or equivalently, an* r*-regular handicap distance antimagic graph with* n *vertices?*

Another direction is to explore the existence of handicap distance d-antimagic graphs for $d > 1$.

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