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The Odd Harmonious Labeling of Dumbbell and Generalized Prism Graphs

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Abstract

A graph $G = (V, E)$ with $|E| = q$ is said to be odd harmonious if there exists an injection $f : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$ such that the induced function $f^* : E(G) \to \{1, 3, 5, \ldots, 2q - 1\}$ defined by $f^*(xy) = f(x) + f(y)$ is a bijection. Then f is said to be odd harmonious labeling of G. A dumbbell graph $D_{n,k,2}$ is a bicyclic graph consisting of two vertex-disjoint cycles C_n, C_k and a path P_2 joining one vertex of C_n with one vertex of C_k. A prism graph $C_n \times P_m$ is a Cartesian product of cycle C_n and path P_m. In this paper we show that the dumbbell graph $D_{n,k,2}$ is odd harmonious for $n \equiv k \equiv 0 \pmod 4$ and $n \equiv k \equiv 2 \pmod 4$, generalized prism graph $C_n \times P_m$ is odd harmonious for $n \equiv 0 \pmod 4$ and for any m, and generalized prism graph $C_n \times P_m$ is not odd harmonious for $n \equiv 2 \pmod 4$.

Keywords: dumbbell, generalized prism, odd harmonious graphs, odd harmonious labeling.

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1. Introduction

A graph $G = (V, E)$ with p vertices and q edges is called a $(p, q)-$graph. A graph G is said to be harmonious if there exists an injection $f : V \to \{0, 1, 2, \ldots, q - 1\}$ such that the induced function $f^* : E(G) \to \{0, 1, 2, \ldots, q - 1\}$ defined by $f^*(xy) = (f(x) + f(y)) \pmod q$ is a bijection and f is said to be harmonious labeling of G. A graph G is said to be odd harmonious if there exists an injection $f : V \to \{0, 1, 2, \ldots, 2q - 1\}$ such that the induced function $f^* : E(G) \to \{1, 3, 5, \ldots, 2q - 1\}$ defined by $f^*(xy) = f(x) + f(y)$ is a bijection. The labeling f is said to be odd harmonious labeling of G [2].

Liang and Bai [2] have obtained necessary conditions for the existence of odd harmonious labeling of graphs:

1. If G is an odd harmonious graph, then G is a bipartite graph.
2. If a (p, q)-graph G is odd harmonious, then $2\sqrt{q} \leq p \leq 2q - 1$.

Furthermore, Liang and Bai proved that a cycle C_n is odd-harmonious if and only if $n \equiv 0 \pmod 4$; a complete graph K_n is odd harmonious if and only if $n = 2$; a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is odd harmonious if and only if $k = 2$; a windmill graph is odd harmonious if and only if $n = 2$; caterpillars and lobsters are odd harmonious.

Vaidya and Shah [3] have proved that the shadow graphs of path P_n and star $K_{1,n}$ are odd harmonious. Furthermore, they proved that split graphs of path P_n and star $K_{1,n}$ admit odd harmonious labeling. For a dynamic survey of various graph labeling problems along with extensive bibliography we refer to Gallian [1].

In this paper we prove the existence of odd harmonious labeling for dumbbell $D_{n,k,2}$ when $n \equiv k \equiv 0 \pmod 4$ and $n \equiv k \equiv 2 \pmod 4$, and prism $C_n \times P_m$ for $n \equiv 0 \pmod 4$ and any m.

2. New Results

A dumbbell graph $D_{n,k,2}$ is a bicyclic graph consisting of two vertex-disjoint cycles C_n and C_k, and a path P_2 joining them by connecting one vertex from each cycle [4]. The sets of vertices and edges of a dumbbell graph $D_{n,n,2}$ are given by $V(D_{n,k,2}) = \{u_i | 0 \leq i \leq n-1\} \cup \{v_i | 0 \leq i \leq k-1\}$ and $E(D_{n,k,2}) = \{u_i u_{i+1} | 0 \leq i \leq n-1\} \cup \{v_i v_{i+1} | 0 \leq i \leq k-1\} \cup \{u_0 v_0\}$ where the indices in u_i and v_i are taken modulo n and k, respectively.

A generalized prism graph $C_n \times P_m$ is the Cartesian product of a cycle C_n and a path P_m [1]. A generalized prism $C_n \times P_m$ has the set of vertices $V(C_n \times P_m) = \{v_i^j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and the set of edges $E(C_n \times P_m) = \{\{v_i^j v_{i+1}^j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\} \cup \{v_i^j v_i^{j+1} \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}\}$ where the indices $i+1$ are taken modulo n.

By [2], a graph which has an odd cycle as its subgraph is not odd harmonious. Consequently, a dumbbell graph $D_{n,k,2}$ and a generalized prism graph $C_n \times P_m$ with n or k odd are not odd harmonious graphs. In the following we consider dumbbell graphs and generalized prism graphs with n and k even.

Lemma 2.1. *The dumbbell $D_{n,k,2}$ is odd harmonious for any $n \equiv k \equiv 0 \pmod 4$.*

Proof. We divide the proof into two cases.

Case 1. Let $n = k$.

Let $G = D_{n,n,2}$. Then $p = 2n$ and $q = 2n + 1$. Denote the vertices in the first cycle C_n by $x_0, x_1, \ldots, x_{n-1}$ and the vertices in the second cycle C_n by $y_0, y_1, \ldots, y_{n-1}$ and label them as follows.

$$f(x_i) = \begin{cases} i, & i \text{ is even}, 0 \le i \le n, \\ i, & i \text{ is odd}, 1 \le i \le \frac{n}{2} - 1, \\ i+2, & i \text{ is odd}, \frac{n}{2} + 1 \le i \le n-1. \end{cases}$$

The induced labels for the first cycle are

$$1, 3, 5, ..., 2n+1, 2n+3.$$

Label the vertices of the second cycle C_n as follows.

$$f(y_i) = \begin{cases} 2n+1-i, & i \text{ is even}, 0 \le i \le n, \\ 2n+1-i, & i \text{ is odd}, 1 \le i \le \frac{n}{2} - 1, \\ 2n-1-i, & i \text{ is odd}, \frac{n}{2} + 1 \le i \le n-1. \end{cases}$$

The induced labels for the second cycle are

$$2n+3, 2n+5, ..., 3n-1, 3n+1, 3n+3, ..., 4n-1.$$

The label for x_0y_0 is $f^*(x_0y_0) = 2n+1$. Thus, dumbbell graph $D_{n,n,2}$ has an odd harmonious labeling.

Case 2. Let $n \ne k$ and assume that $n < k$.

Let $n = 4s,\ k = 4t$. Denote the vertices in C_n by $x_0, \dots, x_{n-1}$ and the vertices in C_k by $y_0, \dots, y_{k-1}$ and label them as follows. For C_n let

$$f(x_i) = i \text{ for } i = 0, 2, \dots, n-2;\ f(x_{n-1}) = n + 2k + 4.$$

With a similar process as in Case 1 we obtain the induced edge labels as follows:

$$1, 3, 5, \dots, 2n-7, 2n-5, 2n+2k+1, n+2k+3.$$

For C_k let

$$f(y_{2i}) = \begin{cases} 2i+n-1 & \text{for } 0 \le t-1, \\ 2i+n+1 & \text{for } t+1 \le i \le 2t-s+1, \\ 2i+n+3 & \text{for } 2t-s+2 \le i \le 2t-1, \end{cases}$$

$$f(y_{2i-1}) = 2i+n-2 \text{ for } 1 \le i \le 2t.$$

The induced edge labels are then

$$2n-1,\ 2n+1,\ \dots,\ 2n+k-5,\ 2n+k-1,\ 2n+k+1,\ \dots,\ n+2k+1,\ n+2k+5,$$

$$n+2k+7,\ \dots,\ 2n+2k-1,\ \dots,\ 2n+k-3.$$

The only unused label is $2n+1$ and we get it by joining vertices x_n and y_4 labeled $n-2$ and $n+3$, respectively. Thus, the dumbbell graph $D_{n,k,2}$ has an odd harmonious labeling. □

Lemma 2.2. *The dumbbell $D_{n,k,2}$ is odd harmonious for any $n \equiv k \equiv 2 \pmod 4$, $n, k \geq 6$.*

Proof. We divide the proof into three cases.

Case 1. $n = k$.

Let $n = k = 4s + 2$. Denote the vertices in the two cycles by $x_0, x_1, \dots, x_{n-1}$ and $y_0, y_1, \dots, y_{n-1}$, respectively, and label them as follows. For the first cycle let

$$f(x_i) = i, \text{ for } 0 \leq i \leq n-2;; \ f(x_{n-1}) = 3n+1.$$

The induced edge labels are then

$$1,\ 3,\ 5, \dots,\ 2n-7,\ 2n-5,\ 4n-1,\ 3n+1.$$

For the second cycle let

$$f(y_{2i}) = \begin{cases} 2i+n, & \text{for } 0 \leq i \leq s, \\ 2i+n+4, & \text{for } s+1 \leq i \leq 2s, \end{cases}$$

$$f(y_{2i-1}) = \begin{cases} 2i+n-3, & \text{for } 1 \leq i \leq 2s, \\ 2i+n-1, & \text{for } i = 2s+1. \end{cases}$$

The induced edge labels are then

$$2n-1, 2n+1, \dots, 3n-3, 3n+3, \dots, 4n-5, 4n-3, 4n+1, 3n-1.$$

The only unused label is $2n-3$ and we get it by joining vertices x_{n-2} and y_1 labeled $n-2$ and $n-1$, respectively.

Case 2. $n = 6, k \geq 10$.

Let $n = 4s + 2$ and $k = 4t + 2$. Denote the vertices of the two cycles by $x_0, x_1, \dots, x_6$ and $y_0, y_1, \dots, y_{k-1}$, respectively, and label them as follows.

$$f(x_i) = i, \text{ for } 0 \leq i \leq 4;\ f(5) = 2k+9.$$

The induced edge labels are then

$$1, 3, 5, 7, 2k+13, 2k+9.$$

For the second cycle let

$$f(y_{2i}) = 2i+5, \text{for } 0 \leq i \leq 2t,$$

$$f(y_{2i-1}) = \begin{cases} 2i+4, & \text{for } 1 \leq i \leq t+1, \\ 2i+6, & \text{for } t+2 \leq i \leq 2t. \end{cases}$$

The induced edge labels are then

$$2n-1,\ 2n+1, \dots,\ 3n-3,\ 3n+3, \dots,\ 4n-5,\ 4n-3,\ 4n+1,\ 3n-1.$$

The only unused label is $2n-3$ and we get it by joining the vertices x_{n-2} and y_1 labeled $n-2$ and $n-1$, respectively.

Case 3. $10 \le n < k$.

Let $n = 4s+2$ and $k = 4t+2$. Denote the vertices in the two cycles by $x_0, x_1, \ldots, x_{n-1}$ and $y_0, y_1, \ldots, y_{n-1}$, respectively, and label them as follows. For the first cycle let

$$f(x_i) = i, \text{ for } 0 \le i \le n-2, \text{ and } f(x_{n-1}) = n+2k+3.$$

The induced edge labels are then

$$1,\ 3,\ \ldots,\ 2n-5,\ 2n+2k+1,\ n+2k+3.$$

For the second cycle let

$$f(y_{2i}) = n+2i-3, \text{ for } 0 \le i \le t+1,$$

$$f(y_{2i-1}) = \begin{cases} n+2i-1, & \text{for } t+2 \le i \le 2t-s+1, \\ n+2i+1, & \text{for } 2t-s+2 \le i \le 2t+2s+3. \end{cases}$$

The induced edge labels are then

$$2n-1,\ 2n+1, \ldots,\ 2n+k-1,\ 2n+k+3, \ldots,\ n+2k+1,$$

$$n+2k+5, \ldots,\ 2n+2k-3,\ 2n+2k-1,\ 2n+k+1.$$

The only unused label is $2n-3$ and we get it by joining the vertices x_{n-3} and y_0 labeled $n-2$ and $n-1$, respectively. Thus the dumbbell $D_{n,k,2}$is odd harmonious for any $n \equiv k \equiv 2 \pmod 4$. □

By the above results, we obtain a full characterization of odd harmonious dumbbell graphs $D_{n,k,2}$.

Theorem 2.3. *The dumbbell $D_{n,k,2}$ is odd harmonious if and only if $n \equiv k \equiv 2 \pmod 4$ or $n \equiv k \equiv 0 \pmod 4$.*

Next, we discuss the existence of odd harmonious labeling for generalized prism graphs. Again by [2], the generalized prism $C_n \times P_m$ with n odd cannot be odd harmonious. We first exclude one more class of generalized prisms.

Lemma 2.4. *The generalized prism $C_n \times P_m$ with $n \equiv 2 \pmod 4$ is not odd harmonious for any m.*

Proof. Suppose there exists an odd harmonious labeling f. Let $G = C_n \times P_m$, $n = 4k+2$ and $m > 1$. Then G has $p = nm = 4km + 2m$ vertices and $q = (2m-1)(4k+2) =$

$2(2m-1)(2k+1)$ edges. For simplicity denote $q = 2s$. Now the sum of all induced edge labels is

$$1 + 3 + 5 + \cdots + (2q-1) = q^2 = 4s^2.$$

Every vertex x of degree four contributes $4f(x)$ to the above sum and hence the total contribution of all vertices of degree four is a multiple of four.

There are $4k+2$ vertices in each of the cycles C_n of $C_n \times P_m$ and all of them have degree three. Exactly $4k+2 = 2(2k+1)$ of them have even labels. Hence, their total contribution is again a multiple of four. Finally, there are $4k+2 = 2(2k+1)$ vertices with odd labels, each of them of degree three. Call them x_i where the label of x_i is $f(x_i) = 2t_i + 1$ for some $t_i \in Z^+$, and $i = 1, 2, \ldots, 4k+2$. So the contribution of each such a vertex x_i is $3f(x_i) = 3(2t_i+1) = 2z_i + 1$ for some z_i and the sum of $4k+2$ odd numbers $(2z_1+1) + (2z_2+1) + \cdots + (2z_{4k+2}+1)$ is congruent to 2 modulo 4. Hence the total contribution of all vertex labels is also congruent to 2 modulo 4, which is impossible. □

Now we construct an odd harmonious labeling for the remaining class of generalized prisms.

Lemma 2.5. *The generalized prism $C_n \times P_m$ with $n \equiv 0 \pmod 4$ is odd harmonious for any m.*

Proof. Suppose the generalized prism $C_n \times P_m$ has the set of vertices $V(C_n \times P_m) = \{v_i^j \mid 1 \le i \le n,\ 1 \le j \le m\}$ and the set of edges $E(C_n \times P_m) = \{\{v_i^j v_{i+1}^j \mid 1 \le i \le n,\ 1 \le j \le m\} \cup \{v_i^j v_i^{j+1} \mid 1 \le i \le n,\ 1 \le j \le m\}\}$ where the indices are taken modulo n.

The labels of the vertices of $C_n \times P_m$ are given below.

1. For $j = 1$ and $j = 2$,

$$f(v_i^1) = \begin{cases} i-1, & \text{for } i \text{ odd, } 1 \le i \le n-1, \\ i-1, & \text{for } i \text{ even, } 2 \le i \le \frac{n}{2}, \\ i+1, & \text{for } i \text{ even, } \frac{n}{2}+2 \le i \le n. \end{cases}$$

$$f(v_i^2) = \begin{cases} 3n+1, & \text{for } i = 1, \\ 2n+i-2, & \text{for } i \text{ even, } 2 \le i \le n, \\ 2n+i-2, & \text{for } i \text{ odd, } 3 \le i \le \frac{n}{2}+1, \\ 2n+i, & \text{for } i \text{ odd, } \frac{n}{2}+3 \le i \le n-1. \end{cases}$$

2. If j is odd, $3 \le j \le m$,

$$f(v_i^j) = \begin{cases} (2j-1)n - j + i, & \text{for } i \text{ odd, } 1 \le i < j, \\ (2j-1)n - (j-2) + i, & \text{for } i \text{ even, } 2 \le i < j, \\ (2j-2)n - j + i, & \text{for } i \text{ odd, } j \le i \le n-1, \\ (2j-2)n - j + i, & \text{for } i \text{ even, } j+1 \le i \le \frac{n}{2} + (j-1), \\ (2j-2)n - (j-2) + i, & \text{for } i \text{ even,} \frac{n}{2} + (j+1) \le i \le n. \end{cases}$$

3. If j is even, $4 \le j \le m$,

$$f(v_i^j) = \begin{cases} (2j-1)n-(j-2)+i, & \text{for } i \text{ odd, } 1 \le i < j, \\ (2j-1)n-j+i, & \text{for } i \text{ even,} 2 \le i < j, \\ (2j-2)n-j+i, & \text{for } i \text{ even, } j \le i \le n, \\ (2j-2)n-j+i, & \text{for } i \text{ odd, } j+1 \le i \le \frac{n}{2}+(j-1), \\ (2j-2)n-(j-2)+i, & \text{for } i \text{ odd, } \frac{n}{2}+(j+1) \le i \le (n-1). \end{cases}$$

Therefore, f is an injection from $V(C_n \times P_m)$ to $\{0, 1, 2, \ldots, (4m-2)n-1\}$.

It follows that we have $f^*(E(C_n \times P_m)) = \{1, 3, 5, \ldots, (4m-2)n-1\}$. Hence $C_n \times P_m$ with $n \equiv 0 \pmod 4$ is odd harmonious for any m. □

The previous results give a full characterization of odd harmonious generalized prisms.

Theorem 2.6. *The generalized prism $C_n \times P_m$ is odd harmonious if and only if $n \equiv 0$ (mod 4).*

The sun $C_n \odot K_1$ is a subgraph of the prism $C_n \times P_2$ consisting of the cycle C_n^1 and the edges $v_i^1 v_i^2$. We can get sun $C_n \odot K_1$ by deleting the edges of the second cycle of $C_n \times P_2$.

Notice that the induced labeling f^* described above assigns the edge labels $1, 3, \ldots, 2n-1$ to the edges of C_n^1, labels $2n+1, 2n+3, \ldots, 4n-1$ to the "spokes", and labels $4n+1, 4n+3, \ldots, 6n-1$ to the edges of C_n^2. labels. Therefore, the following observation holds.

Observation 2.7. *The sun $C_n \odot K_1$ with $n \equiv 0 \pmod 4$ is odd harmonious graph.*

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