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The λ-backbone colorings of graphs with tree backbones

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Abstract

The λ-backbone coloring is one of the various problems of vertex colorings in graphs. Given an integer $\lambda \geq 2$, a graph $G = (V, E)$, and a spanning subgraph (backbone) $H = (V, E_H)$ of G, a λ-*backbone coloring* of (G, H) is a proper vertex coloring $V \to \{1, 2, ...\}$ of G in which the colors assigned to adjacent vertices in H differ by at least λ. *The* λ*-backbone coloring number* $BBC_\lambda(G, H)$ of (G, H) is the smallest positive integer l for which there exists a λ-backbone coloring $f : V \to \{1, 2, ..., l\}$ of (G, H). For a graph G with chromatic number $\chi(G) = k$, the λ-backbone coloring number is denoted by $BBC_\lambda(G(k), H)$. In this paper, we consider λ-backbone colorings of graphs with tree backbones. We determine the relation between $\chi(G)$ and $BBC_\lambda(G(k), H)$ of (G, H) with a tree backbone H for $\lambda \geq 3$.

Keywords: chromatic number, tree backbone, λ-backbone coloring,

λ-backbone coloring number.

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1. Introduction

The backbone coloring was first introduced by Broersma et al. [2]. They studied this topic motivated by the coloring problems related to frequency assignment. In this application, vertices of the graph represent the transmitters and the edges represent the interference effect between two corresponding transmitters when they broadcast on the similar frequency channels. Generally, the problem is to assign the frequency channels to the transmitters such that interference is kept at an acceptable level. One of the various models that can be used to solve this problem is the λ-backbone coloring.

Let $G = (V, E)$ be a graph with finite vertex set V and the edge set E. A function $f : V \to \{1, 2, \ldots\}$ is a *vertex coloring* of G if $|f(u) - f(v)| \geq 1$ for every two adjacent vertices $u, v \in V$. If a vertex coloring f uses k colors such that $f : V \to \{1, 2, \ldots, k\}$, then f is called a *k-coloring*. The smallest integer k such that there exists a k-coloring in G is called the *chromatic number* of G and is denoted by $\chi(G)$.

Let $H = (V, E_H)$ be a *spanning subgraph (backbone)* of G. Given an integer $\lambda \geq 2$, a vertex coloring f of G is a *λ-backbone coloring* of (G, H) if $|f(u) - f(v)| \geq \lambda$ for every edge $uv \in E_H$. The *λ-backbone coloring number* $BBC_\lambda(G, H)$ of (G, H) is the smallest positive integer l for which there exists a λ-backbone coloring $f : V \to \{1, 2, ..., l\}$ of (G, H). For a graph G with chromatic number $\chi(G) = k$, the λ-backbone coloring number is denoted by $BBC_\lambda(G(k), H)$.

We refer to several results concerning λ-backbone colorings. Salman [5] has investigated the computational complexity of λ-backbone coloring of graphs with n complete backbones. Broersma et al. [1] determined the value of the λ-backbone coloring number of any graph with either star or matching backbones where $\lambda \geq 2$. Some researchers also apply this problem to split graph which is a graph whose vertex set can be partitioned into a clique and an independent set, with possibly edges in between. The λ-backbone coloring number of split graphs with either path or tree backbones where $\lambda \geq 2$ can be seen in [2, 4]. Broersma et al. [3] also obtained the good bounds of the λ-backbone coloring number of split graph with either star or matching backbones where $\lambda \geq 2$.

In this paper, we consider the λ-backbone coloring number of any graph G whose chromatic number $\chi(G) = k$, with tree backbones T. We determine the relation between $\chi(G)$ and $BBC_\lambda(G(k), T)$. In order to analyze the maximum difference between these two numbers, we define:

$$\mathcal{T}_\lambda(k) = \max\{BBC_\lambda(G(k), T) \mid T \text{ is a tree backbone of } G\}.$$

Broersma et al. [2] have determined $\mathcal{T}_\lambda(k)$ for $\lambda = 2$.

Theorem 1.1. [2] $\mathcal{T}_2(k) = 2k - 1$ *for every* $k \geq 1$.

Now, we determine good bounds for $\mathcal{T}_\lambda(k)$ when $\lambda \geq 3$. We also give bounds of λ-backbone coloring number for planar graphs with tree backbone.

2. λ-Tree Backbones for Any Graph

Theorem 2.1. *For* $\lambda \geq 3$ *and* $k \geq 1$,

$$\mathcal{T}_\lambda(k) = \begin{cases} 1 + (k-1)\lambda, & \text{for } k \in \{1, 2\}; \\ \lambda + 2k - 2, & \text{for } 3 \leq k \leq \lambda; \\ 2\lambda + k - 1, & \text{for } \lambda + 1 \leq k \leq 2\lambda - 2; \\ 2k, & \text{for } k \geq 2\lambda - 1. \end{cases}$$

Proof. Let $G = (V, E)$ be a connected graph with $\chi(G) = k$, and $T = (V, E_T)$ be a tree backbone of G. Let g be a k-coloring of G and $V_i = \{v \in V \mid g(v) = i\}$ for $1 \leq i \leq k$. Now, we construct a λ-backbone coloring f of (G, T). If $k = 1$, then we obtain the trivial case of f. If $k = 2$, then G is a bipartite graph and we are using colors 1 and $1 + \lambda$ in f.

For $k \geq 3$, we define the color set

$$C_i = \begin{cases} \{i, \lambda + k + i - 1\}, & \text{for } 1 \leq i \leq k-1 \text{ and } 3 \leq k \leq \lambda; \\ \{k, k-1+\lambda\}, & \text{for } i = k \text{ and } 3 \leq k \leq \lambda; \\ \{i, 2\lambda + i - 1\}, & \text{for } 1 \leq i \leq k \text{ and } \lambda + 1 \leq k \leq 2\lambda - 2; \\ \{i, i+k\}, & \text{for } 1 \leq i \leq k \text{ and } k \geq 2\lambda - 1. \end{cases}$$

Let $c_i(1)$ and $c_i(2)$ be the first and the second color of C_i, respectively. Let $u \in V_i$ and $v \in V_j$ for $i, j \in \{1, 2, \ldots, k\}$ and $uv \in E_T$. We assume that v is colored after u. We distinguish three cases.

Case (i): $3 \leq k \leq \lambda$

If the color of u is $c_i(1)$, then we color v with $c_j(2)$, otherwise we color v with $c_j(1)$. Since $c_l(2) - c_l(1) \geq \lambda - 1$ for $1 \leq l \leq k$, we obtain that $|f(u) - f(v)| \geq \lambda$. Therefore, f is a λ-backbone coloring of (G, T) with $\mathcal{T}_\lambda(k) \leq \lambda + 2k - 2$.

Case (ii): $\lambda + 1 \leq k \leq 2\lambda - 2$

For $|i - j| < \lambda$, we use the coloring method in case (i) to color u and v. Since $c_l(2) - c_l(1) \geq 2\lambda - 1$ for $1 \leq l \leq k$, we obtain that $|f(u) - f(v)| \geq \lambda$.

For $|i - j| \geq \lambda$, if the color of u is $c_i(1)$, then we color v with $c_j(1)$, otherwise we color v with $c_j(2)$. Since $|c_i(p) - c_j(p)| \geq \lambda$ for $1 \leq p \leq 2$, we obtain that $|f(u) - f(v)| \geq \lambda$.

Therefore, f is a λ-backbone coloring of (G, T) with $\mathcal{T}_\lambda(k) \leq 2\lambda + k - 1$.

Case (iii): $k \geq 2\lambda - 1$

If $|i - j| < \lambda$, we use the coloring method in the first case of case (ii) to color u and v, otherwise we use the coloring method in the second case of case (ii) to color u and v. Since $c_l(2) - c_l(1) = k \geq 2\lambda - 1$ for $1 \leq l \leq k$, we obtain that $|f(u) - f(v)| \geq \lambda$. Therefore, f is a λ-backbone coloring of (G, T) with $\mathcal{T}_\lambda(k) \leq 2k$.

We now proceed to prove the reverse inequality. Since we obtain the trivial case for $k = 1$, we assume that $k \geq 2$. For $k = 2$, let us consider a path P_2. Let $uv \in E(P_2)$. If the color of u is 1, then the color of v must be $1 + \lambda$.

For $k \geq 3$, let $G = (V, E)$ be a k-complete partite graph. Let $V_1, V_2, \ldots, V_k$ be the independent sets of V with $|V_i| = k^{k-1}$ for $1 \leq i \leq k$. Now, we construct the spanning tree T of G as follows. Let T_1 be a star with center $v_1 \in V_1$ and $k - 1$ leaves in the $k - 1$ sets $V_2, ..., V_k$, one in each set. For $2 \leq j \leq k$, the tree T_j is constructed from the tree T_{j-1}, by creating $k - 1$ new vertices for every vertex v in T_{j-1} and joining them to v. If $v \in V_q$, then every independent set V_i with $i \neq q$, contains exactly one of these new vertices. Note that, all newly created vertices are leaves in the tree T_j. Therefore, $V(T_{j-1}) \subset V(T_j)$. Now, we define $T = T_k$. We distinguish three cases.

Case (i): $3 \le k \le \lambda$

Suppose that $f : V(G) \to \{1, 2, \ldots, \lambda + 2k - 3\}$ is a λ-backbone coloring of (G, T). For $k \le c \le \lambda + k - 2$, we define X_c as the set of all colors d satisfying $|d - c| \ge \lambda$ and $d \in \{1, 2, \ldots, \lambda + 2k - 3\}$. Since $|X_c| < k - 1$, we obtain $f(v) \in \{1, 2, \ldots, k - 1, \lambda + k - 1, \lambda + k, \ldots, \lambda + 2k - 3\}$ for every vertex v in T_{k-1}.

Now, we define the color sets $W_1 = \{1, ..., k - 1\}$ and $W_2 = \{\lambda + k - 1, ..., \lambda + 2k - 3\}$. Let $v_1 \in V_1$ be the center vertex of T_1. Without loss of generality, let $f(v_1) \in W_1$. Let v_1 be adjacent to $v_j \in V_j \cap V(T_1)$ for $2 \le j \le k$. So, we have $f(v_j) \in W_2$ and for every two distinct numbers $p, q \in \{2, 3, \ldots, k\}$, $f(v_p) \ne f(v_q)$. Let v_j be adjacent to $v_{(j,p)} \in V_p \cap V(T_2)$ where $p \in \{2, 3, \ldots, k\}$ and $p \ne j$. So, we obtain that $f(v_{(j,p)}) \in W_1$ and for every two distinct numbers $q, r \in \{2, 3, \ldots, k\}$, $f(v_{(j,q)}) \ne f(v_{(j,r)})$. Let us consider the k vertices $v_1, v_{(3,2)}, v_{(2,3)}, v_{(2,4)}, \ldots, v_{(2,k)}$. Since they are adjacent to each other and $|W_1| = k - 1$, those k vertices cannot be labeled with k different colors. Contradiction.

Case (ii): $\lambda + 1 \le k \le 2\lambda - 2$

For $1 \le i \le k$ and $1 \le j \le k$, we define the independent sets $V_i^j = V_i \cap V(T_j)$. We call $V_1^j, V_2^j, \ldots, V_k^j$ as the independent sets of $V(T_j)$.

Let $W = \{1, ..., 2\lambda + k - 2\}$. Suppose that $f : V \to W$ be a λ-backbone coloring of (G, T). We consider the vertices in T_{k-3}. Note that, $k - 3 \ge 1$. If there exists n independent sets of $V(T_{k-3})$ with $n \in \{1, 2, \ldots, k\}$, such that all vertices in each independent set have the same color, then we need at least $n\lambda + 2(k - n)$ colors in f. Since $n\lambda + 2(k - n) > 2\lambda + k - 2$, every independent set of $V(T_{k-3})$ must contain at least two vertices with different color.

For every $p \in W$, we define $Y_p = \{d \mid |d - p| \ge \lambda\}$. We also define the color sets

$$W_1 = \{1, 2, \ldots, \lambda - 1\} \cup \{\lambda + k, \lambda + k + 1, \ldots, 2\lambda + k - 2\}$$
$$W_2 = \{\lambda, \lambda + 1, \ldots, \lambda + k - 1\}.$$

Since $|W_1| < 2k$, the vertices in T_{k-3} are colored by the color in W_1 or W_2. Note that, for every $p \in W_2$, we obtain that $|Y_p| = k - 1$. So, if we color a vertex $v \in V_i^{k-3}$ by p for $i \in \{1, 2, \ldots, k\}$, then $k - 1$ other vertices from $k - 1$ independent sets $V_1^{k-2}, V_2^{k-2}, \ldots, V_{i-1}^{k-2}, V_{i+1}^{k-2}, \ldots, V_k^{k-2}$, one vertex from each set, which are adjacent to v, are colored by $k - 1$ colors of Y_p, one for each vertex.

Without loss of generality, let $v_1 \in V_1^{k-3}$. For $2 \le i \le k$, let $v_i \in V_i^{k-2}$ be $k - 1$ distinct vertices which are adjacent to v_1. For $2 \le i \le k$ and $1 \le j \le k$ with $j \ne i$, let $v_{(i,j)} \in V_j^{k-1}$ be $k - 1$ different vertices which are adjacent to v_i. For $2 \le i \le k$, $1 \le j \le k$ with $j \ne i$, and $1 \le p \le k$ with $p \ne j$, let $v_{(i,j,p)} \in V_p^k$ be $k - 1$ distinct vertices which are adjacent to $v_{(i,j)}$.

Without loss of generality, let $f(v_1) \in W_2$. We distinguish two cases.

- There exists a vertex $x \in V(T_k)$ such that $f(x) \in \{\lambda, \lambda + k - 1\}$.

  Let $f(v_1) = \lambda + k - 1$, and for $2 \le i \le k$, $f(v_i) = i - 1$.

  (a) Let $k = \lambda + 1$.
  Since $f(v_k) = k - 1$, for $2 \le j \le k - 1$, $f(v_{(k,j)}) \in \{\lambda + k, \lambda + k + 1, \ldots, 2\lambda + k - 2\}$. Note that, for $q \in \{\lambda + 1, \lambda + 2, \ldots, \lambda + k - 2\}$, $|Y_q| = k - 1$ and there exists $j \in \{2, 3, \ldots, k - 1\}$ such that $f(v_j), f(v_{k,j}) \in Y_q$. Therefore, the colors $\{\lambda + 1, \lambda + 2, \ldots, \lambda + k - 2\}$ are not used in f, which implies V_k^{k-1} only use 1 color. Since every independent set must contain at least two different colors, we obtain a contradiction.

  (b) Let $\lambda + 2 \le k \le 2\lambda - 2$.
  Since $f(v_k) = k-1$, for $k-\lambda \le j \le k-1$, $f(v_{k,j}) \in \{\lambda+k, \lambda+k+1, \ldots, 2\lambda+k-2\}$. Since $f(v_{k-1}) = k - 2$ and $f(v_{k,k-1}) \in Y_{k-2}$, we cannot use the color $k - 2$ on f, a contradiction.

  We also obtain a contradiction if $f(v_1) = \lambda$ by using a similar argument in case of $f(v_1) = \lambda + k - 1$.

- Every vertex $x \in V(T_k)$ satisfies $f(x) \in \{\lambda + 1, \lambda + 2, \ldots, \lambda + k - 2\}$.

  Let $f(v_1) = c$ where $c \in \{\lambda + 1, \lambda + 2, \ldots, \lambda + k - 2\}$. For $2 \le i \le c - \lambda + 1$ and $c - \lambda + 2 \le j \le k$, without loss of generality, let $f(v_i) = i - 1$ and $f(v_j) = 2\lambda + j - 2$. Note that, for every $p \in \{\lambda + 1, \lambda + 2, \ldots, \lambda + k - 2\}$ and $p \ne c$, all colors in $Y_p \backslash \{1, 2\lambda+k-2\}$ must be used to color some vertices in $V_3^k \cup V_4^k \cup \ldots V_{k-1}^k$. Therefore, V_2^k and V_k^k are using one color each. Since every independent set must contain at least two different colors, we obtain a contradiction.

Case (iii): $k \ge 2\lambda - 1$

Suppose that $f : V(G) \to \{1, 2, \ldots, 2k - 1\}$ be a λ-backbone coloring of (G, T). Then there exists an independent set containing only one color. Note that, if its color is c, then every color $d \in \{1, 2, \ldots, 2k - 1\}$ satisfying $0 < |d - c| < \lambda$ cannot be used to color all vertices in the other independent sets.

For $1 \le i \le k$, we define the independent sets $V_i^{k-1} = V_i \cap V(T_{k-1})$. We call $V_1^{k-1}, V_2^{k-1}, \ldots, V_k^{k-1}$ as the independent sets of $V(T_{k-1})$.

Without loss of generality, let V_1^{k-1} be colored by only one color. So, there are at most $2k - 1 - \lambda$ remaining colors to color all vertices in $k - 1$ other independent sets. However, we must have another $\lambda - 1$ independent sets containing one color each. So, now we have $2k - 1 - \lambda^2$ remaining colors to color all vertices in $k - \lambda$ other independent sets. Again, we will obtain another independent set which must contain one color. If the process is continued, then the number of remaining colors will be less than the number of uncolored independent sets. Generally, the process above can be seen in the table below.

Step	A	B	C
0	k	0	$2k-1$
1	$k-1$	1	$2k-1-\lambda$
2	$k-(1+(\lambda-1))$	$\lambda=1+(\lambda-1)$	$2k-1-\lambda(1+(\lambda-1))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$k-\sum_{p=0}^{n}(\lambda-1)^p$	$\sum_{p=0}^{n}(\lambda-1)^p$	$2k-1-\lambda\left(\sum_{p=0}^{n}(\lambda-1)^p\right)$

A : The number of uncolored independent sets of T_{k-1}

B : The number of independent sets of T_{k-1} colored by only one color

C : Maximum number of remaining colors for the next step

The process above will stop when $n \geq \left\lceil \frac{\ln\left(\frac{(k-1)(\lambda-2)}{(\lambda-1)}+1\right)}{\ln(\lambda-1)} \right\rceil$, which is $A \geq C$. If $A = C$, then we obtain that $k = \sum_{p=0}^{n}(\lambda-1)^p$ and all of independent sets are containing one color each. It means that we need at least $1+(k-1)\lambda$ colors such that f is a λ-backbone coloring of (G,T). However, $1+(k-1)\lambda > 2k-1$, a contradiction. If $A > C$, then there exists an independent set of T_{k-1} with no color, a contradiction. □

3. λ-Tree Backbones for Planar Graph

There are many open problems about backbone colorings. In this section we only focus on some open problems for planar graphs. By using Theorem 1.1 and Theorem 2.1 and also the Four-Color Theorem, generally we obtain that for any planar graph $\hat{G}$ with a tree backbone T,

$$BBC_\lambda\left(\hat{G},T\right) \leq \begin{cases} 7 & \text{, if } \lambda=2; \\ 9 & \text{, if } \lambda=3; \\ \lambda+6 & \text{, if } \lambda\geq 4. \end{cases}$$

However, it seems that the upper bound is not best possible. Let the greatest value of λ-backbone coloring number of any graph G with chromatic number $\chi(G)=k$ is l. Can we improve that $BBC_\lambda(\hat{G},T) \leq l-1$? In Figure 1, the planar graph G^* demonstrates that this bound cannot be improved to $l-2$. Note that graph G^* consists of four copies of K_4 having a $K_{1,3}$ as a spanning tree.

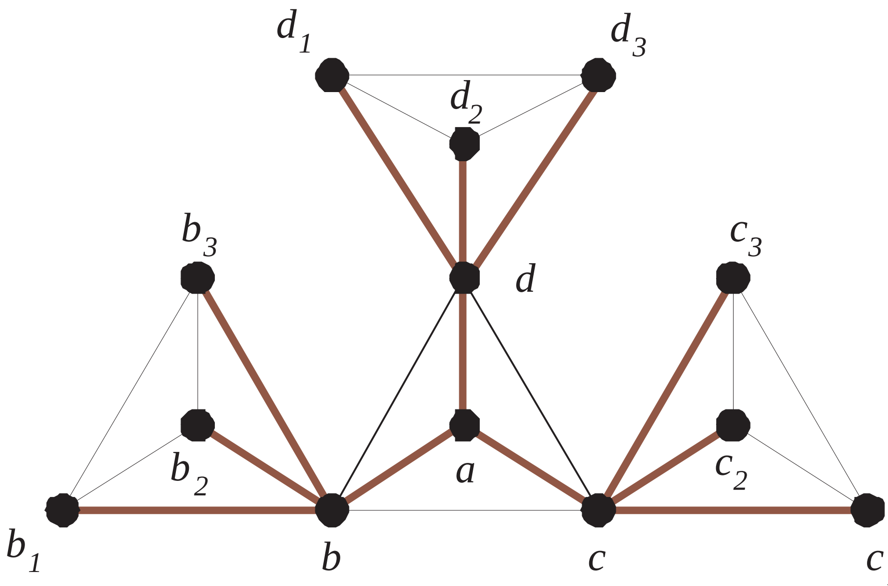

Figure 1: A graph G^* with a tree backbone T^* (bold edges) such that $BBC_\lambda(G^*, T^*) \geq BBC_\lambda(G(k), T) - 1$

Case 1. $\lambda = 2$

Suppose that we only use colors 1, 2, 3, 4, and 5. Then the colors 2, 3, and 4 cannot be used to color the vertices a, b, c, and d. However, the number of remaining colors is not enough to color a, b, c, and d, a contradiction.

Case 2. $\lambda \geq 3$

Suppose that we only use colors $1, 2, \ldots, \lambda + 4$. Then the colors $3, 4, \ldots, \lambda + 2$ cannot be used to color the vertices a, b, c, and d. Let $A = \{1, 2, \lambda + 3, \lambda + 4\}$. Since for every $u \in A$ there exists $v \in A$ satisfying $|u - v| = 1$, if the vertex a is colored by u, then one of b, c, or d must be colored by v, contradiction.

By observations above, we can say that $BBC_\lambda(G^*, T^*) \geq l - 1$

The following problem is still open.

Problem 3.1. *Does any planar graph $\hat{G}$ with a tree backbone T satisfy*

$$BBC_\lambda\left(\hat{G}, T\right) \leq \begin{cases} 6 & , \textit{if } \lambda = 2; \\ 8 & , \textit{if } \lambda = 3; \\ \lambda + 5 & , \textit{if } \lambda \geq 4? \end{cases}$$

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