AKCE Int. J. Graphs Comb., 10, No. 2 (2013), pp. 237-243

Some open problems on graph labelings

S. Arumugam
National Center for Advanced Research in Discrete Mathematics (n-CARDMATH)
Kalasalingam University
Anand Nagar, Krishnankoil-626126, Tamil Nadu, India
e-mail: *s.arumugam.klu@gmail.com*

Martin Bača
Department of Applied Mathematics and Informatics
Technical University
Letna 9, 042 00 Kosice, Slovakia
e-mail: *martin.baca@tuke.sk*

Dalibor Froncek
Department of Mathematics and Statistics
University of Minnesota Duluth
1117 University Drive
Duluth, MN 55812-3000, U.S.A
e-mail: *dalibor@d.umn.edu*

Joe Ryan
Graph Theory and Applications (GTA) Research Group
School of Electrical Engineering and Computer Science
The University of Newcastle, Australia
e-mail: *joe.ryan@newcastle.edu.au*

and

Kiki A. Sugeng
Department of Mathematics
Faculty of Mathematics and Natural Sciences
University of Indonesia, Depok
16424, Indonesia
e-mail: *kiki@sci.ui.ac.id*

Abstract

In this note we present a few open problems on various aspects of graph labelings, which have not been included in any of the other papers appearing in this volume.

Keywords: antimagic total labeling, graceful labeling, α-labeling, distance antimagic labeling, irregular labeling, chromatic number.

2000 Mathematics Subject Classification: 05C78.

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and size of $|E|$ are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [6]. For basic definitions of various types of graph labelings we refer to the dynamic survey of Gallian [9].

1. On (a, d)-edge antimagic total labelings (Posed by Martin Bača)

An (a, d)-*edge-antimagic total labeling* of a (p, q)-graph G is a bijective function $h : V(G) \cup E(G) \to \{1, 2, \dots, p + q\}$ with the property that the edge-weights $\{wt_h(uv) : wt_h(uv) = h(u) + h(uv) + h(v), uv \in E(G)\}$, form an arithmetic progression $\{a, a + d, a + 2d, \dots, a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

Furthermore, h is a *super* (a, d)-*edge-antimagic total labeling* of G if the vertex labels are the integers $1, 2, \dots, p$.

These labelings, introduced by Simanjuntak et al. in [17] are natural extensions of the concept of magic valuation, studied by Kotzig and Rosa [14], and the concept of super edge-magic labeling, defined by Enomoto et al. in [7].

Sugeng et al. [18] proved that all caterpillars $S_{n_1,n_2,\dots,n_r}$ have a super (a, d)-edge-antimagic total labeling for $d \in \{0, 1, 2\}$, where $S_{n_1,n_2,\dots,n_r}$ represents the caterpillar of diameter $r + 2$ whose spine's vertices have degrees $n_1, n_2, \dots, n_r$, respectively. In [18] it is conjectured that if $||A| - |B|| > 1$ there is no super $(a, 3)$-edge-antimagic total labeling of $S_{n_1,n_2,\dots,n_r}$, where $\{A, B\}$ is the bipartition of the vertex set of the caterpillar. As a consequence of the results, which have been proved for edge-antimagicness of trees in [2], it follows that any caterpillar with $||A| - |B|| \leq 1$ admits a super $(a, 3)$-edge-antimagic total labeling. Moreover in [2] is exhibited a super $(a, 3)$-edge-antimagic total labeling of the caterpillar $S_{n,2,n}$, where $|A| = 2$, $|B| = 2n - 1$ and $||A| - |B|| = 2n - 3 \geq 5$, which provides a counterexample for the conjecture posed by Sugeng et al. Thus Bača and Miller propose the following open problem.

Problem 1.1. *For the caterpillar $S_{n_1,n_2,\dots,n_r}$, determine feasible pairs (A, B), $|A| \neq |B|$ and $||A| - |B|| \neq 1$, which make a super $(a, 3)$-edge-antimagic total labeling impossible.*

Bača et al. [4] studied the super edge-antimagicness of a disjoint union of m copies of a caterpillar denoted by $mS_{n_1,n_2,\dots,n_r}$. They proved that the graph $mS_{n_1,n_2,\dots,n_r}$, for $n_1 = n_2 = \cdots = n_r = t$, has a super (a, d)-edge-antimagic total labeling for (i) $d \in \{0, 2\}$, if mr is odd and $t \geq 1$, (ii) $d = 1$, if either mrt is odd, or m is even and r is odd, $t \geq 1$, or $t = 2$ and $m, r \geq 2$, (iii) $d = 3$, if $m, r \geq 2$ and $t = 2$.

For the graph $mS_{n_1,n_2,\dots,n_r}$, for $n_1 = n_2 = \cdots = n_r = t \neq 2$, so far they have not found any super $(a, 3)$-edge-antimagic total labeling. So, they propose the following open problem.

Problem 1.2. *For the graph $mS_{n_1,n_2,\dots,n_r}$, for $n_1 = n_2 = \cdots = n_r = t$, determine if there is a super $(a, 3)$-edge-antimagic total labeling, for every $m \geq 2$, $r \geq 2$ and $t \neq 2$.*

In the case when graph $mS_{n_1,n_2,\ldots,n_r}$ does not have any restriction for the values of $n_1, n_2, \ldots, n_r$, the problem to find a super (a,d)-edge-antimagic total labeling seems to be difficult. For further investigation the authors in [4] suggest the following

Problem 1.3. *Find, if possible, some structural characteristics of a graph $mS_{n_1,n_2,\ldots,n_r}$ which make a super (a,d)-EAT labeling impossible.*

A *graceful labeling* of a (p,q) graph G is an injection $f : V(G) \rightarrow \{1, 2, \ldots, q+1\}$ such that, when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels (or edge-weights) are distinct numbers from the set $\{1, 2, \ldots, q\}$. A graph that admits a graceful labeling is said to be *graceful*.

A graceful labeling f of a graph G is said to be an *α-labeling* if there exists an integer λ such that for each edge uv of G either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$. A graph that admits an α-labeling is called an *α-graph*. Graceful and α-labelings were introduced by Rosa in [15].

Koh et al. in [13] provide a method for constructing bigger graceful tree from a given pair of graceful trees. Let H and G be two graphs and let $\{w_1, w_2, \ldots, w_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ be their corresponding vertex sets. Let x be an arbitrary fixed vertex in G. Based upon the graph H, an isomorphic copy G_i of G is adjoined to each vertex w_i, $i = 1, 2, \ldots, m$, by identifying x^i and w_i, where x^i is the vertex corresponding to x in G_i. All the m copies of G just introduced are pairwise disjoint and no extra edges are added. The obtained graph is denoted by $H \triangle G$. Koh et al. proved that if H and G are both graceful trees then $H \triangle G$ is also graceful.

Ahmad et al. [1] used Koh's construction to obtain an α-tree from smaller α-trees and graceful trees. They proved that if H is a balanced α-tree and G is a graceful tree, where x is its an arbitrary fixed vertex, then $H \triangle G$ is an α-tree. Let us mention that if $\{A, B\}$ is the bipartition of the vertex set of the α-tree with $|A| = |B|$ (respectively, $|A| \neq |B|$) then the α-tree is called *balanced* (respectively, *unbalanced*). We propose the following open problem.

Problem 1.4. *For the graph $H \triangle G$, determine if there is an α-labeling for a graceful tree G and an unbalanced α-tree H.*

For an α-tree H, where $\{A_H, B_H\}$ is the bipartition of its vertex set, Ahmad et al. [1] proved that if H and G are both α-trees with $||A_H| - |B_H|| \leq 1$ then $H \triangle G$ is also an α-tree.

We may suppose that there exist an α-tree H with $||A_H| - |B_H|| > 1$ and an α-tree G such that $H \triangle G$ admits an α-labeling. Therefore we propose the following open problem.

Problem 1.5. *For every α-tree G and for the α-tree H with the bipartition of its vertex set $\{A_H, B_H\}$, determine feasible pairs (A_H, B_H), $|A_H| \neq |B_H|$ and $||A_H| - |B_H|| \neq 1$, which make an α-labeling of $H \triangle G$ impossible.*

2. Distance Antimagic Graphs (Posed by Dalibor Froncek)

Definition 2.1. *An* ordered distance antimagic labeling *of a graph $G(V,E)$ with n vertices is a bijection $\vec{f}: V \to \{1,2,\ldots,n\}$ with the property that $\vec{f}(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an increasing arithmetic progression with difference one. A graph G is an* ordered distance antimagic graph *if it allows an ordered distance antimagic labeling.*

Froncek [8] constructed a family of ordered distance antimagic graphs using magic rectangle sets. However, the graphs do not cover the whole spectrum of possible orders. Therefore, he proposes the following problem.

Problem 2.2. *For what pairs (n,r) there exist r-regular ordered distance antimagic graphs with n vertices?*

3. Graph Labelings and Colorings (Posed by S. Arumugam)

Let $G = (V,E)$ be a graph. Let $f: E \to \{1,2,\ldots,k\}$ be an edge labeling of G (The function f need not be a bijection). We define the weight of a vertex $v \in V$ by $wt(v) = \sum_{v \in e} f(e)$. The labeling f is said to be an *irregular labeling* if all the weights $wt(v)$ are distinct. The *strength* of f is defined by $s(f) = \max_{e \in E}\{f(e)\}$. The *irregularity strength* $s(G)$ of G is defined by $s(G) = \min\{s(f) : f \text{ is an irregular labeling of } G\}$. The concept $s(G)$ was introduced by Chartrand et al. [5].

Bača et al. [3] defined two parameters *total vertex irregularity strength* $tvs(G)$ and *total edge irregularity strength* $tes(G)$.

Let $G = (V,E)$ be a graph. Let $f: V \cup E \to \{1,2,\ldots,k\}$ be a total labeling of G. The weight of a vertex v and the weight of an edge uv are defined by

$$wt(v) = f(v) + \sum_{vw \in E} f(vw) \text{ and } wt(uv) = f(u) + f(v) + f(uv).$$

A total labeling $f: V \cup E \to \{1,2,\ldots,k\}$ is said to be an edge irregular total k-labeling if $wt(e_1) \neq wt(e_2)$ for any two distinct edges e_1 and e_2. Also f is called a vertex irregular total k-labeling if $wt(u) \neq wt(v)$ for any two distinct vertices u, v. The total vertex irregularity strength $tvs(G)$ and the total edge irregularity strength $tes(G)$ are defined by

$$tvs(G) = \min\{s(f) : f \text{ is a total vertex irregular labeling of } G\} \text{ and}$$

$$tes(G) = \min\{s(f) : f \text{ is a total edge irregular labeling of } G\}.$$

Thus for the concept of irregularity strength, all the vertices (edges) receive different weights. It is natural to consider edge labeling where we require that only adjacent vertices have different weights, or equivalently, the induced vertex weighting is a proper vertex coloring of G. Such an edge labeling is called a chromatic edge labeling. One can similarly define chromatic vertex labeling and chromatic total labeling. This concept has been investigated in Karonski et al. [12]. In this context the following coloring parameters arise naturally.

Edge label vertex chromatic number $\chi_{cl}(G)$ = Minimum number of colors in a vertex coloring of G induced by a chromatic edge labeling of G.

Vertex label chromatic index $\chi'_l(G)$ = Minimum number of colors in an edge coloring of G induced by a chromatic vertex labeling of G.

Total label vertex chromatic number $\chi_{tl}(G)$ = Minimum number of colors in a vertex coloring of G induced by a chromatic total labeling of G.

Total label chromatic index $\chi'_{tl}(G)$ = Minimum number of colors in an edge coloring of G induced by a chromatic total labeling of G.

The study of these coloring parameters is open. We have initiated a study of $\chi_{cl}(G)$ and have obtained a few basic results. The following problems are natural.

Problem 3.1. *Characterize graphs G for which*

(a) $\chi(G) = \chi_{cl}(G)$

(b) $\chi(G) = \chi_{tl}(G)$.

Problem 3.2. *Characterize graphs G for which*

(a) $\chi'(G) = \chi'_l(G)$

(b) $\chi'(G) = \chi'_{tl}(G)$.

Problem 3.3. [12] *Is it possible to label the edges of any non-trivial graph with the integers $\{1, 2, 3\}$ such that the resultant vertex weighting is a proper coloring of G?*

4. Sum and Exclusive Sum Labelings (Posed by Kiki A. Sugeng)

Sum labeling was introduced by Harary in 1990 [11]. A graph $G(V, E)$ is called sum graph if there exists an injective function f from V to a set of positive integers S such that $uv \in E$ if and only if there is vertex w such that $f(w) = f(u) + f(v)$. Furthermore, we call vertex w as a working vertex and the function f as sum labeling. A graph G will need at least one isolated vertex to be a sum graph. The minimum number of additional isolated vertices required by a graph to support a sum labeling is called sum number of G, and is denoted by $\sigma(G)$.

A sum labeling f is called exclusive sum labeling if all of its working vertices are isolated vertices, otherwise it said to be inclusive. The minimum number of isolated vertices that need to be added to a graph G to support an exclusive sum labeling is called the exclusive sum number of G, and is denoted by $\epsilon(G)$.

Let δ and Δ be the minimum degree and maximum degree of vertices in a graph G. Then $\sigma(G) \geq \delta(G)$ and $\epsilon(G) \geq \Delta(G)$. Note that if $\epsilon(G) = \Delta(G)$, then the graph G is called Δ-optimum exclusive sum graph. The exclusive sum number is never smaller than the corresponding sum number, since every exclusive sum graph is a sum graph. The exclusive sum number of caterpillars is given in [16]. Tuga and Miller [19] determined the exclusive sum number of several families of graphs with radius 1 such as fans, multifans and friendship graphs. For details of graphs whose sum numbers are known we refer to the dynamic survey by Gallian ([9], Page 150). Similarly the exclusive sum number is known for several families of graphs and for details we refer to ([9], Page 149).

There are many constructions of sum and exclusive sum labelings on various classes of graphs, including the disjoint union of some classes of graphs.

Problem 4.1. *Find the sum number and/or exclusive sum number for various families of graphs.*

Problem 4.2. *Find a class of graphs which is Δ-optimum exclusive sum graph.*

Problem 4.3. *Find the exclusive sum number for disjoint union of graphs.*

Gould and Rodl [10] showed that there exist graphs that require a number of isolates of the order of n^2 in order to support a sum labeling.

Problem 4.4. *Find an example of graph with (exclusive) sum number in the order of n^2.*

References

[1] A. Ahmad, M. Bača, A. Semaničová- Feňovčíková, and M.K. Siddiqui, Construction of alpha-tree from smaller graceful trees, *Utilitas Math.* in press.

[2] M. Bača and C. Barrientos, Graceful and edge-antimagic labelings, *Ars Combin.*, **96** (2010), 505–513.

[3] M. Bača, S. Jendroľ, M. Miller and J. Ryan, On irregular total labelings, *Discrete Math.*, **307** (2007), 1378–1388.

[4] M. Bača, Dafik, M. Miller and J. Ryan, Edge-antimagic total labeling of disjoint union of caterpillars, *J. Combin. Math. Combin. Comput.*, **65** (2008), 61–70.

[5] G. Chartrand, M.S. Jocobson, J. Lehel, O.R. Oellermann, S. Ruiz and F. Saba, Irregular Networks, *Congr. Numer.*, **64** (1988), 197–210.

[6] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman and Hall, CRC, 4^{th} edition, 2005.

[7] H. Enomoto, A.S. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998), 105–109.

[8] D. Froncek, *Handicap distance antimagic graphs and incomplete tournaments*, submitted.

[9] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, **19** (2012), # DS6.

[10] R. Gould and V. Rodl, *Bounds on the number of isolated vertices in sum graphs*, Graph Theory, Combinatorics and Applications (edited by Y. Alevi, G. Chartrand, O. R.Oellermann and A. J. Schwenk), John Wiley and Sons (1991), 553–562.

[11] F. Harary, Sum graphs and difference graphs,*Congr. Numer.*,**72**(1990), 101–108.

[12] M. Karoński, T. Łuczak and A. Thomason, Edge weights and vertex colours, *J. Combin. Theory Ser. B*, **91** (2004), 151–157.

[13] K.M. Koh, D.G. Rogers and T. Tan, Two theorems on graceful trees, *Discrete Math.* **25** (1979), 141–148.

[14] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[15] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Symposium, Rome, July 1966, Gordon and Breach, N.Y. and Dunod Paris* (1967), 349–355.

[16] J. Ryan, Exclusive sum labeling of graphs: A survey, *AKCE J. Graphs. Combin.*, **6**(1) (2009), 113–126.

[17] R. Simanjuntak, F. Bertault and M. Miller, Two new (a, d)-antimagic graph labelings, *Proc. of Eleventh Australian Workshop of Combinatorial Algorithm* (2000), 179–189.

[18] K.A. Sugeng, M. Miller, Slamin and M. Bača , (a, d)-edge-antimagic total labelings of caterpillars, *Lecture Notes in Computer Science* **3330** (2005), 169–180.

[19] M. Tuga and M. Miller, Δ-*optimum exclusive sum labeling of certain graphs with radius one*, Proceedings of IJCCGGT 2003, Bandung, Indonesia, (2003), 216–225.