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Enlarging the classes of super edge-magic 2-regular graphs

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Abstract

A graph G is called super edge-magic if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$ and $f(u) + f(v) + f(uv)$ is a constant for each $uv \in E(G)$. A graph G with isolated vertices is called pseudo super edge-magic if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that the set $\{f(u) + f(v) \,|uv \in E(G)\} \cup \{2f(u) \,|\deg_G u = 0\}$ consists of $|E(G)| + |\{u \in V(G) \,|\deg_G u = 0\}|$ consecutive integers. In this paper, we enlarge the classes of super edge-magic 2-regular graphs by presenting some constructions that allow us to generate large classes of super edge-magic 2-regular graphs from previously known super edge-magic 2-regular graphs or pseudo super edge-magic graphs.

Keywords: edge-magic labeling, super edge-magic labeling, pseudo super edge-magic labeling, 2-regular graph.

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1. Introduction

For most of the graph theory terminology and notation used throughout this paper, we will follow Chartrand and Lesniak [4]. The vertex set of a graph G is denoted by $V(G)$, while the edge set is denoted by $E(G)$. For two graphs G_1 and G_2 with disjoint vertex sets, the union $G \cong G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph G consists of m disjoint copies of a graph H, then we write $G \cong mH$.

For two integers a and b, we will denote the greatest common divisor and the least common multiple of integers a and b by (a, b) and $[a, b]$, respectively.

In 1970, Kotzig and Rosa [20] initiated the study of what they called magic valuations. This concept was later named edge-magic labelings by Ringel and Lladó [21] and this has become the popular term. A graph G is called *edge-magic* if there exists a bijective function $f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(u)+f(v)+f(uv)$ is a constant (the valence of f) for each $uv \in E(G)$. Such a function is called an *edge-magic labeling.*

In 1998, Enomoto et al. [5] introduced a particular type of edge-magic labelings, namely, super edge-magic labelings. They defined an edge-magic labeling of a graph G to be a *super edge-magic labeling* with the additional property that $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling. According to the latest version of the survey on graph labelings by Gallian [12] available to the authors, Hegde and Shetty [15] showed that the concepts of super edge-magic graphs and strongly indexable graphs (see [1] for the definition of a strongly indexable graph) are equivalent.

Due to the following lemma found in [6], we are allowed to think of super edge-magic labelings as vertex labelings instead of total labelings.

Lemma 1.1. *A graph G is super edge-magic if and only if there exists a bijective function $f: V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}$ such that the set $S = \{f(u) + f(v) \,|uv \in E(G)\}$ consists of $|E(G)|$ consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k = |V(G)| + |E(G)| + s$, where $s = \min(S)$.*

In [20], Kotzig and Rosa include an edge-magic labeling of the cycle C_n and propose the problem of finding necessary and sufficient conditions to determine whether any 2-regular graph is edge-magic. It is worth the while to mention that all cycles were independently shown to be edge-magic by Goldbold and Slater [13] as well.

In [5], Enomoto et al. found the following characterization regarding super edge-magic cycles.

Theorem 1.2. *Let n be an integer with $n \geq 3$. Then the cycle C_n is super edge-magic if and only if n is odd.*

In [7], Figueroa-Centeno et al. generalized Theorem 1.2 by proving that the 2-regular graph mC_n is super edge-magic if and only if m and n are odd. Later, they established in [9] an even more general result which is stated next.

Theorem 1.3. *If G is a (super) edge-magic bipartite or tripartite graph, and m is odd, then mG is (super) edge-magic.*

We now introduce a new concept, which extends the concept of a super edge-magic graph to a graph with isolated vertices. A graph G with isolated vertices is called *pseudo super edge-magic* if there exists a bijective function $f: V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}$ such that the set $\{f(u) + f(v) \,|uv \in E(G)\} \cup \{2f(u) \,|\deg_G u = 0\}$ consists of $|E(G)| + |\{u \in V(G) \,|\deg_G u = 0\}|$ consecutive integers. Such a function is called a *pseudo super edge-magic labeling.*

The goal of this paper is to add to what is known about super edge-magic 2-regular graphs by presenting some constructions to generate large classes of super edge-magic 2-regular graphs from previously known super edge-magic 2-regular graphs or certain pseudo super edge-magic graphs.

For more information on super edge-magic 2-regular graphs and related topics, see [2, 10, 11, 14, 16, 17, 18, 19].

Additive types of graph labelings have been studied extensively in the intervening years. Excellent sources for more information in this area are found in the survey by Gallian [12], and the books by Bača and Miller [3] and Wallis [22].

2. Main result

The proof of the following theorem gives us a heretofore unknown way of obtaining super edge-magic 2-regular graphs from other super edge-magic 2-regular graphs of smaller order.

Theorem 2.1. *Assume that* $G \cong \bigcup_{i=1}^{k} C_{n_i}$ *is any super edge-magic* 2*-regular graph, and let* m *be odd. Then* $H \cong \bigcup_{i=1}^{k} (m, n_i)\, C_{[m,n_i]}$ *is a super edge-magic* 2*-regular graph.*

Proof. Our construction requires the super edge-magic labeling of C_m, where m is odd, found by Enomoto et al. [5]. Thus, let

$$V(C_m) = \{u_j | 1 \leq j \leq m\} \quad \text{and} \quad E(C_m) = \{u_1 u_m\} \cup \{u_j u_{j+1} | 1 \leq j \leq m-1\},$$

and consider the vertex labeling $f : V(C_m) \to \{1, 2, \ldots, m\}$ such that

$$f(x) = \begin{cases} j & \text{if } x = u_{2j-1} \text{ and } 1 \leq j \leq (m+1)/2 \\ (m+1)/2 + j & \text{if } x = u_{2j} \text{ and } 1 \leq j \leq (m-1)/2. \end{cases}$$

Then, by Lemma 1.1, f extends to a super edge-magic labeling of C_m with valence $(5m+3)/2$.

Also, let $G \cong \bigcup_{i=1}^{k} C_{n_i}$ be the 2-regular graph with

$$V(G) = \left\{ v_j^i | 1 \leq i \leq k, 1 \leq j \leq n_i \right\}$$

and

$$E(G) = \left\{ v_1^i v_{n_i}^i \, | 1 \leq i \leq k \right\} \cup \left\{ v_j^i v_{j+1}^i \, | 1 \leq i \leq k, 1 \leq j \leq n_i - 1 \right\}.$$

Define $\alpha_s = p(f(u_s) - 1)$ for every integer s with $1 \leq s \leq m$, where $p = |V(G)|$. Moreover, let g be any super edge-magic labeling of G.

Now, consider the 2-regular graph $H \cong \bigcup_{i=1}^{k} (m, n_i) C_{[m,n_i]}$ with

$$V(H) = \left\{ w^i_{(j-1)[m,n_i]+l} | 1 \le i \le k, 1 \le j \le (m, n_i), 1 \le l \le [m, n_i] \right\}$$

and

$$\begin{aligned} E(H) &= \left\{ w^i_{(j-1)[m,n_i]+1} w^i_{(j-1)[m,n_i]+[m,n_i]} | 1 \le i \le k, 1 \le j \le (m, n_i) \right\} \\ &\cup \left\{ w^i_{(j-1)[m,n_i]+l} w^i_{(j-1)[m,n_i]+l+1} | \right. \\ &\qquad \left. 1 \le i \le k, 1 \le j \le (m, n_i), 1 \le l \le [m, n_i] - 1 \right\}. \end{aligned}$$

Next, construct a vertex labeling h of H as follows. For a given integer i with $1 \le i \le k$, define

$$h\left(w^i_{(j-1)[m,n_i]+l} \right) = \alpha_s + g\left(v^i_t \right),$$

where $s = l \pmod{m}$, $t = j + l - 1 \pmod{n_i}$, $1 \le j \le (m, n_i)$ and $1 \le l \le [m, n_i]$.

To show that h is actually a super edge-magic labeling of H, it is necessary to verify the following claims:

Claim 1. Let i be an integer with $1 \le i \le k$. Then the function $F^i : \mathbb{Z}_{(m,n_i)} \times \mathbb{Z}_{[m,n_i]} \to \mathbb{Z}_m \times \mathbb{Z}_{n_i}$ defined by $F^i(j, l) = (\phi(j, l), \psi(j, l))$ is an injective function, where

$$\begin{aligned} \phi(j, l+1) &= \phi(j, l) + 1; & \psi(j, l+1) &= \psi(j, l) + 1; \\ \phi(j+1, l) &= \phi(j, l); & \psi(j+1, l) &= \psi(j, l) + 1. \end{aligned}$$

Claim 2. Let i be an integer with $1 \le i \le k$, and consider the function $G^i : \mathbb{Z}_m \times \mathbb{Z}_{n_i} \to V\left((m, n_i) C_{[m,n_i]} \right)$ defined by $G^i(s, t) = p(f(u_s) - 1) + g\left(v^i_t \right)$. Then $G^i \circ F^i$ is an injective function.

Claim 3. Let i_1, i_2, j_1, j_2, l_1 and l_2 be integers such that $1 \le i_1, i_2 \le k$ with $i_1 \ne i_2$, $1 \le j_1 \le (m, n_{i_1})$, $1 \le j_2 \le (m, n_{i_2})$, $1 \le l_1 \le [m, n_{i_1}]$ and $1 \le l_2 \le [m, n_{i_2}]$. Then

$$h\left(w^{i_1}_{(j_1-1)[m,n_{i_1}]+l_1} \right) \ne h\left(w^{i_2}_{(j_2-1)[m,n_{i_2}]+l_2} \right).$$

Claim 4. Let i, j and l be integers such that $1 \le i \le k$, $1 \le j \le (m, n_i)$ and $1 \le l \le [m, n_i]$. Then

$$1 \le h\left(w^i_{(j-1)[m,n_i]+l} \right) \le pm.$$

Claim 5. Let i be an integer with $1 \leq i \leq k$, and consider the function $H^i : \mathbb{Z}_m \times \mathbb{Z}_{n_i} \to E\left((m, n_i)\, C_{[m,n_i]}\right)$ defined by $H^i(s,t) = p\left(f(u_s) + f(u_{s+1}) - 2\right) + g\left(v^i_t\right) + g\left(v^i_{t+1}\right)$. Then $H^i \circ F^i$ is an injective function.

Claim 6. Let i_1, i_2, j_1, j_2, l_1 and l_2 be integers such that $1 \leq i_1, i_2 \leq k$ with $i_1 \neq i_2$, $1 \leq j_1 \leq (m, n_{i_1})$, $1 \leq j_2 \leq (m, n_{i_2})$, $1 \leq l_1 \leq [m, n_{i_1}]$ and $1 \leq l_2 \leq [m, n_{i_2}]$. Then

$$\begin{aligned} & h\left(w^{i_1}_{(j_1-1)[m,n_{i_1}]+l_1}\right) + h\left(w^{i_1}_{(j_1-1)[m,n_{i_1}]+l_1+1}\right) \\ & \neq h\left(w^{i_2}_{(j_2-1)[m,n_{i_2}]+l_2}\right) + h\left(w^{i_2}_{(j_2-1)[m,n_{i_2}]+l_2+1}\right), \end{aligned}$$

where $[m, n_{i_1}] + 1 = 1$ and $[m, n_{i_2}] + 1 = 1$.

Claim 7. Let i, j and l be integers such that $1 \leq i \leq k$, $1 \leq j \leq (m, n_i)$ and $1 \leq l \leq [m, n_i]$. Then

$$\begin{aligned} & \max\left\{h\left(w^i_{(j-1)[m,n_i]+l}\right) + h\left(w^i_{(j-1)[m,n_i]+l+1}\right)\right\} \\ & - \min\left\{h\left(w^i_{(j-1)[m,n_i]+l}\right) + h\left(w^i_{(j-1)[m,n_i]+l+1}\right)\right\} \\ & = pm - 1, \end{aligned}$$

where $[m, n_i] + 1 = 1$.

Let $F^i(j,l) = F^i(j + r, l + r')$. Then

$$\left(\phi(j,l), \psi(j,l)\right) = \left(\phi(j,l) + r', \psi(j,l) + r + r'\right).$$

This gives $\phi(j,l) = \phi(j,l) + r'$ and $\psi(j,l) = \psi(j,l) + r + r'$. This implies in turn that $r' \equiv 0 \pmod{m}$ and $r + r' \equiv 0 \pmod{n_i}$, which implies that $r \equiv 0 \pmod{(m, n_i)}$. This together with $r + r' \equiv 0 \pmod{n_i}$ implies that $r' \equiv 0 \pmod{n_i/(m, n_i)}$. Now, the last congruence with $r' \equiv 0 \pmod{n_i/(m, n_i)}$ leads to $r' \equiv 0 \pmod{mn_i/(m, n_i)}$. Thus, it follows that $r' \equiv 0 \pmod{[m, n_i]}$, which shows that F^i is an injective function and completes the verification of Claim 1.

In light of Claim 1, it is sufficient to show that G^i is an injective function. For all integers s_1, s_2, t_1 and t_2 with $1 \leq s_1, s_2 \leq m$ and $1 \leq t_1, t_2 \leq n_i$ such that $(s_1, t_1) \neq (s_2, t_2)$, assume, to the contrary, that $G^i(s_1, t_1) = G^i(s_2, t_2)$. Then

$$p\left(f(u_{s_1}) - f(u_{s_2})\right) = g\left(v^i_{t_2}\right) - g\left(v^i_{t_1}\right).$$

Now, consider two cases according to the value of $f(u_{s_1}) - f(u_{s_2})$. If $f(u_{s_1}) - f(u_{s_2}) \neq 0$, then

$$\left|g\left(v^i_{t_2}\right) - g\left(v^i_{t_1}\right)\right| \geq p.$$

However, $\left|g\left(v_{t_2}^i\right) - g\left(v_{t_1}^i\right)\right| < p$, since g is a super edge-magic labeling of G. This clearly produces a contradiction. On the other hand, if $f\left(u_{s_1}\right) - f\left(u_{s_2}\right) = 0$, then $g\left(v_{t_1}^i\right) = g\left(v_{t_2}^i\right)$ for some positive integer i. This contradicts the fact that g is a super edge-magic labeling of G. Thus, G^i is an injective function, which completes the verification of Claim 2.

Let

$$h\left(w^{i_1}_{(j_1-1)[m,n_{i_1}]+l_1}\right) = \alpha_{s_1} + g\left(v_{t_1}^{i_1}\right) \quad \text{and} \quad h\left(w^{i_2}_{(j_2-1)[m,n_{i_2}]+l_2}\right) = \alpha_{s_2} + g\left(v_{t_2}^{i_2}\right),$$

and assume, to the contrary, that

$$h\left(w^{i_1}_{(j_1-1)[m,n_{i_1}]+l_1}\right) = h\left(w^{i_2}_{(j_2-1)[m,n_{i_2}]+l_2}\right).$$

Then

$$p\left(f\left(u_{s_1}\right) - f\left(u_{s_2}\right)\right) = g\left(v_{t_2}^{i_2}\right) - g\left(v_{t_1}^{i_1}\right).$$

Now, note that the result of Claim 3 is readily obtained by a similar argument to the one used for the verification of Claim 2.

Since f and g are super edge-magic labelings of C_m and G, respectively, the result of Claim 4 readily follows from the last two equations, and Claims 2 and 3 together with

$$\begin{aligned}&\min\left\{h\left(w^i_{(j-1)[m,n_i]+l}\right) | 1 \le i \le k, 1 \le j \le (m, n_i), 1 \le l \le [m, n_i]\right\}\\ &= \min\left\{p\left(f\left(u_s\right) - 1\right) + g\left(v_t^i\right) | 1 \le i \le k, 1 \le s \le m, 1 \le t \le n_i\right\} \ge 1\end{aligned}$$

and

$$\begin{aligned}&\max\left\{h\left(w^i_{(j-1)[m,n_i]+l}\right) | 1 \le i \le k, 1 \le j \le (m, n_i), 1 \le l \le [m, n_i]\right\}\\ &= \max\left\{p\left(f\left(u_s\right) - 1\right) + g\left(v_t^i\right) | 1 \le i \le k, 1 \le s \le m, 1 \le t \le n_i\right\} \le pm.\end{aligned}$$

In light of Claim 1, it is sufficient to show that H^i is an injective function. For all integers s_1, s_2, t_1 and t_2 with $1 \le s_1, s_2 \le m$ and $1 \le t_1, t_2 \le n_i$ such that $(s_1, t_1) \ne (s_2, t_2)$, assume, to the contrary, that $H^i(s_1, t_1) = H^i(s_2, t_2)$. Then

$$\begin{aligned}&p\left\{\left(f\left(u_{s_1}\right) + f\left(u_{s_1+1}\right)\right) - \left(f\left(u_{s_2}\right) + f\left(u_{s_2+1}\right)\right)\right\}\\ &= \left(g\left(v_{t_2}^i\right) + g\left(v_{t_2+1}^i\right)\right) - \left(g\left(v_{t_1}^i\right) + g\left(v_{t_1+1}^i\right)\right).\end{aligned}$$

Now, consider two cases according to the value of

$$\left(f\left(u_{s_1}\right) + f\left(u_{s_1+1}\right)\right) - \left(f\left(u_{s_2}\right) + f\left(u_{s_2+1}\right)\right).$$

If $\left(f\left(u_{s_1}\right) + f\left(u_{s_1+1}\right)\right) - \left(f\left(u_{s_2}\right) + f\left(u_{s_2+1}\right)\right) \ne 0$, then

$$\left|\left(g\left(v_{t_2}^i\right) + g\left(v_{t_2+1}^i\right)\right) - \left(g\left(v_{t_1}^i\right) + g\left(v_{t_1+1}^i\right)\right)\right| \ge p.$$

However,

$$\left|\left(g\left(v_{t_2}^i\right)+g\left(v_{t_2+1}^i\right)\right)-\left(g\left(v_{t_1}^i\right)+g\left(v_{t_1+1}^i\right)\right)\right|<p,$$

since g is a super edge-magic labeling of G. This clearly produces a contradiction. On the other hand, if $(f(u_{s_1})+f(u_{s_1+1}))-(f(u_{s_2})+f(u_{s_2+1}))=0$, then

$$g\left(v_{t_1}^i\right)+g\left(v_{t_1+1}^i\right)=g\left(v_{t_2}^i\right)+g\left(v_{t_2+1}^i\right)$$

for some positive integer i. This contradicts the fact that g is a super edge-magic labeling of G. Thus, H^i is an injective function, which completes the verification of Claim 5.

Assume, to the contrary, that

$$\begin{aligned}&h\left(w_{(j_1-1)[m,n_{i_1}]+l_1}^{i_1}\right)+h\left(w_{(j_1-1)[m,n_{i_1}]+l_1+1}^{i_1}\right)\\&=h\left(w_{(j_2-1)[m,n_{i_2}]+l_2}^{i_2}\right)+h\left(w_{(j_2-1)[m,n_{i_2}]+l_2+1}^{i_2}\right).\end{aligned}$$

Then

$$\begin{aligned}&p\left\{(f(u_{s_1})+f(u_{s_1+1}))-(f(u_{s_2})+f(u_{s_2+1}))\right\}\\&=\left(g\left(v_{t_2}^{i_2}\right)+g\left(v_{t_2+1}^{i_2}\right)\right)-\left(g\left(v_{t_1}^{i_1}\right)+g\left(v_{t_1+1}^{i_1}\right)\right).\end{aligned}$$

Next, consider two cases according to the value of

$$(f(u_{s_1})+f(u_{s_1+1}))-(f(u_{s_2})+f(u_{s_2+1})).$$

Now, note that the result of Claim 6 is readily obtained by a similar argument to the one used for the verification of Claim 5.

Notice that

$$\begin{aligned}&\max\left\{h\left(w_{(j-1)[m,n_i]+l}^i\right)+h\left(w_{(j-1)[m,n_i]+l+1}^i\right)\right\}\\&=\max\left\{p\left(f(u_s)+f(u_{s+1})-2\right)+g\left(v_t^i\right)+g\left(v_{t+1}^i\right)\right\}\\&\leq p\cdot\max\left\{f(u_s)+f(u_{s+1})\right\}+\max\left\{g\left(v_t^i\right)+g\left(v_{t+1}^i\right)\right\}-2p\end{aligned}$$

and

$$\begin{aligned}&\min\left\{h\left(w_{(j-1)[m,n_i]+l}^i\right)+h\left(w_{(j-1)[m,n_i]+l+1}^i\right)\right\}\\&=\min\left\{p\left(f(u_s)+f(u_{s+1})-2\right)+g\left(v_t^i\right)+g\left(v_{t+1}^i\right)\right\}\\&\geq p\cdot\min\left\{f(u_s)+f(u_{s+1})\right\}+\min\left\{g\left(v_t^i\right)+g\left(v_{t+1}^i\right)\right\}-2p.\end{aligned}$$

This implies that the difference of

$$\max\left\{h\left(w_{(j-1)[m,n_i]+l}^i\right)+h\left(w_{(j-1)[m,n_i]+l+1}^i\right)\right\}$$

and

$$\min\left\{h\left(w^{i}_{(j-1)[m,n_i]+l}\right)+h\left(w^{i}_{(j-1)[m,n_i]+l+1}\right)\right\}$$

is bounded above by $pm-1$, which is equal to $|E(H)|-1$. Hence, the result of Claim 7 immediately follows from Claims 5 and 6.

Therefore, we conclude from Lemma 1.1 that h extends to a super edge-magic labeling of H, completing the proof. □

The following is an example that illustrates the construction given in the proof of Theorem 2.1.

Consider a super edge-magic labeling of $G \cong C_4 \cup C_5$: $1-5-2-7-1$ and $6-4-9-3-8-6$. Now, by letting $m=5$, we obtain a super edge-magic labeling of $H \cong C_{20} \cup 5C_5$: $1-32-11-43-19-5-29-16-37-23-2-34-10-41-20-7-28-14-38-25-1$, $6-31-18-39-26-6$, $4-36-12-44-24-4$, $9-30-17-42-22-9$, $3-35-15-40-27-3$ and $8-33-13-45-21-8$.

3. Applications

In this section, we will use Theorem 2.1 proved in the previous section to enlarge the classes of super edge-magic 2-regular graphs with k components from certain super edge-magic 2-regular graphs with k components.

Theorem 3.1. *Assume that $G \cong \bigcup_{i=1}^{k} C_{n_i}$ is any super edge-magic 2-regular graph, and let m be odd such that $(m,n_i)=1$ or $(m,n_i)=n_i$ for all positive integers i with $1 \le i \le k$. Then $H' \cong \bigcup_{i=1}^{k} C_{mn_i}$ is a super edge-magic 2-regular graph.*

Proof. If $(m,n_i)=1$ for some positive integer i with $1 \le i \le k$, then the result is an immediate consequence of Theorem 2.1. Thus, assume that $(m,n_i)=n_i$ for some positive integer i with $1 \le i \le k$. Then define the 2-regular graph $H \cong \bigcup_{i=1}^{k}(m,n_i)\,C_{[m,n_i]}$ as in the proof of Theorem 2.1. Now, consider the 2-regular graph $H' \cong \bigcup_{i=1}^{k} C_{mn_i}$ obtained from H as follows: $V(H')=V(H)$ and $E(H')=(E(H)-S)\cup T$, where

$$S=\left\{w^{i}_{(j-1)m+1}w^{i}_{(j-1)m+m}\,|1\le j\le n_i\right\}$$

and

$$\begin{aligned} T &= \left\{ w^i_{(n_i-3-2j)m+1} w^i_{(n_i-1-2j)m+m} \,|0 \le j \le (n_i-3)/2 \right\} \\ &\cup \left\{ w^i_{(n_i-2-2j)m+1} w^i_{(n_i-2j)m+m} \,|1 \le j \le (n_i-3)/2 \right\} \\ &\cup \left\{ w^i_{(n_i-2)m+1} w^i_{m} \right\} \cup \left\{ w^i_{(n_i-1)m+1} w^i_{m+m} \right\} \end{aligned}$$

for a given integer i with $1 \le i \le k$ such that $(m, n_i) = n_i$. Also, define the vertex labeling $h' : V(H') \to \{1, 2, \ldots, m(n_1 + n_2 + \cdots + n_k)\}$ to be such that $h'(v) = h(v)$ for each $v \in V(H)$, where h is the super edge-magic labeling of H provided in Theorem 2.1. Consequently, we have

$$\begin{aligned} &\left\{ h'(x) + h'(y) \,|xy \in T \right\} \\ &= \left\{ p(f(u_1)-1) + g\left(v^i_{n_i-2-2j}\right) + p(f(u_m)-1) + g\left(v^i_{n_i-1-2j}\right) |0 \le j \le (n_i-3)/2 \right\} \\ &\cup \left\{ p(f(u_1)-1) + g\left(v^i_{n_i}\right) + p(f(u_m)-1) + g\left(v^i_{1}\right) \right\} \\ &\cup \left\{ p(f(u_1)-1) + g\left(v^i_{n_i-1}\right) + p(f(u_m)-1) + g\left(v^i_{n_i}\right) \right\} \\ &\cup \left\{ p(f(u_1)-1) + g\left(v^i_{n_i-1-2j}\right) + p(f(u_m)-1) + g\left(v^i_{n_i-2j}\right) |1 \le j \le (n_i-3)/2 \right\}, \end{aligned}$$

whereas we have

$$\begin{aligned} &\left\{ h(x) + h(y) \,|xy \in S \right\} \\ &= \left\{ p(f(u_1)-1) + g\left(v^i_{n_i-1-2j}\right) + p(f(u_m)-1) + g\left(v^i_{n_i-2-2j}\right) |0 \le j \le (n_i-3)/2 \right\} \\ &\cup \left\{ p(f(u_1)-1) + g\left(v^i_{1}\right) + p(f(u_m)-1) + g\left(v^i_{n_i}\right) \right\} \\ &\cup \left\{ p(f(u_1)-1) + g\left(v^i_{n_i}\right) + p(f(u_m)-1) + g\left(v^i_{n_i-1}\right) \right\} \\ &\cup \left\{ p(f(u_1)-1) + g\left(v^i_{n_i-2j}\right) + p(f(u_m)-1) + g\left(v^i_{n_i-1-2j}\right) |1 \le j \le (n_i-3)/2 \right\}. \end{aligned}$$

This implies that $\{h(x) + h(y) \,|xy \in S\} = \{h'(x) + h'(y) \,|xy \in T\}$ and these sets are of cardinality n_i for each positive integer i with $1 \le i \le k$ such that $(m, n_i) = n_i$.

Therefore, we conclude from Lemma 1.1 that h' extends to a super edge-magic labeling of H'. $\square$

The following is an example that illustrates the construction given in the proof of Theorem 3.1.

Consider a super edge-magic labeling of $G \cong C_4 \cup C_5$: $1-5-2-7-1$ and $6-4-9-3-8-6$. Now, by letting $m = 5$, we obtain a super edge-magic labeling of $H \cong C_{20} \cup 5C_5$: $1-32-11-43-19-5-29-16-37-23-2-34-10-41-20-7-28-14-38-25-1$, $6-31-18-39-26-6$, $4-36-12-44-24-4$, $9-30-17-42-22-9$, $3-35-15-40-27-3$ and $8-33-13-45-21-8$, whereas we obtain a super edge-magic labeling of $H' \cong C_{20} \cup C_{25}$: $1-32-11-43-19-5-29-16-37-23-2-34-10-41-20-7-28-14-38-25-1$

and $6-31-18-39-26-3-35-15-40-27-4-36-12-44-24-8-33-13-45-21-9-30-17-42-22-6$.

Observe that our construction of the super edge-magic labeling of H given in the proof of Theorem 2.1 is highly depending on the choice of α_s defined in the proof of the same theorem. In the proof of the following theorem, we take account of this observation.

Theorem 3.2. *Assume that $G \cong \bigcup_{i=1}^{k} C_{n_i}$ is any super edge-magic 2-regular graph, and let m be odd such that $(m, n_i) = 1$ or $(m, n_i) = m$ for all positive integers i with $1 \le i \le k$. Then $H' \cong \bigcup_{i=1}^{k} C_{mn_i}$ is a super edge-magic 2-regular graph.*

Proof. First, define the odd cycle C_m, the 2-regular graph $G \cong \bigcup_{i=1}^{k} C_{n_i}$ and the graph $H \cong \bigcup_{i=1}^{k} (m, n_i)\, C_{[m,n_i]}$ as in the proof of Theorem 2.1. Also, consider the super edge-magic labeling g of G as in the proof of the same theorem, and define

$$\beta_s^i = m\left(g\left(v_s^i\right) - 1\right)$$

for every integer s with $1 \le s \le n_i$. Furthermore, recall the super edge-magic labeling f of the odd cycle C_m from the proof of the same theorem. Now, for a given integer i with $1 \le i \le k$, define h to be a vertex labeling of H such that

$$h\left(w^i_{(j-1)[m,n_i]+l}\right) = \beta_s^i + f\left(u_t\right),$$

where $s = l \pmod{n_i}$, $t = j + l - 1 \pmod{m}$, $1 \le j \le (m, n_i)$ and $1 \le l \le [m, n_i]$.

To show that h is actually a super edge-magic labeling of H, it is necessary to establish the following claims. However, the verification of these are essentially the same as the ones given in the proof of Theorem 2.1. Hence, we omit them for the sake of brevity.

Claim 1. Let i be an integer with $1 \le i \le k$. Then the function $F^i : \mathbb{Z}_{(m,n_i)} \times \mathbb{Z}_{[m,n_i]} \to \mathbb{Z}_m \times \mathbb{Z}_{n_i}$ defined by $F^i(j,l) = (\phi(j,l), \psi(j,l))$ is an injective function, where

$$\begin{array}{rclrcl} \phi(j,l+1) & = & \phi(j,l)+1; & \psi(j,l+1) & = & \psi(j,l)+1; \\ \phi(j+1,l) & = & \phi(j,l)+1; & \psi(j+1,l) & = & \psi(j,l). \end{array}$$

Claim 2. Let i be an integer with $1 \le i \le k$, and consider the function $G^i : \mathbb{Z}_m \times \mathbb{Z}_{n_i} \to V\left((m, n_i)\, C_{[m,n_i]}\right)$ defined by $G^i(t,s) = m\left(g\left(v_s^i\right) - 1\right) + f\left(u_t\right)$. Then $G^i \circ F^i$ is an injective function.

Claim 3. Let i_1, i_2, j_1, j_2, l_1 and l_2 be integers such that $1 \leq i_1, i_2 \leq k$ with $i_1 \neq i_2$, $1 \leq j_1 \leq (m, n_{i_1})$, $1 \leq j_2 \leq (m, n_{i_2})$, $1 \leq l_1 \leq [m, n_{i_1}]$ and $1 \leq l_2 \leq [m, n_{i_2}]$. Then

$$h\left(w^{i_1}_{(j_1-1)[m,n_{i_1}]+l_1}\right) \neq h\left(w^{i_2}_{(j_2-1)[m,n_{i_2}]+l_2}\right).$$

Claim 4. Let i, j and l be integers such that $1 \leq i \leq k$, $1 \leq j \leq (m, n_i)$ and $1 \leq l \leq [m, n_i]$. Then $1 \leq h\left(w^i_{(j-1)[m,n_i]+l}\right) \leq pm$, where $p = |V(G)|$.

Claim 5. Let i be an integer with $1 \leq i \leq k$, and consider the function $H^i : \mathbb{Z}_m \times \mathbb{Z}_{n_i} \to E\left((m, n_i) C_{[m,n_i]}\right)$ defined by $H^i(t, s) = m\left(g\left(v^i_s\right) + g\left(v^i_{s+1}\right) - 2\right) + f(u_t) + f(u_{t+1})$. Then $H^i \circ F^i$ is an injective function.

Claim 6. Let i_1, i_2, j_1, j_2, l_1 and l_2 be integers such that $1 \leq i_1, i_2 \leq k$ with $i_1 \neq i_2$, $1 \leq j_1 \leq (m, n_{i_1})$, $1 \leq j_2 \leq (m, n_{i_2})$, $1 \leq l_1 \leq [m, n_{i_1}]$ and $1 \leq l_2 \leq [m, n_{i_2}]$. Then

$$\begin{aligned} & h\left(w^{i_1}_{(j_1-1)[m,n_{i_1}]+l_1}\right) + h\left(w^{i_1}_{(j_1-1)[m,n_{i_1}]+l_1+1}\right) \\ & \neq h\left(w^{i_2}_{(j_2-1)[m,n_{i_2}]+l_2}\right) + h\left(w^{i_2}_{(j_2-1)[m,n_{i_2}]+l_2+1}\right), \end{aligned}$$

where $[m, n_{i_1}] + 1 = 1$ and $[m, n_{i_2}] + 1 = 1$.

Claim 7. Let i, j and l be integers such that $1 \leq i \leq k$, $1 \leq j \leq (m, n_i)$ and $1 \leq l \leq [m, n_i]$. Then

$$\begin{aligned} & \max\left\{h\left(w^i_{(j-1)[m,n_i]+l}\right) + h\left(w^i_{(j-1)[m,n_i]+l+1}\right)\right\} \\ & - \min\left\{h\left(w^i_{(j-1)[m,n_i]+l}\right) + h\left(w^i_{(j-1)[m,n_i]+l+1}\right)\right\} \\ & = pm - 1, \end{aligned}$$

where $p = |V(G)|$ and $[m, n_i] + 1 = 1$.

For the case where $(m, n_i) = 1$ for some positive integer i with $1 \leq i \leq k$, the desired result readily follows from the fact that h is a super edge-magic labeling of H. Hence, assume that $(m, n_i) = m$ for some positive integer i with $1 \leq i \leq k$. In this case, consider the 2-regular graph $H' \cong \bigcup_{i=1}^{k} C_{mn_i}$ obtained from H as follows: $V(H') = V(H)$ and $E(H') = (E(H) - S) \cup T$, where

$$S = \left\{w^i_{(j-1)n_i+1} w^i_{(j-1)n_i+n_i} \,|1 \leq j \leq m\right\}$$

and

$$\begin{aligned} T &= \left\{ w^i_{(m-3-2j)n_i+1} w^i_{(m-1-2j)n_i+n_i} \,|0 \le j \le (m-3)/2 \right\} \\ &\cup \left\{ w^i_{(m-2-2j)n_i+1} w^i_{(m-2j)n_i+n_i} \,|1 \le j \le (m-3)/2 \right\} \\ &\cup \left\{ w^i_{(m-2)n_i+1} w^i_{n_i} \right\} \cup \left\{ w^i_{(m-1)n_i+1} w^i_{n_i+n_i} \right\} \end{aligned}$$

for a given integer i with $1 \le i \le k$ such that $(m, n_i) = m$. Then define the vertex labeling $h' : V(H') \to \{1, 2, \ldots, m(n_1 + n_2 + \cdots + n_k)\}$ such that $h'(v) = h(v)$ for each $v \in V(H)$, where h is a super edge-magic labeling of H. As a result of this, we can obtain that $\{h(x) + h(y) \,|xy \in S\} = \{h'(x) + h'(y) \,|xy \in T\}$ in the same manner of the proof of Theorem 3.6. These sets have the same cardinality, namely, m for each integer i with $1 \le i \le k$ such that $(m, n_i) = m$.

Therefore, we conclude from Lemma 1.1 that h' extends to a super edge-magic labeling of H'. □

The following is an example that illustrates the construction given in the proof of Theorem 3.2.

Consider a super edge-magic labeling of $G \cong C_4 \cup C_5$: $1-5-2-7-1$ and $6-4-9-3-8-6$. In the case where $m = 5$, we obtain a super edge-magic labeling of $H \cong C_{20} \cup 5C_5$: $1-24-7-35-3-21-9-32-5-23-6-34-2-25-8-31-4-22-10-33-1$, $26-19-42-15-38-26$, $29-17-45-13-36-29$, $27-20-43-11-39-27$, $30-18-41-14-37-30$ and $28-16-44-12-40-28$, whereas we obtain a super edge-magic labeling of $H' \cong C_{20} \cup C_{25}$: $1-24-7-35-3-21-9-32-5-23-6-34-2-25-8-31-4-22-10-33-1$ and $26-19-42-15-38-40-12-44-16-28-36-13-45-17-29-37-14-41-18-30-27-20-43-11-39-26$.

As a corollary of Theorem 3.2, we have the following result.

Corollary 3.3. *Assume that* $G \cong \bigcup_{i=1}^{k} C_{n_i}$ *is any super edge-magic 2-regular graph, and let m be an odd prime. Then* $H \cong \bigcup_{i=1}^{k} C_{mn_i}$ *is a super edge-magic 2-regular graph.*

Every odd m greater than 1 can be expressed as a product of primes (with perhaps only one factor). Thus, this together with Corollary 3.3 gives us the following result.

Corollary 3.4. *Assume that* $G \cong \bigcup_{i=1}^{k} C_{n_i}$ *is any super edge-magic 2-regular graph, and let m be odd. Then* $H \cong \bigcup_{i=1}^{k} C_{mn_i}$ *is a super edge-magic 2-regular graph.*

4. Pseudo super edge-magic 2-regular graphs

In this section, we describe how pseudo super edge-magic labelings of 2-regular graphs is useful to generate new classes of super edge-magic 2-regular graphs.

Theorem 4.1. *Assume that* $G \cong \left(\bigcup_{i=1}^{k_1} C_{n_i}\right) \cup k_2K_1$ *is any pseudo super edge-magic graph, and let* m *be odd. Then* $H \cong \left(\bigcup_{i=1}^{k_1} (m, n_i)\, C_{[m,n_i]}\right) \cup k_2C_m$ *is a super edge-magic graph.*

Proof. Throughout the proof of this theorem, define the odd cycle C_m as in the proof of Theorem 2.1 with the super edge-magic labeling f of C_m given in the proof of the same theorem. Now, let $G \cong \left(\bigcup_{i=1}^{k_1} C_{n_i}\right) \cup k_2K_1$ be the graph with

$$V\left(G\right) = \left\{v_j^i \,|1 \le i \le k_1 + k_2, 1 \le j \le n_i\right\}$$

and

$$E\left(G\right) = \left\{v_1^i v_{n_i}^i \,|1 \le i \le k_1\right\} \cup \left\{v_j^i v_{j+1}^i \,|1 \le i \le k_1, 1 \le j \le n_i - 1\right\}.$$

Define $\alpha_s = p\left(f\left(u_s\right) - 1\right)$ for every integer s with $1 \le s \le m$, where $p = |V\left(G\right)|$. Also, let g be any pseudo super edge-magic labeling of G.

Next, consider the 2-regular graph $H \cong \left(\bigcup_{i=1}^{k_1} (m, n_i)\, C_{[m,n_i]}\right) \cup k_2C_m$ with

$$V\left(H\right) = \left\{w_{(j-1)[m,n_i]+l}^i \,|1 \le i \le k_1 + k_2, 1 \le j \le \left(m, n_i\right), 1 \le l \le [m, n_i]\right\}$$

and

$$\begin{aligned} E\left(H\right) &= \left\{w_{(j-1)[m,n_i]+1}^i w_{(j-1)[m,n_i]+[m,n_i]}^i \,|1 \le i \le k_1 + k_2, 1 \le j \le \left(m, n_i\right)\right\} \\ &\cup \left\{w_{(j-1)[m,n_i]+l}^i w_{(j-1)[m,n_i]+l+1}^i | \right. \\ &\qquad \left. 1 \le i \le k_1 + k_2, 1 \le j \le \left(m, n_i\right), 1 \le l \le [m, n_i] - 1\right\}, \end{aligned}$$

and construct a vertex labeling h of H as follows. For a given integer i with $1 \le i \le k_1 + k_2$, define

$$h\left(w_{(j-1)[m,n_i]+l}^i\right) = \begin{cases} \alpha_s + g\left(v_t^i\right) & \text{if } 1 \le i \le k_1 \\ \alpha_s + g\left(v_1^i\right) & \text{if } k_1 + 1 \le i \le k_1 + k_2, \end{cases}$$

where $s = l \pmod m$, $t = j + l - 1 \pmod{n_i}$, $1 \le j \le (m, n_i)$ and $1 \le l \le [m, n_i]$.

To show that h is indeed a super edge-magic labeling of H, it is necessary to establish the following claims:

Claim 1. Let i be an integer such that $1 \leq i \leq k_1$. Then the function $F^i : \mathbb{Z}_{(m,n_i)} \times \mathbb{Z}_{[m,n_i]} \to \mathbb{Z}_m \times \mathbb{Z}_{n_i}$ defined by $F^i(j,l) = (\phi(j,l), \psi(j,l))$ is an injective function, where

$$\begin{array}{rclrcl} \phi(j,l+1) & = & \phi(j,l)+1; & \psi(j,l+1) & = & \psi(j,l)+1; \\ \phi(j+1,l) & = & \phi(j,l); & \psi(j+1,l) & = & \psi(j,l)+1. \end{array}$$

Claim 2. Let i be an integer such that $1 \leq i \leq k_1$, and consider the function $G^i : \mathbb{Z}_m \times \mathbb{Z}_{n_i} \to V\left((m,n_i)\, C_{[m,n_i]}\right)$ defined by $G^i(s,t) = p(f(u_s) - 1) + g\left(v_t^i\right)$. Then $G^i \circ F^i$ is an injective function.

Claim 3. Let i, l_1 and l_2 be integers such that $k_1 + 1 \leq i \leq k_1 + k_2$ and $1 \leq l_1, l_2 \leq m$ with $l_1 \neq l_2$. Then

$$h\left(w_{l_1}^i\right) \neq h\left(w_{l_2}^i\right).$$

Claim 4. Let i_1, i_2, j_1, j_2, l_1 and l_2 be integers such that $1 \leq i_1, i_2 \leq k_1 + k_2$ with $i_1 \neq i_2$, $1 \leq j_1 \leq (m, n_{i_1})$, $1 \leq j_2 \leq (m, n_{i_2})$, $1 \leq l_1 \leq [m, n_{i_1}]$ and $1 \leq l_2 \leq [m, n_{i_2}]$. Then

$$h\left(w^{i_1}_{(j_1-1)[m,n_{i_1}]+l_1}\right) \neq h\left(w^{i_2}_{(j_2-1)[m,n_{i_2}]+l_2}\right).$$

Claim 5. Let i, j and l be integers such that $1 \leq i \leq k_1 + k_2$, $1 \leq j \leq (m, n_i)$ and $1 \leq l \leq [m, n_i]$. Then

$$1 \leq h\left(w^i_{(j-1)[m,n_i]+l}\right) \leq pm.$$

Claim 6. Let i be an integer such that $1 \leq i \leq k_1$, and consider the function $H^i : \mathbb{Z}_m \times \mathbb{Z}_{n_i} \to E\left((m,n_i)\, C_{[m,n_i]}\right)$ defined by $H^i(s,t) = p(f(u_s) + f(u_{s+1}) - 2) + g\left(v_t^i\right) + g\left(v_{t+1}^i\right)$. Then $H^i \circ F^i$ is an injective function.

Claim 7. Let i, l_1 and l_2 be integers such that $k_1 + 1 \leq i \leq k_1 + k_2$ and $1 \leq l_1, l_2 \leq m$ with $l_1 \neq l_2$. Then $h\left(w_{l_1}^i\right) + h\left(w_{l_1+1}^i\right) \neq h\left(w_{l_2}^i\right) + h\left(w_{l_2+1}^i\right)$, where $m + 1 = 1$.

Claim 8. Let i_1, i_2, j_1, j_2, l_1 and l_2 be integers such that $1 \leq i_1, i_2 \leq k_1 + k_2$ with $i_1 \neq i_2$, $1 \leq j_1 \leq (m, n_{i_1})$, $1 \leq j_2 \leq (m, n_{i_2})$, $1 \leq l_1 \leq [m, n_{i_1}]$ and $1 \leq l_2 \leq [m, n_{i_2}]$. Then

$$\begin{aligned} & h\left(w^{i_1}_{(j_1-1)[m,n_{i_1}]+l_1}\right) + h\left(w^{i_1}_{(j_1-1)[m,n_{i_1}]+l_1+1}\right) \\ \neq\ & h\left(w^{i_2}_{(j_2-1)[m,n_{i_2}]+l_2}\right) + h\left(w^{i_2}_{(j_2-1)[m,n_{i_2}]+l_2+1}\right), \end{aligned}$$

where $[m, n_{i_1}] + 1 = 1$ and $[m, n_{i_2}] + 1 = 1$.

Claim 9. Let i, j and l be integers such that $1 \leq i \leq k_1 + k_2$, $1 \leq j \leq (m, n_i)$ and $1 \leq l \leq [m, n_i]$. Then

$$\begin{aligned}&\max\left\{h\left(w^i_{(j-1)[m,n_i]+l}\right) + h\left(w^i_{(j-1)[m,n_i]+l+1}\right)\right\}\\&-\min\left\{h\left(w^i_{(j-1)[m,n_i]+l}\right) + h\left(w^i_{(j-1)[m,n_i]+l+1}\right)\right\}\\&= pm - 1,\end{aligned}$$

where $[m, n_i] + 1 = 1$.

Since the results of Claims 1, 2, 5, 6 and 9 are readily obtained by using similar arguments to those used for the verification of Claims 1, 2, 4, 5 and 7 in Theorem 2.1, respectively, we omit them for the sake of brevity. Notice also that Claim 4 follows from Claims 2 and 3, while Claim 8 follows from Claims 6 and 7. Thus, it is sufficient to verify Claims 3 and 7. To do so, define $n_i = 1$ for all integers i with $k_1 + 1 \leq i \leq k_1 + k_2$.

For the verification of Claim 3, let $h\left(w^i_{l_1}\right) = \alpha_{s_1} + g\left(v^i_1\right)$ and $h\left(w^i_{l_2}\right) = \alpha_{s_2} + g\left(v^i_1\right)$, and suppose, to the contrary, that

$$h\left(w^i_{l_1}\right) = h\left(w^i_{l_2}\right).$$

Then

$$p\left(f\left(u_{s_1}\right) - 1\right) + g\left(v^i_1\right) = p\left(f\left(u_{s_2}\right) - 1\right) + g\left(v^i_1\right).$$

This implies that

$$p\left(f\left(u_{s_1}\right) - f\left(u_{s_2}\right)\right) = 0,$$

which leads to $f\left(u_{s_1}\right) = f\left(u_{s_2}\right)$, that is, $s_1 = s_2$. Since $s = l \pmod{m}$ and $1 \leq l \leq [m, n_i] = m$, it follows that $s_1 = l_1$ and $s_2 = l_2$. This produces a contradiction, and completes the verification of Claim 3.

For the verification of Claim 7, suppose, to the contrary, that

$$h\left(w^i_{l_1}\right) + h\left(w^i_{l_1+1}\right) = h\left(w^i_{l_2}\right) + h\left(w^i_{l_2+1}\right).$$

Then

$$p\left\{\left(f\left(u_{s_1}\right) + f\left(u_{s_1+1}\right)\right) - \left(f\left(u_{s_2}\right) + f\left(u_{s_2+1}\right)\right)\right\} = 0.$$

This leads to $f\left(u_{s_1}\right) + f\left(u_{s_1+1}\right) = f\left(u_{s_2}\right) + f\left(u_{s_2+1}\right)$, that is, $s_1 = s_2$. Since $s = l \pmod{m}$ and $1 \leq l \leq [m, n_i] = m$, it follows that $s_1 = l_1$ and $s_2 = l_2$. This produces a contradiction, and completes the verification of Claim 7.

Therefore, we conclude from Lemma 1.1 that h extends to a super edge-magic labeling of H. □

The following serves as an example that illustrates the construction provided in the proof of Theorem 4.1.

Consider the graph $G \cong C_3 \cup C_4 \cup 2K_1$, and label the vertices in its cycles with 4−5−8−4, 1−6−2−9−1 and its isolated vertices 3 and 7. If we let $m = 5$, then we obtain a super edge-magic labeling of $H \cong C_{15} \cup C_{20} \cup 2C_5$: 4−32−17−40−23−8−31−14−44−22−5−35−13−

$41-26-4$, $1-33-11-45-19-6-29-18-37-24-2-36-10-42-20-9-28-15-38-27-1$, $3-30-12-39-21-3$ and $7-34-16-43-25-7$.

Combining Theorems 3.1, 3.2 and 4.10 with Corollaries 3.3 and 3.4, we now obtain the following result.

Corollary 4.2. *Assume that* $G \cong \left(\bigcup_{i=1}^{k_1} C_{n_i}\right) \cup k_2 K_1$ *is any pseudo super edge-magic graph, and let* m *be odd. Then* $H \cong \left(\bigcup_{i=1}^{k_1} C_{mn_i}\right) \cup k_2 C_m$ *is a super edge-magic graph.*

5. Conclusions

In this paper, we present some methods to construct new classes of super edge-magic 2-regular graphs from certain super edge-magic 2-regular graphs or pseudo super edge-magic graphs. The techniques used in our constructions significantly extend the previously existing results on super edge-magic 2-regular graphs. By virtue of relationships among other classes of labelings established in [6, 8, 17, 18], the 2-regular graphs obtained from our constructions are also harmonious, sequential, felicitous and equitable (see [12] for the definitions of harmonious, sequential, felicitous and equitable labelings). Moreover, our results in this paper adds credence to the conjecture of Holden et al. [16] that all 2-regular graphs of odd order with the exception of $C_3 \cup C_4$, $3C_3 \cup C_4$ and $2C_3 \cup C_5$ possess strong vertex-magic total labelings, which are equivalent to super edge-magic labelings for 2-regular graphs.

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