Wheel-supermagic labelings for a wheel $k$-multilevel corona with a cycle

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Abstract

Let $k$ be a positive integer. A graph $G$ $k$-multilevel corona with a graph $H$, denoted by $G \odot^k H$, is a graph that is defined by $(G \odot^{k-1} H) \odot H$ for $k \geq 2$ and by $G \odot H$ for $k = 1$ where $G \odot H$ is a graph obtained from $G$ and $|V(G)|$ copies of $H$, namely $H_1, H_2, ..., H_{|V(G)|}$, and joined every $v_i$ in $V(G)$ to all vertices in $V(H_i)$. A graph $G = (V, E)$ is said to be $H$-magic if every edge of $G$ belongs to at least one subgraph isomorphic to $H$ and there is a total labeling $f : V(G) \cup E(G) \to \{1, 2, ..., |V(G)| + |E(G)|\}$ such that for each subgraph $H' = (V', E')$ of $G$ isomorphic to $H$, the sum of all vertex labels in $V'$ plus the sum of all edge labels in $E'$ is a constant. Additionally, $G$ is said to be $H$-supermagic, if $f(V(G)) = \{1, 2, ..., |V(G)|\}$. We prove that a wheel $W_n$ $k$-multilevel corona with a cycle $C_n$ is $W_n$-supermagic.

Keywords: $k$-multilevel corona, cycle, $H$-covering, $H$-supermagic, labeling, wheel.

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1. Introduction

Let $G = (V, E)$ be a finite simple graph with the number of vertices in $G$ is $|V(G)|$ and the number of edges in $G$ is $|E(G)|$. Let $n \geq 3$ be an integer. A cycle $C_n$ is a connected graph with order $n$ whose every vertex has degree two. A wheel $W_n$ is a graph obtained from a cycle $C_n$ by adding a new vertex and $n$ edges joining it to all the vertices of the cycle.

A graph $G$ corona a graph $H$, denoted by $G \odot H$, is a graph that is obtained from $G$ and $|V(G)|$ copies of $H$, namely $H_1, H_2, ..., H_{|V(G)|}$, and joined every $v_i$ in $V(G)$ to all vertices in $V(H_i)$. Let $k$ be a positive integer. A graph $G$ $k$-multilevel corona with a graph $H$, denoted by $G \odot^k H$, is a graph that is defined by $(G \odot^{k-1} H) \odot H$ for $k \geq 2$ and by $G \odot H$ for $k = 1$.

A graph $G = (V, E)$ is said to be $H$-magic, if every edge of $G$ belongs to at least one subgraph isomorphic to $H$ and there is a total labeling $f : V(G) \cup E(G) \to \{1, 2, ..., |V(G)| + |E(G)|\}$ such that for each subgraph $H' = (V', E')$ of $G$ isomorphic to

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In 2005, Gutiérrez and Lladó [3] introduced the notion of an $H$-magic labeling of a graph. They proved that a complete bipartite graph $K_{m,n}$ is a star $K_{1,h}$-supermagic and a path $P_n$ and a cycle $C_n$ are $P_h$-supermagic. Maryati et al [8] proved that shackles and amalgamations are $H$-supermagic for some $H$. Jeyanthi and Selvagopal [4] showed that an edge amalgamation to any 2-connected simple graph $H$ is $H$-supermagic. Some other results about $H$-supermagic labelings can be seen in [1], [2], [5], [6], [7], [9], [10], [11], and [12].

In this paper, we study an $H$-supermagic labeling of a graph $G_k$-multilevel corona with a graph $H$. We prove that a wheel $W_n$ corona $k$-multilevel with a cycle $C_n$ has a $W_n$-supermagic labeling.

2. Main Result

To assist in a $W_n$-supermagic labeling for a graph $W_n \odot^k C_n$, we give a name for every part of the graph as follows:

- $W_n$ is named with $W$,
- For $i \in [1,k]$, there are $(n+1)^i$ cycles on $W_n \odot^k C_n$ that is named respectively $C_i^j$ for $1 \leq j \leq (n+1)^i$. For example, a wheel $W_3$ 2-multilevel corona with a cycle $C_3$ can be seen in Figure 1.

In this paper, we use the notation $[a, b]$ to mean $\{x \in \mathbb{N} | a \leq x \leq b\}$. Let $k$ be a positive integer and $X$ be a set that contains some positive integers, we use the notation $\sum X$ to mean $\sum_{x \in X} x$ and $k + X$ to mean $\{k + x : x \in X\}$. We say that $X$ is $k$-equipartition, if there exist $k$ subsets of $X$, say $X_1, X_2, \ldots, X_k$, such that $\bigcup_{i=1}^k X_i = X$ and $|X_i| = \frac{|X|}{k}$ for every $i \in [1,k]$. Additionally, $X$ is said $k$-balanced, if $X$ is $k$-equipartition and $\sum X_i$ is constant for every $i \in [1,k]$.

**Lemma 2.1.** [4] Let $h$ be an even integer and $k$ be a positive integer, then $X = [1, hk]$ is $k$-equipartition such that $\sum X_i = \frac{h(k+1)}{2}$ for every $i \in [1,k]$.

**Lemma 2.2.** [4] Let $h$ be an even integer and $k \geq 3$ be an integer, then $X = [1, hk]$ is $k$-equipartition such that $\sum X_i = \frac{(h-1)(hk+k+1)}{2} + i$ for every $i \in [1,k]$.

**Lemma 2.3.** Let $h$ be an odd integer and $k \geq 3$ be an integer, then $X = [1, hk]$ is $k$-equipartition such that $\sum X_i = \frac{h(k+1)}{2}$ for every $i \in [1,k]$.

**Proof.** Partition $X = [1, hk]$ into two sets $Y = [1, (h-1)k]$ and $Z = [(h-1)k + 1, hk]$. Since $h$ is odd, by using Lemma 2.2, we obtain that $Y$ is $k$-equipartition such that
Figure 1: A wheel $W_3$ 2-multilevel corona with a cycle $C_3$.
Lemma 2.4. Let \( k > 3 \) be an even integer, then \( X = [1, 4k] \) is \( k \)-equipartition such that \( \sum X_i = 8k + 2 \) for every \( i \in [1, k] \).

Proof. For every \( i \in [1, k] \), define \( X_i = \{a_i, b_i, c_i, d_i\} \) where

\[
a_i = \begin{cases} 
\frac{k}{2} + 1 - i & \text{for } 1 \leq i \leq \frac{k}{2}, \\
\frac{3k}{2} - i + 1 & \text{for } \frac{k}{2} + 1 \leq i \leq k;
\end{cases}
\]

\[
b_i = \frac{3k}{2} + i;
\]

\[
c_i = \begin{cases} 
3k - i + 1 & \text{for } 1 \leq i \leq \frac{k}{2}, \\
\frac{k}{2} + i & \text{for } \frac{k}{2} + 1 \leq i \leq k;
\end{cases}
\]

\[
d_i = \begin{cases} 
3k + i & \text{for } 1 \leq i \leq \frac{k}{2}, \\
4k + \frac{k}{2} + 1 - i & \text{for } \frac{k}{2} + 1 \leq i \leq k.
\end{cases}
\]

Let

\[
A = \{a_i | 1 \leq i \leq k\} = [1, k],
\]

\[
B = \{b_i | 1 \leq i \leq k\} = [k + \frac{k}{2} + 1, 2k + \frac{k}{2}],
\]

\[
C = \{c_i | 1 \leq i \leq k\} = [k + 1, k + \frac{k}{2}] \cup [2k + \frac{k}{2} + 1, 3k],
\]

\[
D = \{d_i | 1 \leq i \leq k\} = [3k + 1, 4k].
\]

Since \( A \cup B \cup C \cup D = X \), \( \bigcup_{i=1}^{k} X_i = X \). We find that \( |X_i| = 4 \) and \( \sum X_i = 8k + 2 \) for every \( i \in [1, k] \).

Theorem 2.5. Let \( k \) be a positive integer. A graph wheel \( k \)-multilevel corona with a cycle \( W_n \circ^k C_n \) is a \( W_n \)-supermagic.

Proof. Since \( |V(W_n \circ^k C_n)| = (n + 1)^{k+1} \) and \( |E(W_n \circ^k C_n)| = 2n \left( \sum_{i=0}^{k} (n + 1)^i \right) \), the set of labels used to label all vertices and edges of \( W_n \circ^k C_n \) is

\[
X = \left[ 1, (n + 1)^{k+1} + 2n \left( \sum_{i=0}^{k} (n + 1)^i \right) \right].
\]
Partition $X$ into two sets as follows:

$$Y = [1, (n + 1)^{k+1}]$$

and

$$Z = (n + 1)^{k+1} + \left[ 1, 2n \left( \sum_{i=0}^{k} (n+1)^i \right) \right].$$

The elements of $Y$ are used to label all vertices of $W_n \odot^k C_n$ and the elements of $Z$ are used to label all edges of $W_n \odot^k C_n$. We divide the proof into two cases.

- **Case 1** $n$ is even.
  1. For $l = 1$, define $Y^l = [1, (n + 1)^{k+1}]$. By Lemma 2.3, $Y^1$ is $(n + 1)^k$-equipartition such that $|Y_i| = n + 1$ and $\sum Y^1_i = \frac{(n+1)((n+1)^{k+1}+1)}{2}$ for $i \in [1, (n + 1)^k]$.
  2. For $2 \leq l < k+1$, define $Y^l = \sum_{j=2}^{l} \frac{n}{2} (n+1)^{k-j+2} + [1, (n+1)^{k-l+2}]$. We obtain that $Y^l$ is a subset of $Y^{l-1}$. By Lemma 2.3, $Y^l$ is $(n + 1)^{k+1-l}$-equipartition such that $|Y^l_i| = n + 1$ and $\sum Y^l_i = \frac{(n+1)((n+1)^{k+1}+1)}{2}$ for $1 \leq i \leq k+1 - l$.
  3. For $l = k+1$, define $Y^{k+1} = \sum_{j=2}^{k+1} \frac{n}{2} (n+1)^{k-j+2} + [1, (n+1)]$. We obtain that $Y^{k+1}$ is a subset of $Y^{k}$, $|Y^{k+1}| = n + 1$, and $\sum Y^{k+1} = \frac{(n+1)((n+1)^{k+1}+1)}{2}$.

- **Case 2** $n$ is odd.
  1. For $l = 1$, define $Y^1 = [1, (n + 1)^{k+1}]$. For $n = 3$, define $Y^1 = Q^1$. For $n > 3$, partition $Y^1$ into two sets as follows:

$$P^1 = [1, \frac{n-3}{2} (n + 1)^k] \cup \left[ \frac{n+5}{2} (n + 1)^k + 1, (n + 1)^{k+1} \right]$$

and

$$Q^1 = \left[ \frac{n-3}{2} (n + 1)^k + 1, \frac{n+5}{2} (n + 1)^k \right].$$

For every $i \in [1, (n + 1)^k]$, define $P^1_i = \{a^i_j, b^i_j | 1 \leq j \leq \frac{n-3}{2}\}$ where

$$a^i_j = (j-1)(n+1)^k + i \quad \text{and} \quad b^i_j = (n+2-j)(n+1)^k - i + 1.$$

So, for every $i \in [1, (n + 1)^k]$, we obtain:

$$|P^1_i| = n - 3 \quad \text{and} \quad \sum P^1_i = \frac{n-3}{2} ((n + 1)^{k+1} + 1).$$
By Lemma 2.4, $Q^1$ is $(n+1)^k$-equipartition such that $\sum Q^1_i = 2(n+1)^{k+1}+2$ for $1 \leq i \leq (n+1)^k$. Then define $Y^1_i = P^1_i \cup Q^1_i$ for $1 \leq i \leq (n+1)^k$. So, we obtain:

$$|Y^1_i| = n+1 \text{ and } Y^1_i = \frac{(n+1)((n+1)^{k+1}+1)}{2}.$$

2. For $2 \leq l < k+1$, define $Y^l = \sum_{j=2}^{l} \left( \frac{n(n+1)^{k-j+2}}{2} + [1, (n+1)^{k-l+2}] \right)$. For $n = 3$, define $Y^l = Q^l$. For $n > 3$, partition $Y^l$ into two sets as follows:

$$P^l = \sum_{j=2}^{l} \frac{n(n+1)^{k-j+2}}{2} + \left[ 1, \frac{n-3}{2}(n+1)^{k-l+1} \right] \cup \left[ \frac{n+5}{2}(n+1)^{k-l+1} + 1, (n+1)^{k-l+2} \right]$$

and

$$Q^l = \sum_{j=2}^{l} \frac{n(n+1)^{k-j+2}}{2} + \left[ \frac{n-3}{2}(n+1)^{k-l+1} + 1, \frac{n+5}{2}(n+1)^{k-l+1} \right].$$

For every $i \in [1, (n+1)^k]$, define $P^l_i = \{a^i_j, b^i_j | 1 \leq j \leq \frac{n-3}{2}\}$ where

$$a^i_j = \sum_{m=2}^{l} \frac{n(n+1)^{k-m+2}}{2} + (j-1)(n+1)^{k-j+2} + i \text{ and}$$

$$b^i_j = \sum_{m=2}^{l} \frac{n(n+1)^{k-m+2}}{2} + (n+2-j)(n+1)^{k-j+2} - i + 1.$$

We obtain $|P^l_i| = n-3$ and

$$\sum P^l_i = \frac{n-3}{2} \sum_{j=2}^{l} \frac{n(n+1)^{k-j+2} n-3}{2} (n+1)^{k+1} + 1.$$

By using Lemma 2.4, $Q^l = \frac{n-3}{2}(n+1)^k + [1, 4(n+1)^{k-l+1}]$ is $(n+1)^{k-l+1}$-equipartition such that for $i \in [1, (n+1)^{k-l+1}]$,

$$\sum Q^l_i = 4 \sum_{j=2}^{l} \frac{n(n+1)^{k-j+2}}{2} + 2(n+1)^{k-l+2} + 2.$$
For $1 \leq i \leq (n+1)^k$, define $Y^l_i = P^l_i \cup Q^l_i$. We can get $Y^l$ is $(n+1)^k$-equipartition such that $|Y^l| = n+1$ and

$$\sum Y^l = \frac{(n+1)((n+1)^{k+1}+1)}{2}.$$ 

3. For $l = k+1$, define $Y^{k+1} = \sum_{j=2}^{k+1} \frac{n(n+1)^{k-j+2}}{2} + [1, (n+1)]$. We can check that $Y^{k+1}$ is a subset of $Y^k$, $|Y^{k+1}| = n+1$, and $\sum Y^{k+1} = \frac{(n+1)((n+1)^{k+1}+1)}{2}$.

We can conclude that $W_n \circ^k C_n$ is $W_n$-supermagic.

For illustration, we can see a $W_3$-supermagic labeling of $W_n \circ^k C_n$ in Figure 2.
Figure 2: $W_3$-supermagic labeling of $W_3 \circ C_3$ with $f(W_3) = 895$
References


