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Wheel-supermagic labelings for a wheel k-multilevel corona with a cycle

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Abstract

Let k be a positive integer. A graph G k-multilevel corona with a graph H, denoted by $G \odot^k H$, is a graph that is defined by $(G \odot^{k-1} H) \odot H$ for $k \geq 2$ and by $G \odot H$ for $k = 1$ where $G \odot H$ is a graph obtained from G and $|V(G)|$ copies of H, namely $H_1, H_2, ..., H_{|V(G)|}$, and joined every v_i in $V(G)$ to all vertices in $V(H_i)$. A graph $G = (V, E)$ is said to be H-magic if every edge of G belongs to at least one subgraph isomorphic to H and there is a total labeling $f : V(G) \cup E(G) \to \{1, 2, ..., |V(G)| + |E(G)|\}$ such that for each subgraph $H' = (V', E')$ of G isomorphic to H, the sum of all vertex labels in V' plus the sum of all edge labels in E' is a constant. Additionally, G is said to be H-supermagic, if $f(V(G)) = \{1, 2, ..., |V(G)|\}$. We prove that a wheel W_n k-multilevel corona with a cycle C_n is W_n-supermagic.

Keywords: k-multilevel corona, cycle, H-covering, H-supermagic, labeling, wheel.

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1. Introduction

Let $G = (V, E)$ be a finite simple graph with the number of vertices in G is $|V(G)|$ and the number of edges in G is $|E(G)|$. Let $n \geq 3$ be an integer. A cycle C_n is a connected graph with order n whose every vertex has degree two. A wheel W_n is a graph obtained from a cycle C_n by adding a new vertex and n edges joining it to all the vertices of the cycle.

A graph G corona a graph H, denoted by $G \odot H$, is a graph that is obtained from G and $|V(G)|$ copies of H, namely $H_1, H_2, ..., H_{|V(G)|}$, and joined every v_i in $V(G)$ to all vertices in $V(H_i)$. Let k be a positive integer. A graph G k-multilevel corona with a graph H, denoted by $G \odot^k H$, is a graph that is defined by $(G \odot^{k-1} H) \odot H$ for $k \geq 2$ and by $G \odot H$ for $k = 1$.

A graph $G = (V, E)$ is said to be H-magic, if every edge of G belongs to at least one subgraph isomorphic to H and there is a total labeling $f : V(G) \cup E(G) \to \{1, 2, ..., |V(G)| + |E(G)|\}$ such that for each subgraph $H' = (V', E')$ of G isomorphic to

H, the sum of all vertex labels in V' plus the sum of all edge labels in E' is constant. Additionally, G is said to be H-supermagic, if $f(V(G)) = \{1, 2, ..., |V(G)|\}$.

In 2005, Gutiérrez and Lladó [3] introduced the notion of an H-magic labeling of a graph. They proved that a complete bipartite graph $K_{m,n}$ is a star $K_{1,h}$-supermagic and a path P_n and a cycle C_n are P_h-supermagic. Maryati et al [8] proved that shackles and amalgamations are H-supermagic for some H. Jeyanthi and Selvagopal [4] showed that an edge amalgamation to any 2-connected simple graph H is H-supermagic. Some other results about H-supermagic labelings can be seen in [1], [2], [5], [6], [7], [9], [10], [11], and [12].

In this paper, we study an H-supermagic labeling of a graph G k-multilevel corona with a graph H. We prove that a wheel W_n corona k-multilevel with a cycle C_n has a W_n-supermagic labeling.

2. Main Result

To assist in a W_n-supermagic labeling for a graph $W_n \odot^k C_n$, we give a name for every part of the graph as follows:

- W_n is named with W,
- For $i \in [1, k]$, there are $(n+1)^i$ cycles on $W_n \odot^k C_n$ that is named respectively C_j^i for $1 \leq j \leq (n+1)^i$.

For example, a wheel W_3 2-multilevel corona with a cycle C_3 can be seen in Figure 1.

In this paper, we use the notation $[a, b]$ to mean $\{x \in N | a \leq x \leq b\}$. Let k be a positive integer and X be a set that contains some positive integers, we use the notation $\sum X$ to mean $\sum_{x \in X} x$ and $k + X$ to mean $\{k + x : x \in X\}$. We say that X is k-*equipartition*, if there exist k subsets of X, say $X_1, X_2, ...X_k$, such that $\bigcup_{i=1}^k X_i = X$ and $|X_i| = \frac{|X|}{k}$ for every $i \in [1, k]$. Additionally, X is said k-*balanced*, if X is k-equipartition and $\sum X_i$ is constant for every $i \in [1, k]$.

Lemma 2.1. *[4] Let h be an even integer and k be a positive integer, then* $X = [1, hk]$ *is k-equipartition such that* $\sum X_i = \frac{h(hk+1)}{2}$ *for every* $i \in [1, k]$.

Lemma 2.2. *[4] Let h be an even integer and* $k \geq 3$ *be an integer, then* $X = [1, hk]$ *is k-equipartition such that* $\sum X_i = \frac{(h-1)(hk+k+1)}{2} + i$ *for every* $i \in [1, k]$.

Lemma 2.3. *Let h be an odd integer and* $k \geq 3$ *be an integer, then* $X = [1, hk]$ *is k-equipartition such that* $\sum X_i = \frac{h(hk+1)}{2}$ *for every* $i \in [1, k]$.

Proof. Partition $X = [1, hk]$ into two sets $Y = [1, (h-1)k]$ and $Z = [(h-1)k + 1, hk]$. Since h is odd, by using Lemma 2.2, we obtain that Y is k-equipartition such that

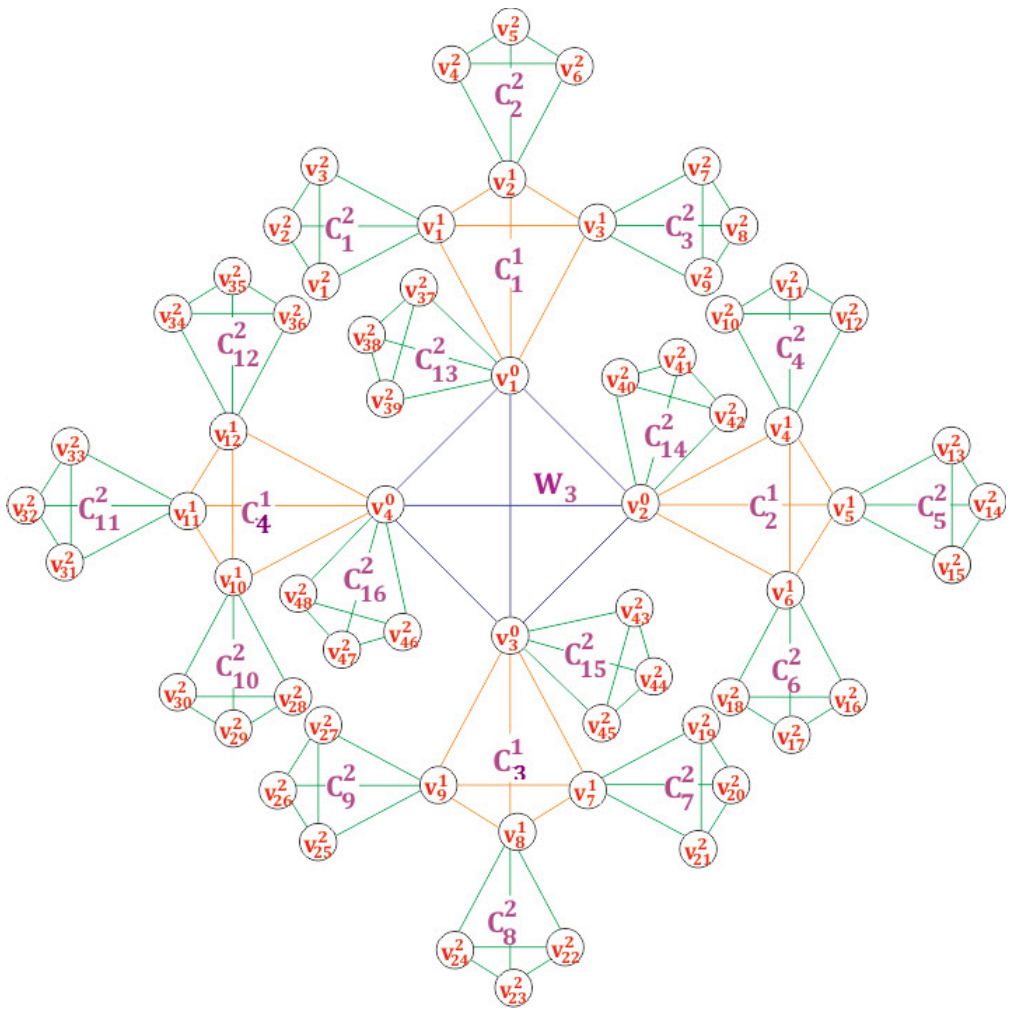

Figure 1: A wheel W_3 2-multilevel corona with a cycle C_3

$\sum Y_i = \frac{(h-2)(hk+1)}{2} + i$ and $|Y_i| = h-1$ for $i \in [1,k]$. Then define $X_i = Y_i \cup \{hk-i+1\}$. So, we obtain $\bigcup_{i=1}^{k} X_i = X$, $\sum X_i = \frac{h(hk+1)}{2}$, and $|X_i| = h$ for every $i \in [1,k]$. □

Lemma 2.4. *Let $k > 3$ be an even integer, then $X = [1, 4k]$ is k-equipartition such that $\sum X_i = 8k+2$ for every $i \in [1,k]$.*

Proof. For every $i \in [1,k]$, define $X_i = \{a_i, b_i, c_i, d_i\}$ where

$$a_i = \begin{cases} \frac{k}{2} + 1 - i & \text{for } 1 \le i \le \frac{k}{2}; \\ \frac{3k}{2} - i + 1 & \text{for } \frac{k}{2} + 1 \le i \le k; \end{cases}$$

$$b_i = \frac{3k}{2} + i;$$

$$c_i = \begin{cases} 3k - i + 1 & \text{for } 1 \le i \le \frac{k}{2}; \\ \frac{k}{2} + i & \text{for } \frac{k}{2} + 1 \le i \le k; \end{cases}$$

$$d_i = \begin{cases} 3k + i & \text{for } 1 \le i \le \frac{k}{2}; \\ 4k + \frac{k}{2} + 1 - i & \text{for } \frac{k}{2} + 1 \le i \le k. \end{cases}$$

Let

$$\begin{aligned} A &= \{a_i | 1 \le i \le k\} = [1, k], \\ B &= \{b_i | 1 \le i \le k\} = [k + \frac{k}{2} + 1, 2k + \frac{k}{2}], \\ C &= \{c_i | 1 \le i \le k\} = [k+1, k + \frac{k}{2}] \cup [2k + \frac{k}{2} + 1, 3k], \\ D &= \{d_i | 1 \le i \le k\} = [3k+1, 4k]. \end{aligned}$$

Since $A \cup B \cup C \cup D = X$, $\bigcup_{i=1}^{k} X_i = X$. We find that $|X_i| = 4$ and $\sum X_i = 8k+2$ for every $i \in [1,k]$. □

Theorem 2.5. *Let k be a positive integer. A graph wheel k-multilevel corona with a cycle $W_n \odot^k C_n$ is a W_n-supermagic.*

Proof. Since $|V(W_n \odot^k C_n)| = (n+1)^{k+1}$ and $|E(W_n \odot^k C_n)| = 2n\left(\sum_{i=0}^{k}(n+1)^i\right)$, the set of labels used to label all vertices and edges of $W_n \odot^k C_n$ is

$$X = \left[1, (n+1)^{k+1} + 2n\left(\sum_{i=0}^{k}(n+1)^i\right)\right].$$

Partition X into two sets as follows:

$$Y = [1, (n+1)^{k+1}]$$

and

$$Z = (n+1)^{k+1} + \left[1, 2n\left(\sum_{i=0}^{k}(n+1)^i\right)\right].$$

The elements of Y are used to label all vertices of $W_n \odot^k C_n$ and the elements of Z are used to label all edges of $W_n \odot^k C_n$. We divide the proof into two cases.

- **Case 1** n is even.
  1. For $l = 1$, define $Y^1 = [1, (n+1)^{k+1}]$. By Lemma 2.3, Y^1 is $(n+1)^k$-equipartition such that $|Y_i| = n+1$ and $\sum Y_i^1 = \frac{(n+1)((n+1)^{k+1}+1)}{2}$ for $i \in [1, (n+1)^k]$.
  2. For $2 \le l < k+1$, define $Y^l = \sum_{j=2}^{l} \frac{n}{2}(n+1)^{k-j+2} + [1, (n+1)^{k-l+2}]$. We obtain that Y^l is a subset of Y^{l-1}. By Lemma 2.3, Y^l is $(n+1)^{k+1-l}$-equipartition such that $|Y_i^l| = n+1$ and $\sum Y_i^l = \frac{(n+1)((n+1)^{k+1}+1)}{2}$ for $1 \le i \le k+1-l$.
  3. For $l = k+1$, define $Y^{k+1} = \sum_{j=2}^{k+1} \frac{n}{2}(n+1)^{k-j+2} + [1, (n+1)]$. We obtain that Y^{k+1} is a subset of Y^k, $|Y^{k+1}| = n+1$, and $\sum Y^{k+1} = \frac{(n+1)((n+1)^{k+1}+1)}{2}$.
- **Case 2** n is odd.
  1. For $l = 1$, define $Y^1 = [1, (n+1)^{k+1}]$. For $n = 3$, define $Y^1 = Q^1$. For $n > 3$, partition Y^1 into two sets as follows:

$$P^1 = [1, \frac{n-3}{2}(n+1)^k] \cup [\frac{n+5}{2}(n+1)^k + 1, (n+1)^{k+1}]$$

and

$$Q^1 = \left[\frac{n-3}{2}(n+1)^k + 1, \frac{n+5}{2}(n+1)^k\right].$$

For every $i \in [1, (n+1)^k]$, define $P_i^1 = \{a_j^i, b_j^i | 1 \le j \le \frac{n-3}{2}\}$ where

$$a_j^i = (j-1)(n+1)^k + i \text{ and } b_j^i = (n+2-j)(n+1)^k - i + 1.$$

So, for every $i \in [1, (n+1)^k]$, we obtain:

$$|P_i^1| = n-3 \text{ and } \sum P_i^1 = \frac{n-3}{2}\left((n+1)^{k+1} + 1\right).$$

By Lemma 2.4, Q^1 is $(n+1)^k$-equipartition such that $\sum Q_i^1 = 2(n+1)^{k+1}+2$ for $1 \le i \le (n+1)^k$. Then define $Y_i^1 = P_i^1 \cup Q_i^1$ for $1 \le i \le (n+1)^k$. So, we obtain:

$$|Y_i^1| = n+1 \text{ and } Y_i^1 = \frac{(n+1)((n+1)^{k+1}+1)}{2}.$$

2. For $2 \le l < k+1$, define $Y^l = \sum_{j=2}^{l} \left(\frac{n(n+1)^{k-j+2}}{2}\right) + \left[1, (n+1)^{k-l+2}\right]$. For $n = 3$, define $Y^l = Q^l$. For $n > 3$, partition Y^l into two sets as follows:

$$P^l = \sum_{j=2}^{l} \frac{n(n+1)^{k-j+2}}{2} + \left[1, \frac{n-3}{2}(n+1)^{k-l+1}\right] \cup \left[\frac{n+5}{2}(n+1)^{k-l+1} + 1, (n+1)^{k-l+2}\right]$$

and

$$Q^l = \sum_{j=2}^{l} \frac{n(n+1)^{k-j+2}}{2} + \left[\frac{n-3}{2}(n+1)^{k-l+1} + 1, \frac{n+5}{2}(n+1)^{k-l+1}\right].$$

For every $i \in [1, (n+1)^k]$, define $P_i^l = \{a_j^i, b_j^i | 1 \le j \le \frac{n-3}{2}\}$ where

$$a_j^i = \sum_{m=2}^{l} \frac{n(n+1)^{k-m+2}}{2} + (j-1)(n+1)^{k-j+2} + i \text{ and}$$

$$b_j^i = \sum_{m=2}^{l} \frac{n(n+1)^{k-m+2}}{2} + (n+2-j)(n+1)^{k-j+2} - i + 1.$$

We obtain $|P_i^l| = n-3$ and

$$\sum P_i^l = \frac{n-3}{2} \sum_{j=2}^{l} \frac{n(n+1)^{k-j+2}}{2} \frac{n-3}{2} \left((n+1)^{k+1} + 1\right).$$

By using Lemma 2.4, $Q^l = \frac{n-3}{2}(n+1)^k + [1, 4(n+1)^{k-l+1}]$ is $(n+1)^{k-l+1}$-equipartition such that for $i \in [1, (n+1)^{k-l+1}]$,

$$\sum Q_i^l = 4 \sum_{j=2}^{l} \frac{n(n+1)^{k-j+2}}{2} + 2(n+1)^{k-l+2} + 2.$$

For $1 \leq i \leq (n+1)^k$, define $Y_i^l = P_i^l \cup Q_i^l$. We can get Y^l is $(n+1)^{k-l+1}$-equipartition such that $|Y_i^l| = n+1$ and

$$\sum Y_i^l = \frac{(n+1)((n+1)^{k+1}+1)}{2}.$$

3. For $l = k+1$, define $Y^{k+1} = \sum_{j=2}^{k+1} \frac{n(n+1)^{k-j+2}}{2} + [1,(n+1)]$. We can check that Y^{k+1} is a subset of Y^k, $|Y^{k+1}| = n+1$, and $\sum Y^{k+1} = \frac{(n+1)((n+1)^{k+1}+1)}{2}$.

We obtain for $l \in [1,k+1]$, Y^l is a balanced subset of Y. Label all vertices of W with the elements of Y^{k+1}. For $j \in [1,n+1]$, label all vertices of C_j^1 by using elements of set $Y^k \backslash Y^{k+1}$. For $i \in [2,k]$ and $j \in [1,(n+1)^i]$, label all vertices of C_j^i using the elements of set $Y^{i-l} \backslash Y^i$. We can check that the sum of all vertex labels of every subgraph W_n of $W_n \odot^k C_n$ is

$$f(V(W_n)) = \frac{(n+1)((n+1)^{k+1}+1)}{2}.$$

Now, we label all edges of $W_n \odot^k C_n$ with elements of Z. By using Lemma 2.1, Z is $\sum_{i=0}^{k}(n+1)^i$-equipartition, so that for $i \in [1, \sum_{i=0}^{k}(n+1)^i]$, the sum of Z_i is

$$\sum Z_i = 2n(n+1)^{k+1} + \frac{2n(2n(\sum_{i=0}^{k}(n+1)^k)+1)}{2}.$$

It mean that the sum of all edge labels of every subgraph W_n of $W_n \odot^k C_n$ is $f(E(W_n)) = \sum Z_i$.

Hence, the sum of all vertex and edge labels of every subgraph W_n of $W_n \odot^k C_n$ is constant, namely

$$f(W_n) = \frac{(9n+1)(n+1)^{k+1} - n + 1}{2}.$$

We can conclude that $W_n \odot^k C_n$ is W_n-supermagic. □

For illustration, we can see a W_3-supermagic labeling of $W_n \odot^k C_n$ in Figure 2.

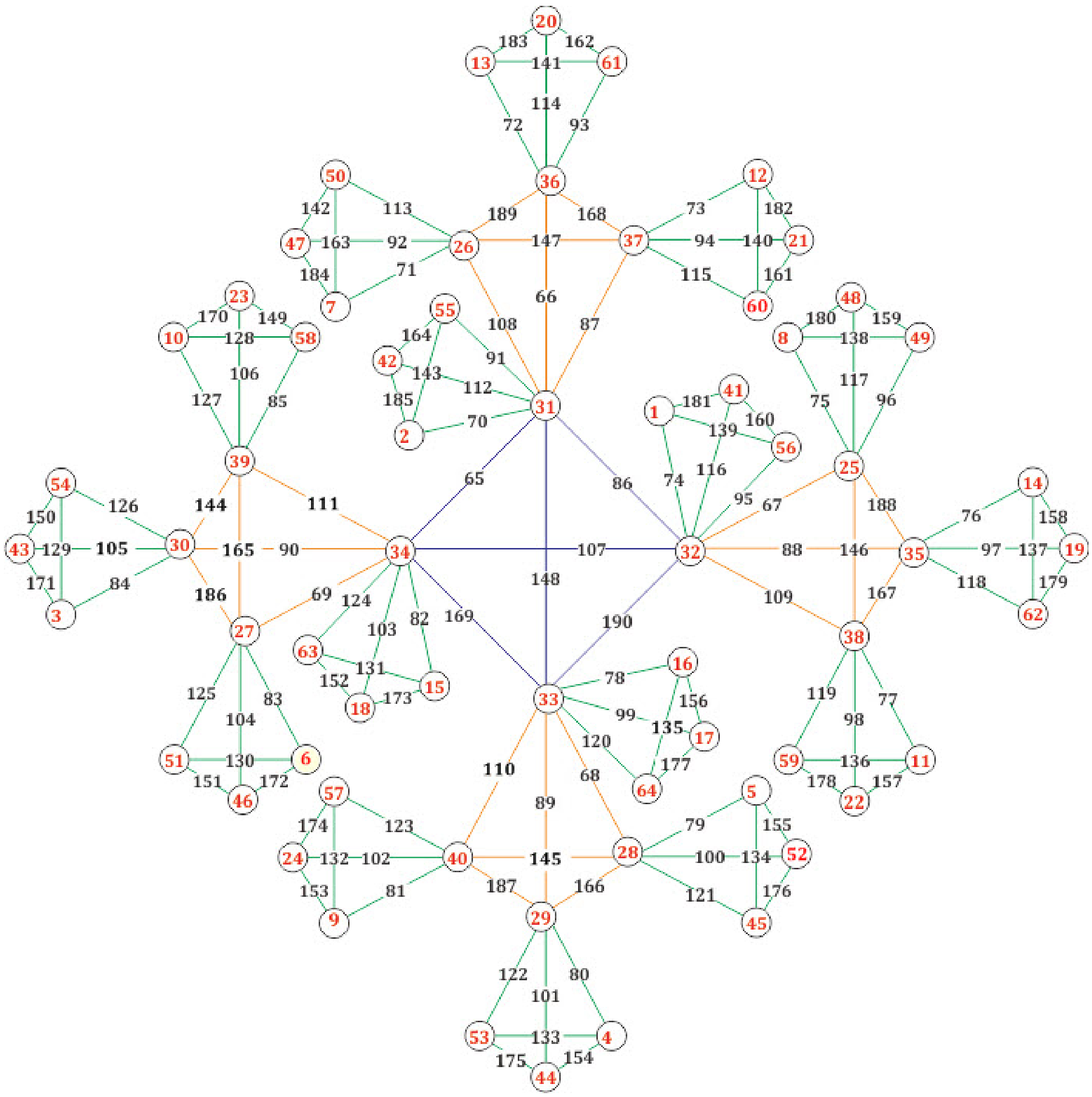

Figure 2: W_3-supermagic labeling of $W_3 \odot^2 C_3$ with $f(W_3) = 895$

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