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Sparse graphs with vertex antimagic edge labelings

Mirka Miller[a,b,c,d], Oudone Phanalasy[a,e], Joe Ryan[f] and Leanne Rylands[g]
[a]School of Mathematical and Physical Sciences, University of Newcastle, Australia
[b]Department of Mathematics, University of West Bohemia, Pilsen, Czech Republic
[c]Department of Informatics, King's College London, UK
[d]Department of Mathematics, ITB Bandung, Indonesia
[e]Department of Mathematics, National University of Laos, Laos
[f]School of Electrical Engineering and Computer Science, University of Newcastle, Australia
[g]School of Computing, Engineering and Mathematics, University of Western Sydney, Australia
e-mail: *mirka.miller@newcastle.edu.au, oudone.phanalasy@gmail.com, joe.ryan@newcastle.edu.au, l.rylands@uws.edu.au*

Abstract

Hartsfeld and Ringel in 1990 introduced the concept of an *antimagic labeling* of a graph, that is, a *vertex antimagic edge labeling* and they also conjectured that every connected graph, except K_2, is antimagic. As a means of providing an incremental advance towards proving the conjecture of Hartsfield and Ringel, in this paper we provide constructions whereby, given any degree sequence pertaining to a tree, we can construct two different vertex antimagic edge trees with the given degree sequence. Moreover, we modify a construction presented for trees to obtain an antimagic unicyclic graph with a given degree sequence pertaining to a unicyclic graph.

Keywords: antimagic labeling, antimagic tree, antimagic unicyclic graph.

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1. Introduction

All graphs considered in this paper are finite, simple, undirected and connected. A labeling of a graph G is a map from some set of graph elements to a set of numbers. Particularly, in this paper we are only interested in labeling of edges of graphs. An *edge labeling* of a graph $G(V,E)$ is a bijection $l : E \longrightarrow \{1, 2, \dots, |E|\}$. For an edge labeling l, the weight of a vertex v is $wt(v) = \sum_{u \in N(v)} l(uv)$, where $N(v)$ is the set of neighbors of v. An edge labeling of G is said to be a *vertex antimagic edge labeling (VAE labeling)* if all vertex weights are pairwise distinct. A graph G is *vertex antimagic edge (VAE)* if it has a VAE labeling.

In 1990, Hartsfield and Ringel [4] introduced the concept of an antimagic labeling of graph, that is, a VAE labeling. They showed that P_n, S_n, C_n, K_n, W_n and $K_{2,n}$, $n \geq 3$, are VAE. They also conjectured that "Every connected graph, except K_2, is VAE". Since then several families of graphs have been proved to be VAE, for example, see [1, 7]. Many other results concerning antimagic labelings of graphs can be found in the dynamic survey by Gallian [3].

In this paper we describe a new construction to obtain a VAE labeling of a tree corresponding to a given degree sequence which pertains to the tree. We prove that this construction and the one described in [6] whereby, given a degree sequence pertaining to tree with at least seven vertices, provide different VAE trees.

As mentioned above, for more than two decades there have been many attempts to prove the conjecture of Hartsfield and Ringel. However, even their weaker conjecture, "Every tree different from K_2 is VAE", is still open. Our results here are just one more step towards proving the weaker Hartsfield and Ringel conjecture. Furthermore, we modify the construction presented in [6] to obtain a VAE labeling of a unicyclic graph with a given degree sequence pertaining to a unicyclic graph.

2. Results

We here recall the definition of a Ferrers diagram [5]. Let r be a positive integer. An n-part partition of r is represented by a sequence of positive integers $d = (d_1, d_2, \ldots, d_n)$, where $r = d_1 + d_2 + \cdots + d_n$ and $d_1 \geq d_2 \geq \cdots \geq d_n > 0$. The corresponding Ferrers diagram, $F(d)$, consists of r boxes, arranged in n left-justified rows. The number of boxes in row i of $F(d)$ is d_i, for $1 \leq i \leq n$. The Ferrers diagram determines a graph with an edge labelling. The ith row of $F(d)$ corresponds to vertex v_i. Vertices v_i and v_j are joined by edge with label k if k appears in both row i and row j of $F(d)$. As the ith row of $F(d)$ contains d_i numbers, vertex v_i is incident with d_i edges and so has degree d_i.

The following theorem has been proved in [6]. However, we recall here the construction from [6] as well as provide a new construction for VAE labelings of trees. We prove in Theorem 2.6 that these two constructions may provide two different VAE trees corresponding to the given degree sequence when there are at least seven vertices.

Theorem 2.1. *For every degree sequence $d = (d_1, d_2, \ldots, d_n)$ of a connected graph, where $n \geq 3$ and $\sum_{i=1}^{n} d_i = 2(n-1)$, there exists a vertex antimagic edge labeling of a tree corresponding to that degree sequence.*

Proof. Assume that $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. Then we can construct the Ferrers diagram $F(d)$. Let c_j, $1 \leq j \leq d_1$, be the number of boxes in column j. We know that $c_1 = n > c_2 \geq c_3 \geq \cdots \geq c_{d_1} \geq 1$.

Construction 1. [6]: We construct an edge labeling of a graph with the degree sequence d, where entries in each row of $F(d)$ are the labels of edges incident with the vertex of the graph corresponding to that row. We replace the boxes in the first column (from the top) with the labels $n-1, n-1, n-2, n-3, \dots, 2, 1$. There are now $n-2$ boxes left that need to be replaced by the labels $1, 2, \dots, n-2$. We replace the box in row i and column j of $F(d)$, for $1 \le i \le c_2$, $2 \le j \le d_i$, with $n-j$ when $i=1$ and $n - \sum_{k=1}^{i-1} d_k + (i-j-1)$ when $2 \le i \le c_2$. It is easy to check that no label occurs twice in the same row. Moreover, it is clear that the weight of each vertex (row), that is, the sum of all labels in that row, is greater than the weight of the vertex below.

Construction 2. This is similar to Construction 1, but we replace the boxes in $F(d)$ column by column. We do the same as in Construction 1 for the first column. There are now $n-2$ boxes left that need to be replaced by the labels $1, 2, \dots, n-2$. We replace the box in column j and row i of $F(d)$, for $2 \le j \le d_1$, $1 \le i \le c_j$, with $n-(i+1)$, when $j=2$; and $n - \sum_{k=2}^{j-1} c_k - (i+1)$, for $3 \le j \le d_1$. It is easy to check that no label occurs twice in the same row. Moreover, it is clear that the weight of each vertex (row), that is, the sum of all edge labels in that row, is greater than the weight of the vertex below.

It is easy to show that the graphs obtained from both constructions are connected. As they have n vertices and $n-1$ edges, they are both trees. Both constructions result in VAE trees with the correct degree sequence. $\square$

Consider the degree sequence $d = (4, 3, 3, 2, 1, 1, 1, 1, 1, 1)$. We first apply Construction 1 to get a VAE labeling of a tree corresponding to that degree sequence as shown in Figure 1.

9	8	7	6
9	5	4	
8	3	2	
7	1		
6			
5			
4			
3			
2			
1			

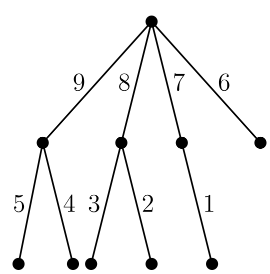

Figure 1: The tree with VAE labeling obtained from $d = (4, 3, 3, 2, 1, 1, 1, 1, 1, 1)$ by Construction 1

By applying Construction 2, we obtain a VAE labeling of a tree corresponding to that degree sequence as shown in Figure 2.

9	8	4	1
9	7	3	
8	6	2	
7	5		
6			
5			
4			
3			
2			
1			

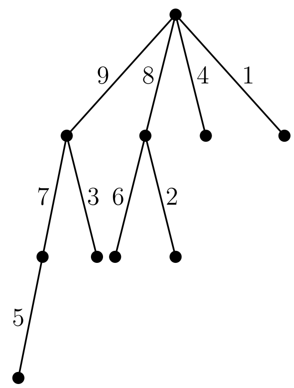

Figure 2: The tree with VAE labeling obtained from $d = (4, 3, 3, 2, 1, 1, 1, 1, 1, 1)$ by Construction 2

There is only one star (resp. path) with a given number vertices. Both stars and paths have been proved to be VAE in [4]. Hence we have

Observation 2.2. *For $c_2 = 1, n-2$, for $n \geq 3$, Constructions 1 and 2 provide the same vertex antimagic edge labeling for S_n and P_n, respectively.*

Lemma 2.3. *If $2 \leq c_2 \leq 3 < n - 2$, then Constructions 1 and 2 provide two different vertex antimagic edge labelings for the same tree.*

Proof. Let T_n and $T_n^{'}$ be the two VAE trees obtained from Constructions 1 and 2 corresponding to $d = (d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$, respectively. We consider two cases.

Case 1. $c_2 = 2$.

In this case, $d_1 \geq d_2 \geq 2$ and $d_i = 1$, for $3 \leq i \leq n$. There are two internal vertices for both T_n and $T_n^{'}$. The edge label $n-1$ is incident with the vertex of degree d_1 in T_n and $T_n^{'}$; and also incident with the vertices of degree d_2 in T_n and $T_n^{'}$. Hence T_n and $T_n^{'}$ are the same tree. It is simple to check that two VAE labelings are different.

Case 2. $c_2 = 3$.

In this case, $d_1 \geq d_2 \geq d_3 \geq 2$ and $d_i = 1$ for $4 \leq i \leq n$. Write v_i for the vertex which corresponds to the degree d_i (the i-th row of $F(d)$). Both T_n and $T_n^{'}$ have three internal vertices: v_1, v_2 and v_3 and in both trees the vertex v_1 is joined to v_2 by edge $n-1$ and to v_3 by edge $n-2$. The degrees of v_1, v_2 and v_3 are d_1, d_2 and d_3 respectively. Therefore $T_n = T_n^{'}$ as trees.

The conditions on the degree sequence force $d_1 > 2$. In T_n the vertex v_1 is joined to a leaf by edge $n-3$ and in $T_n^{'}$ the vertex v_2 is joined to a leaf by $n-3$. Therefore the trees have different labelings. $\square$

Lemma 2.4. *Let $4 \leq c_2 \leq n-3$ and T_n be a VAE tree obtained by Construction 1. Then $diam(T_n) \leq c_2$.*

Proof. As in the proof of Lemma 2.3, the vertex v_i stands for the i-th row of $F(d)$. The conditions on the degree sequence force $d_1 > 2$. There are c_2 internal vertices. By Construction 1, vertex v_1 has degree $d_1 > 2$ and is adjacent to at least v_2, v_3 and v_4. Hence a path in T_n cannot contain all the internal vertices, it can contain at most $c_2 - 1$ internal vertices and so have length at most c_2. Therefore, $diam(T_n) \leq c_2$. □

Lemma 2.5. *Let $4 \leq c_2 \leq n-3$ and $T_n^{'}$ be a VAE tree obtained by Construction 2. Then $diam(T_n^{'}) = c_2 + 1$.*

Proof. As in the proof of Lemma 2.3, the vertex v_i stands for the i-th row of $F(d)$. By Construction 2, there are c_2 internal vertices: $v_1, v_2, \ldots, v_{c_2}$ and $n - c_2$ leaves. The vertex v_1 is joined exactly with two internal vertices v_2 and v_3 by the edge labels $n-1$ and $n-2$, respectively; the vertex v_2 is joined with the vertex v_4 by the edge label $n-3$ and the vertex v_3 is joined with the vertex v_5 by the edge label $n-4$, and so on. In general, the vertex v_i is joined with the vertex v_{i+2}, for $2 \leq i \leq c_2 - 2$, by the edge label $n-(i+1)$. Hence there is a path in $T_n^{'}$ containing all internal vertices. Therefore, $diam(T_n^{'}) = c_2 + 1$. □

Combining Lemmas 2.4 and 2.5, the following theorem is proved.

Theorem 2.6. *Let T_n and $T_n^{'}$ be the VAE trees obtained by Constructions 1 and 2, respectively. If $4 \leq c_2 \leq n-3$, then Constructions 1 and 2 give VAE labelings for two different trees with the same degree sequence.*

A unicyclic graph is a connected graph $G = (V, E)$ that contains exactly one cycle. Alternatively, it is a connected graph $G = (V, E)$ such that $|V| = |E|$. The cycle C_n, $n \geq 3$, is a special case of unicyclic graphs and it has been proved to be VAE in [4, 7]. We modify Construction 1 to prove the following theorem.

Theorem 2.7. *For every degree sequence $d = (d_1, d_2, \ldots, d_n)$, where $d_i \geq 2$, for some $i \geq 3$ and $\sum_{i=1}^{n} d_i = 2n$, there exists a vertex antimagic edge labeling of a unicyclic graph corresponding to that degree sequence.*

Proof. Assume that $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. Then we can construct the Ferrers diagram $F(d)$. We construct an edge labeling of G satisfying that degree sequence, where entries in each row of $F(d)$ are the labels of edges incident with the vertex of G corresponding to that row. There are $2n$ boxes in $F(d)$ to be replaced by the integers from $\{1, 2, \ldots, n\}$. We first replace the boxes in the first column (from the top) with the labels $n, n, n-1, \ldots, 3, 2$. There are now n boxes left that need to be replaced by the labels $1, 2, \ldots, n-1$ (label 1 must appear twice). We replace the box in row i and column j of $F(d)$, for $2 \leq j \leq d_i$, with the label $n-(j-1)$ when $i = 1$; and when $2 \le i \le n$, with the label $n - \sum_{k=1}^{i-1} d_k + (i-j)$; if the label is 0, replace it by 1. That is, the last two boxes are replaced by the label 1. Note that the resulting array is not always the array of an edge labeling. We consider two cases.

Case 1. If there is some $d_i = 2$, for some $i \ge 3$, then the two occurrences of the label 1 must occur in two different rows and no pair of distinct labels occurs in two different rows. So we have an edge labeling.

Case 2. If there is no $d_i = 2$, for $i \ge 3$, then two occurrences of the label 1 must occur in the same row, say row i, where i is the last row in which d_i is the minimal integer greater than 2. We swap one occurrence of the label 1 with the label $d_i - 1$ in the first column of $F(d)$ and then reorder the labels in that column in descending order. We now have an edge labeling of G corresponding to that degree sequence.

In both cases it is clear that the weight of each vertex (row), that is, the sum of the all labels in that row, is greater than the weight of the vertex (row) below. It is easy to show that the graph obtained is connected. As G has n vertices and n edges, it is a unicyclic graph. $\square$

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