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On the total irregularity strength of some Cartesian product graphs

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Abstract

Let $G = (V, E)$ be a graph. A total labeling $f : V \cup E \to \{1, 2, \cdots, k\}$ is called totally irregular total k-labeling of G if every two distinct vertices x and y in V satisfies $wt(x) \neq wt(y)$, and every two distinct edges x_1x_2 and y_1y_2 in E satisfies $wt(x_1x_2) \neq wt(y_1y_2)$, where $wt(x) = f(x) + \sum_{xz \in E(G)} f(xz)$ and $wt(x_1x_2) = f(x_1) + f(x_1x_2) + f(x_2)$. The minimum k for which a graph G has a totally irregular total k-labeling is called the total irregularity strength of G, denoted by $ts(G)$. The Cartesian product $G \square H$ of graphs G and H is a graph such that the vertex set of $G \square H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (u, u') and (v, v') are adjacent in $G \square H$ if and only if either $u = v$ and u' is adjacent with v' in H, or $u' = v'$ and u is adjacent with v in G. The join graph $G + H$ of two graphs G and H is their graph union with all the edges that connect the vertices of G with the vertices of H. In this paper, we consider the total irregularity strength of some Cartesian product graphs, which are $S_n \square P_2$, $(P_n + P_1) \square P_2$, $P_n \square P_2$, and $C_n \square P_2$, where P_n is a path of order n, C_n is a cycle of order n, and S_n is a star of order $n + 1$.

Keywords: Cartesian product graph, cycle, join graph, path, star, total irregularity strength, totally irregular total k-labeling.

2000 Mathematics Subject Classification: 05C78.

1. Introduction

The vertex irregular total k-labeling and the edge irregular total k-labeling were introduced by Bača et al. [1]. A total labeling $f : V \cup E \to \{1, 2, \cdots, k\}$ is called a vertex irregular total k-labeling of G if every two distinct vertices x and y in V satisfies $wt(x) \neq wt(y)$, where $wt(x) = f(x) + \sum_{xz \in E(G)} f(xz)$. Some results about the vertex irregular total k-labeling were given by Nurdin et al. [4], [5] and [8]. A total labeling $f : V \cup E \rightarrow \{1, 2, \cdots, k\}$ is called an edge irregular total k-labeling of G if every two distinct edges x_1x_2 and y_1y_2 in E satisfies $wt(x_1x_2) \neq wt(y_1y_2)$, where $wt(x_1x_2) = f(x_1) + f(x_1x_2) + f(x_2)$. Some results about the edge irregular total k-labeling were given by Nurdin et al. [7] and Jendroľ et al. [2]. Combining both of these notions, Marzuki et al. [3] introduced a new irregular total k-labeling of a graph G called *totally irregular total k-labeling*, denoted by $ts(G)$, which is required to be at the same time both vertex and edge irregular. In the paper, they have given an upper bound and a lower bound of $ts(G)$. Besides that, they determined the total irregularity strength of cycles and paths.

In [1] several bounds and exact values of $tvs(G)$ were established for different types of graphs. In particular, the authors proved that for any graph $G(V, E)$ the following bounds hold.

$$\left\lceil \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right\rceil \leq tvs(G) \leq |V(G)| + \Delta(G) - 2\delta(G) + 1. \tag{1}$$

In the same paper, Bača et al. [1] derived lower and upper bounds of the total edge-irregular strength of any graph $G(V, E)$ as follows.

$$\left\lceil \frac{|E(G)| + 2}{3} \right\rceil \leq tes(G) \leq |E(G)|. \tag{2}$$

Marzuki et al. [3] has given a lower bound of $ts(G)$ as follows.

$$\text{For every graph } G,\ ts(G) \geq max\{tes(G); tvs(G)\}. \tag{3}$$

2. Main Results

2.1 The Total Irregularity Strength of $S_n \square P_2$

The Cartesian product graph $S_n \square P_2$ is a graph with the vertex set $V(S_n \square P_2) = \{u, u_1, u_2, \ldots, u_n, v, v_1, v_2, \ldots, v_n\}$ and the edge set $E(S_n \square P_2) = \{uu_i, vv_i, u_iv_i | i = 1, 2, \ldots, n\} \cup \{uv\}$.

Theorem 2.1. *Let $n \geq 3$, then $ts(S_n \square P_2) = n + 1$.*

Proof. $S_n \square P_2$ has $2n + 2$ vertices and $3n + 1$ edges. The minimum degree of $S_n \square P_2$ is $\delta(S_n \square P_2) = 2$ and the maximum degree of $S_n \square P_2$ is $\Delta(S_n \square P_2) = n + 1$. From (1) and (2), we get $tvs(S_n \square P_2) \geq \left\lceil \frac{2n+2+2}{n+2} \right\rceil$ and $tes(S_n \square P_2) \geq \left\lceil \frac{3n+3}{3} \right\rceil = n + 1$. Therefore, from (3), we get $ts(S_n \square P_2) \geq n + 1$.

Next, we will show that $ts(S_n \square P_2) \leq n + 1$.

Define a total labeling of $S_n \square P_2$ as follows:

$$f(u_i) = f(u_i v_i) = 1, \text{ for } 1 \le i \le n;$$
$$f(v_i) = f(uu_i) = i, \text{ for } 1 \le i \le n;$$
$$f(u) = f(v) = f(uv) = f(vv_i) = n+1, \text{ for } 1 \le i \le n.$$

We can see that f is a labeling from $V(S_n \square P_2) \cup E(S_n \square P_2)$ into $\{1, 2, \cdots, n+1\}$. It is easy to check that:

$$wt(u_i v_i) = 2 + i, \text{ for } 1 \le i \le n;$$
$$wt(uu_i) = n + 2 + i, \text{ for } 1 \le i \le n;$$
$$wt(vv_i) = 2n + 2 + i, \text{ for } 1 \le i \le n;$$
$$wt(uv) = 3n + 3;$$
$$wt(u_i) = 2 + i, \text{ for } 1 \le i \le n;$$
$$wt(v_i) = n + 2 + i, \text{ for } 1 \le i \le n;$$
$$wt(u) = \frac{(n+4)(n+1)}{2};$$
$$wt(v) = (n+1)(n+2).$$

Hence, there are no two vertices of the same weight and there are no two edges of the same weight. So, f is a totally irregular total $(n+1)$-labeling. We conclude that $ts(S_n \square P_2) = n + 1$. □

For ilustration, we give a totally irregular total 5-labeling for $S_4 \square P_2$ in Figure 1(a). The weights of all vertices and the weights of all edges under the totally irregular total 5-labeling are given in Figure 1(b).

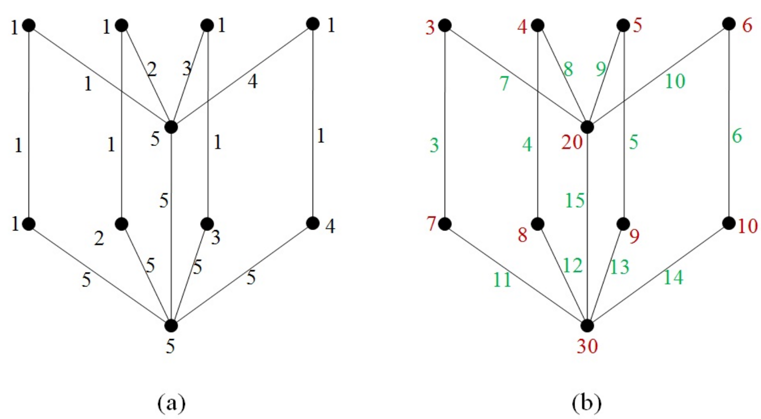


Figure 1: (a) A totally irregular total 5-labeling for $S_4 \times P_2$ (b) The weights of vertices and edges under the labeling

2.2 The Total Irregularity Strength of $(P_n + P_1)\square P_2$

The Cartesian product graph $(P_n + P_1)\square P_2$ is a graph with the vertex set $V((P_n + P_1)\square P_2) = \{u, u_1, u_2, \ldots, u_n, v, v_1, v_2, \ldots, v_n\}$ and the edge set $E((P_n + P_1)\square P_2) = \{uu_i, vv_i, u_iv_i | i = 1, 2, \ldots n\} \cup \{u_iu_{i+1}, v_iv_{i+1} | i = 1, 2, \ldots n-1\} \cup \{uv\}$.

Theorem 2.2. *Let $n \geq 3$, then $ts((P_n + P_1)\square P_2) = \lceil \frac{5n+1}{3} \rceil$.*

Proof. $(P_n + P_1)\square P_2$ has $2n + 2$ vertices and $5n - 1$ edges. The minimum degree of $(P_n + P_1)\square P_2$ is $\delta((P_n + P_1)\square P_2) = 3$ and the maximum degree of $(P_n + P_1)\square P_2$ is $\Delta((P_n + P_1)\square P_2) = n + 1$. From (1) and (2), we get $tvs((P_n + P_1)\square P_2) \geq \left\lceil \frac{2n+5}{n+2} \right\rceil$ and $tes((P_n + P_1)\square P_2) \geq \lceil \frac{5n+1}{3} \rceil$. Therefore, from (3), we get $ts((P_n + P_1)\square P_2) \geq \lceil \frac{5n+1}{3} \rceil$.

Next, we will show that $ts((P_n + P_1)\square P_2) \leq \lceil \frac{5n+1}{3} \rceil$.

Define $r = 2n + 2 - \lceil \frac{5n+1}{3} \rceil$.

Define a total labeling of $(P_n + P_1)\square P_2$ as follows:

$$\begin{aligned} f(u) &= f(uv) = f(uu_i) = 1; \\ f(u_i) &= i, \text{ for } 1 \leq i \leq n; \\ f(v) &= n + 1; \\ f(v_i) &= \left\lceil \frac{5n+1}{3} \right\rceil, \text{ for } 1 \leq i \leq n; \\ f(u_iv_i) &= r, \text{ for } 1 \leq i \leq n; \\ f(vv_i) &= r - 1 + i, \text{ for } 1 \leq i \leq n; \\ f(u_iu_{i+1}) &= n + 2 - i, \text{ for } 1 \leq i \leq n - 1; \\ f(v_iv_{i+1}) &= 2r + n - 2 - i, \text{ for } 1 \leq i \leq n - 1. \end{aligned}$$

We can see that f is a labeling from $V((P_n + P_1)\square P_2)) \cup E((P_n + P_1)\square P_2)$ into $\{1, 2, \cdots, \lceil \frac{5n+1}{3} \rceil\}$. It is easy to check that:

$$\begin{aligned} wt(uu_i) &= 2 + i, \text{ for } 1 \leq i \leq n; \\ wt(uv) &= n + 3; \\ wt(u_iu_{i+1}) &= n + 3 + i, \text{ for } 1 \leq i \leq n - 1; \\ wt(u_iv_i) &= 2n + 2 + i, \text{ for } 1 \leq i \leq n; \\ wt(vv_i) &= 3n + 2 + i, \text{ for } 1 \leq i \leq n; \\ f(v_iv_{i+1}) &= 5n + 2 - i, \text{ for } 1 \leq i \leq n - 1; \\ wt(u) &= n + 2; \\ wt(v) &= n(r + 1) + \frac{(n-1)(n)}{2} + 2; \end{aligned}$$

$$wt(u_i) = \begin{cases} n + r + 3, & \text{for } i = 1; \\ n + r + 4, & \text{for } i = n; \\ 2n + r + 6 - i, & \text{for } 2 \le i \le n - 1; \end{cases}$$

$$wt(v_i) = \begin{cases} 4r + n + \lceil \frac{5n+1}{3} \rceil - 3, & \text{for } i = 1; \\ 4r + n + \lceil \frac{5n+1}{3} \rceil - 2, & \text{for } i = n; \\ 6r + 2n + \lceil \frac{5n+1}{3} \rceil - 4 - i, & \text{for } 2 \le i \le n - 1. \end{cases}$$

Hence, there are no two vertices of the same weight and there are no two edges of the same weight. So, f is a totally irregular total $\lceil \frac{5n+1}{3} \rceil$-labeling. We conclude that $ts((P_n + P_1)\Box P_2) = \lceil \frac{5n+1}{3} \rceil$. □

For ilustration, we give a totally irregular total 9-labeling for $(P_5 + P_1)\Box P_2$ in Figure 2(a). The weights of all vertices and the weights of all edges under the totally irregular total 9-labeling are given in Figure 2(b).

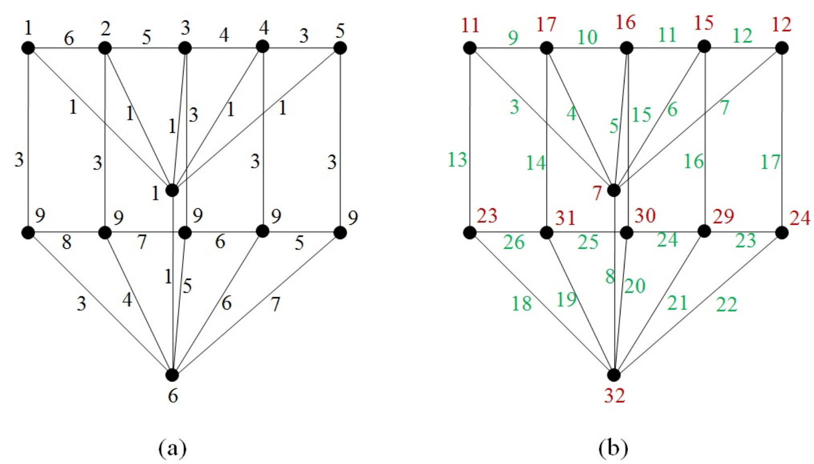


Figure 2: (a) A totally irregular total 9-labeling for $(P_5 + P_1)\Box P_2$ (b) The weights of vertices and edges under the labeling

2.3 The Total Irregularity Strength of $P_n \Box P_2$

The Cartesian product graph $P_n \Box P_2$ is a graph with the vertex set $V(P_n \Box P_2) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and the edge set $E(P_n \Box P_2) = \{u_i v_i | i = 1, 2, \ldots n\} \cup \{u_i u_{i+1}, v_i v_{i+1} | i = 1, 2, \ldots n - 1\}$.

Theorem 2.3. *Let $n \ge 3$, then $ts(P_n \Box P_2) = n$.*

Proof. $P_n \Box P_2$ has $2n$ vertices and $3n - 2$ edges. The minimum degree of $P_n \Box P_2$ is $\delta(P_n \Box P_2) = 2$ and the maximum degree of $P_n \Box P_2$ is $\Delta(P_n \Box P_2) = 3$. From (1) and

(2), we get $tvs(P_n\Box P_2) \geq \left\lceil \frac{2n+2}{4} \right\rceil$ and $tes(P_n\Box P_2) \geq \left\lceil \frac{3n}{3} \right\rceil = n$. Therefore, from (3), we get $ts(P_n\Box P_2) \geq n$.

Next, we will show that $ts(P_n\Box P_2) \leq n$.

Define a total labeling of $P_n\Box P_2$ as follows:

$$f(u_i) = \begin{cases} 2i-1, \text{ for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil; \\ 2(n+1-i), \text{ for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n-1; \\ 1, \text{ for } i = n; \end{cases}$$

$$f(v_i) = \begin{cases} 2i-1, \text{ for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil; \\ 2(n+1-i), \text{ for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n; \end{cases}$$

$$f(u_iu_{i+1}) = \begin{cases} 2i-1, \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor; \\ 2(n-i), \text{ for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n-2; \\ 3, \text{ for } i = n-1; \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 2i, \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor; \\ 2(n-i)+1, \text{ for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n-1; \end{cases}$$

$$f(u_iv_i) = \begin{cases} 1, \text{ for } i \in \{1, n\}; \\ 2i-3, \text{ for } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor; \\ n, \text{ for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1; \\ 2(n-1), \text{ for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n-1. \end{cases}$$

We can see that f is a labeling from $V(P_2 \times P_n) \cup E(P_2 \times P_n)$ into $\{1, 2, \cdots, n\}$. It is easy to check that:

$$wt(u_iu_{i+1}) = \begin{cases} 6i-1, \text{ for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1; \\ 6i-5, \text{ for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and odd } n; \\ 6i-2, \text{ for } i = \frac{n}{2} \text{ and even } n; \\ 6(n-i)+2, \text{ for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n-1; \end{cases}$$

$$wt(v_iv_{i+1}) = \begin{cases} 6i, \text{ for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1; \\ 6i-4, \text{ for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and odd } n; \\ 6i-1, \text{ for } i = \frac{n}{2} \text{ and even } n; \\ 6(n-i)+3, \text{ for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n-1; \end{cases}$$

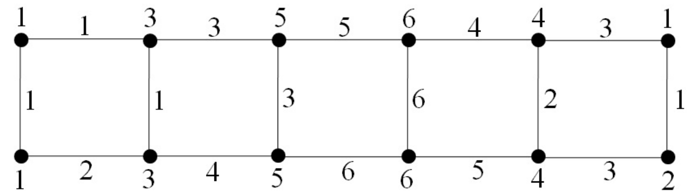
$$wt(u_iv_i) = \begin{cases} 3, \text{ for } i = 1; \\ 6i-5, \text{ for } 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor; \\ 6i-3, \text{ for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and odd } n; \\ 6i-6, \text{ for } i = \frac{n}{2} + 1 \text{ and even } n; \\ 6(n-i)+4, \text{ for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n; \end{cases}$$

$$wt(u_i) = \begin{cases} 3, \text{ for } i = 1; \\ 8(i-1), \text{ for } 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor; \\ 10, \text{ for } i = 2 \text{ and } n = 3; \\ 14, \text{ for } i = 3 \text{ and } n = 4; \\ 8i-7, \text{ for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and odd } n \ge 5; \\ 8i-11, \text{ for } i = \frac{n}{2} + 1 \text{ and even } n \ge 6; \\ 8(n-i)+4, \text{ for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n-2; \\ 5, \text{ for } (i = 3 \text{ and } n = 3) \text{ or } (i = 4 \text{ and } n = 4); \\ 8(n-i)+5, \text{ for } n-1 \le i \le n \text{ and } n \ge 5; \end{cases}$$

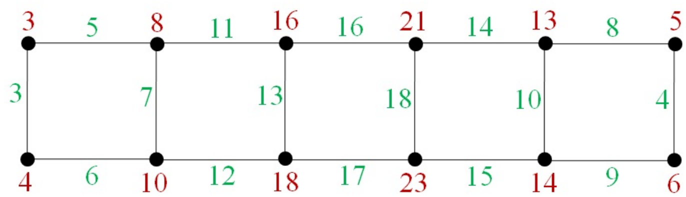
$$wt(v_i) = \begin{cases} 4, \text{ for } i = 1; \\ 8i-6, \text{ for } 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor; \\ 8i-5, \text{ for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and odd } n; \\ 8i-9, \text{ for } i = \frac{n}{2} + 1 \text{ and even } n; \\ 8(n-i)+6, \text{ for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n. \end{cases}$$

Hence, there are no two vertices of the same weight and there are no two edges of the same weight. So, f is a totally irregular total n-labeling. We conclude that $ts(P_2 \Box P_n) = n$. □

For ilustration, we give a totally irregular total 6-labeling for $P_6 \Box P_2$ in Figure 3(a). The weights of all vertices and the weights of all edges under the totally irregular total 6-labeling are given in Figure 3(b).



(a)



(b)

Figure 3: (a) A totally irregular total 6-labeling for $P_6 \times P_2$ (b) The weights of vertices and edges under the labeling

2.4 The Total Irregularity Strength of $C_n \square P_2$

The Cartesian product graph $C_n \square P_2$ is a graph with the vertex set $V(C_n \square P_2) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and the edge set $E(C_n \square P_2) = \{u_i v_i, u_i u_{i+1}, v_i v_{i+1} | i = 1, 2, \ldots n\}$, where $u_{n+1} = u_1$ and $v_{n+1} = v_1$.

Theorem 2.4. *Let $n \geq 3$, then $ts(C_n \square P_2) = n + 1$.*

Proof. $C_n \square P_2$ has $2n$ vertices and $3n$ edges. The minimum and the maximum degree of $C_n \square P_2$ is $\delta(C_n \square P_2) = \Delta(C_n \square P_2) = 3$. From (1) and (2), we get $tvs(C_n \square P_2) \geq \lceil \frac{2n+3}{4} \rceil$ and $tes(C_n \square P_2) \geq \lceil \frac{3n+2}{3} \rceil = n + 1$. Therefore, from (3), we get $ts(C_n \square P_2) \geq n + 1$.

Next, we will show that $ts(C_n \square P_2) \leq n + 1$.

Define a total labeling of $C_n \square P_2$ as follows:

$$f(u_i) = \begin{cases} i, \text{ for } 1 \leq i \leq \lceil \frac{n}{2} \rceil; \\ 2 \lceil \frac{n-i}{2} \rceil + 1, \text{ for } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n; \end{cases}$$

$$f(v_i) = n + 1, \text{ for } 1 \leq i \leq n;$$

$$f(u_i u_{i+1}) = \begin{cases} 1, \text{ for } i = \lceil \frac{n}{2} \rceil \text{ and } (n \equiv 0 \mod 4 \text{ or } n \equiv 1 \mod 4); \\ 2, \text{ for } i = \lceil \frac{n}{2} \rceil \text{ and } (n \equiv 2 \mod 4 \text{ or } n \equiv 3 \mod 4); \\ 1, \text{ for others } i; \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 2i, \text{ for } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor; \\ 2(n-i)+1, \text{ for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n; \end{cases}$$

$$f(u_iv_i) = \begin{cases} i, \text{ for } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil; \\ 2\left\lceil \frac{n-i+1}{2} \right\rceil, \text{ for } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n. \end{cases}$$

We can see that f is a labeling from $V(C_2 \times P_n) \cup E(C_2 \times P_n)$ into $\{1, 2, \cdots, n+1\}$. It is easy to check that:

$$wt(u_iu_{i+1}) = \begin{cases} 2i+2, \text{ for } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor; \\ 2(n-i)+3, \text{ for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n; \end{cases}$$

$$wt(v_iv_{i+1}) = \begin{cases} 2i+2n+2, \text{ for } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor; \\ 2(n-i)+2n+3, \text{ for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n; \end{cases}$$

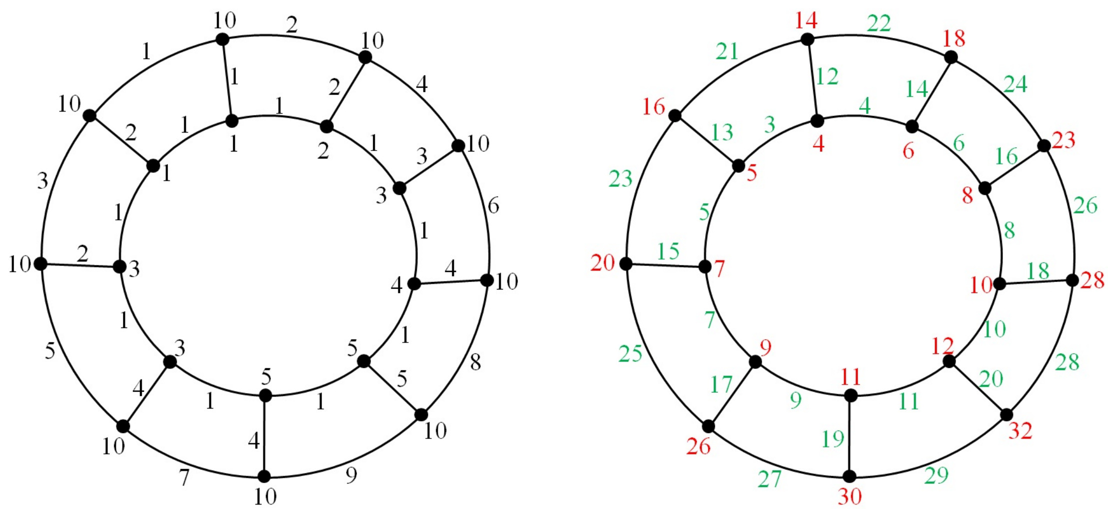
$$wt(u_iv_i) = \begin{cases} 2i+n+1, \text{ for } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil; \\ 2(n-i)+n+4, \text{ for } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n; \end{cases}$$

$$wt(u_i) = \begin{cases} 2i+2, \text{ for } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1; \\ 2\left\lceil \frac{n}{2} \right\rceil + 2, \text{ for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } (n \equiv 0 \mod 4 \text{ or } n \equiv 1 \mod 4); \\ 2\left\lceil \frac{n}{2} \right\rceil + 3, \text{ for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } (n \equiv 2 \mod 4 \text{ or } n \equiv 3 \mod 4); \\ 2\left\lfloor \frac{n}{2} \right\rfloor + 3, \text{ for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } (n \equiv 0 \mod 4 \text{ or } n \equiv 1 \mod 4); \\ 2\left\lfloor \frac{n}{2} \right\rfloor + 4, \text{ for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } (n \equiv 2 \mod 4 \text{ or } n \equiv 3 \mod 4); \\ 2(n-i)+5, \text{ for } \left\lceil \frac{n}{2} \right\rceil + 2 \le i \le n; \end{cases}$$

$$wt(v_i) = \begin{cases} n+5, \text{ for } i = 1; \\ n+5i-1, \text{ for } 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor; \\ n+5\left\lfloor \frac{n}{2} \right\rfloor + 3, \text{ for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and odd } n; \\ n+5\frac{n}{2}, \text{ for } i = \frac{n}{2} + 1 \text{ and } n \equiv 0 \mod 4; \\ n+5\frac{n}{2}+1, \text{ for } i = \frac{n}{2} + 1 \text{ and } n \equiv 2 \mod 4; \\ n+5(n-i)+7, \text{ for odd } i, \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n \text{ and odd } n; \\ n+5(n-i)+6, \text{ for odd } i, \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n \text{ and even } n; \\ n+5(n-i)+6, \text{ for even } i, \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n \text{ and odd } n; \\ n+5(n-i)+7, \text{ for even } i, \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n \text{ and even } n. \end{cases}$$

Hence, there are no two vertices of the same weight and there are no two edges of the same weight. So, f is a totally irregular total $(n+1)$-labeling. We conclude that $ts(C_2 \square P_n) = n+1$. □

For ilustration, we give a totally irregular total 10-labeling for $C_9 \square P_2$ in Figure 4(a). The weights of all vertices and the weights of all edges under the totally irregular total 10-labeling are given in Figure 4(b).



(a) (b)

Figure 4: (a) A totally irregular total 10-labeling for $C_9 \times P_2$ (b) The weights of vertices and edges under the labeling

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