AKCE Int. J. Graphs Comb., 10, No. 3 (2013), pp. 245-252

On the locating-chromatic number of homogeneous lobsters

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Abstract

Let $G = (V, E)$ be a connected graph. Let $\Pi = \{C_1, C_2, \ldots, C_k\}$ be the partition of $V(G)$ induced by a k-coloring c on V. The color code $c_\Pi(v)$ of a vertex v in G is defined as $(d(v, C_1), d(v, C_2), \ldots, d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x) | x \in C_i\}$ for $1 \leq i \leq k$. If all distinct vertices of G have distinct color codes, then c is called a locating k-coloring of G. The locating-chromatic number of G, denoted by $\chi_L(G)$, is the least integer k such that G has a locating k-coloring. In this paper, we determine the locating-chromatic number of a lobster, namely a tree with the property that the removal of the endpoints results a caterpillar.

Keywords: Locating-chromatic number, color code, lobster.

2000 Mathematics Subject Classification: 05C12.

1. Introduction

Let $G = (V, E)$ be a connected graph. The locating-chromatic number of a graph G is defined by Chartrand et al. on 2002 as follows. Let c be a k-*coloring* on G. Let $\Pi = \{C_1, C_2, \ldots, C_k\}$ be the partition of V induced by c. The *color code* $c_\Pi(v)$ of a vertex v in G is defined as $(d(v, C_1), d(v, C_2), \ldots, d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x) | x \in C_i\}$ for $1 \leq i \leq k$. If all distinct vertices of G have distinct color codes, then c is called a *locating* k-*coloring* of G. The least integer k such that G has a locating k-coloring is called the *locating-chromatic number* of G and it is denoted by $\chi_L(G)$.

In 2002, Chartrand et al. [6] determined the locating-chromatic numbers for some well-known classes of trees, namely paths and double stars. Furthermore, in 2003 Chartrand et al. [5] showed that for any $k \in \{3, 4, \ldots, n-2, n\}$, there exists a tree on n vertices with locating-chromatic number k. They also showed that there is no tree on n vertices with locating-chromatic number $n-1$. Recently, the locating-chromatic numbers of classes of trees, namely an amalgamation of stars and a firecracker graph, were determined by Asmiati et al. [1, 2]. Meanwhile, the locating-chromatic number of other classes of graphs are known, *i.e.* kneser graphs [3], join of graphs [4], and halin graphs [7]. In particular, for trees, the locating-chromatic number of a general tree is still not completely solved. Therefore, in this paper, we determine the locating-chromatic number of homogeneous lobsters.

A *lobster* is a tree with the property that the removal of endpoints results a caterpillar. Throughout this paper, for $m \geq 1$ and $n \geq 2$, we denote by $Lb(m,n)$ a *homogeneous lobster*, namely a graph with the vertex set $V = V_1 \cup V_2 \cup V_3$, where

$$V_1 = \{x_i | i \in [1,m]\},$$
$$V_2 = \{y_{ij} | i \in [1,m], j \in [1,n]\},$$
$$V_3 = \{z_{ijk} | i \in [1,m], j,k \in [1,n]\},$$

and the edge set $E = E_1 \cup E_2 \cup E_3$, where

$$E_1 = \{x_i x_{i+1} | i \in [1,m-1]\},$$
$$E_2 = \{x_i, y_{ij} | 1 \in [1,m], j \in [1,n]\},$$
$$E_3 = \{y_{ij}, z_{ijk} | i \in [1,m], j,k \in [1,n]\}.$$

Note that $[1,n]$ is the set of all integers from 1 to n. Figure 1 illustrates a homogeneous lobster $Lb(m,n)$.

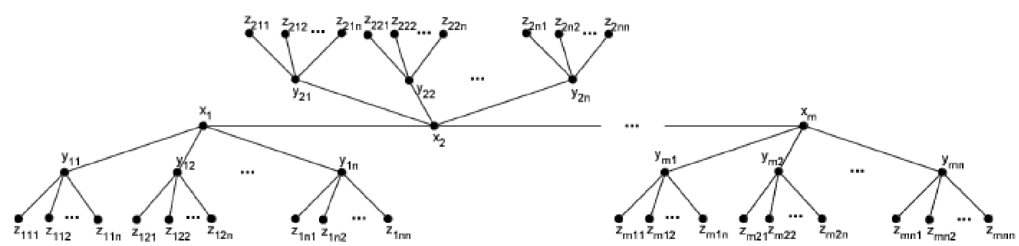

Figure 1: A homogeneous lobster $Lb(m,n)$.

The following results were proved by Chartrand et al. [6].

Theorem 1.1. *Let c be a locating-coloring in a connected graph G. If u and v are distinct vertices of G such that $d(u,w) = d(v,w)$ for all $w \in V(G) - \{u,v\}$, then $c(u) \neq c(v)$. In particular, if u and v are nonadjacent vertices of G such that $N(u) = N(v)$, then $c(u) \neq c(v)$.*

Corollary 1.2. *If G is a connected graph containing a vertex adjacent to k endpoints of G, then $\chi_L(G) \geq k+1$.*

2. Main Results

We will use the following notations. The set of neighbours of a vertex v in G is denoted by $N(v)$. Let c be a k-coloring of G. Then, $c(H) = \{c(s)|s \in H\}$ for any $H \subset V(G)$. For any $i \in [1,m]$, $j \in [1,n]$, define $\triangle_i = \{x_i, y_{ij}|j \in [1,n]\}$ and $\triangle_{ij} = \{y_{ij}, z_{ijk}|k \in [1,n]\}$. For any integers i, j, r and a set $S \subset [1,k]$, define $c(\triangle_{ij}) = \triangle_S^r$ if the following conditions hold:

(i) $c(y_{ij}) = r$, and

(ii) $c(A) = S \backslash \{r\}$, where $A = \{z_{ijk}|1 \leq k \leq n\}$.

The locating-chromatic number of a homogeneous lobster $Lb(m,n)$ is as follows.

Theorem 2.1. $\chi_L(Lb(1,n)) = n+1$ *for* $n \geq 2$.

Proof. First, we show that $\chi_L(Lb(1,n)) \geq n+1$. Since $Lb(1,n)$ contains a vertex adjacent to n distinct endpoints, then by Corollary 1.2 $\chi_L(Lb(1,n)) \geq n+1$.

Next, we will show that $\chi_L(Lb(1,n)) \leq n+1$. Without loss of generality, define a $(n+1)$-coloring $c : V(Lb(1,n)) \to [1, n+1]$ such that

$$c(v) = \begin{cases} 1 & \text{if } v = x_1, \\ j+1 & \text{if } v = y_{1j}, \\ k & \text{if } v = z_{1jk} \text{ and } k \neq j+1, \\ n+1 & \text{if } v = z_{1jk} \text{ and } k = j+1, \end{cases}$$

for all $j, k \in [1,n]$. Then, we have:

- $c_\Pi(x_1) = (0, 1, 1, \ldots, 1)$,
- For $1 \leq j \leq n$, $c_\Pi(y_{1j}) = (1, 1, \ldots, 1, 0, 1, \ldots, 1)$, where 0 in the j^{th}-ordinate,
- For $1 \leq j, k \leq n$, $c_\Pi(z_{1jk}) = (2, 2, \ldots, 2, 0, 1, 2, \ldots, 2)$, where 0 and 1 are in the k^{th} and $(j+1)^{th}$-ordinate, respectively.

Therefore, the color codes of all vertices are distinct. This implies that $\chi_L(Lb(n,1)) \leq n+1$. □

Theorem 2.2. *If* $2 \leq m \leq 3(n+2)+1$ *and* $n \geq 2$, *then* $\chi_L(Lb(m,n)) = n+2$.

Proof. By Corollary 1.2, $\triangle_{ij}$ requires $n+1$ colors for each pair (i,j). If we use only $n+1$ colors in $Lb(m,n)$, then there are at most $n+1$ distinct vertices y_{ij} that can have distinct color codes. This is because each y_{ij} is adjacent to n endpoints. However, since $m \geq 2$, there are at least $2n$ vertices y_{ij} in $Lb(m,n)$. Therefore, to have distinct color codes for all vertices, it requires at least $n+2$ colors; and so $\chi_L(Lb(m,n)) \geq n+2$.

Table 1: The locating 4-coloring of $Lb(m,2)$ for $1 \leq m \leq 13$.

i	$c(x_i)$	$c(y_{i1})$	$c(y_{i2})$	$c(z_{i11})$	$c(z_{i12})$	$c(z_{i21})$	$c(z_{i22})$
1	3	2	4	1	3	2	3
2	2	3	4	2	4	2	3
3	4	2	2	1	4	1	3
4	2	4	1	2	3	2	4
5	4	1	2	2	4	1	4
6	1	4	4	2	3	1	3
7	4	1	3	2	4	1	4
8	1	3	4	1	4	1	3
9	3	1	1	2	4	2	3
10	1	2	3	1	3	1	4
11	3	1	2	2	3	1	3
12	2	3	3	1	4	2	4
13	3	1	2	3	4	3	4

Next, we shall show that $\chi_L(Lb(m,n)) \leq n+2$ for $n \geq 2$ and $2 \leq m \leq 3(n+2)+1$. For $n=2$ and $1 \leq m \leq 13$, define $c : V(G) \to [1,4]$ as in Table 1.

For $n > 2$, let $A = [1, n+2]$, and $A_i = A \backslash \{i\}$, $t = \lfloor \frac{i}{3} \rfloor$. We define a $(n+2)$-coloring $c : V(Lb(m,n)) \to [1, n+2]$ satisfying the following conditions.

1. For $i = 1 \bmod 3$, except for $i = 3(n+2)+1$, define
   - $c(x_i) = n+t+1$,
   - $c(\triangle_{i1}) = \begin{cases} \triangle^{n+t}_{A_{t+1}} & \text{if } n = 3, \\ \triangle^{n+t}_{A_{t+3}} & \text{if } n > 3, \end{cases}$
   - $c(\triangle_{i2}) = \triangle^{n+t}_{A_{n+t+2}}$,
   - $c(\triangle_{ij}) = \triangle^{j+t}_{A_{n+t+2}}$ for $j \in [3, n-1]$,
   - $c(\triangle_{in}) = \triangle^{n+t+2}_{A_{n+t}}$.
2. For $i = 2 \bmod 3$, define
   - $c(x_i) = n+t+2$,
   - $c(\triangle_{ij}) = \triangle^{j+t+1}_{A_{t+1}}$ for $j \in [1, n]$.
3. For $i = 3 \bmod 3$, define
   - $c(x_i) = n+t$,
   - $c(y_{ij}) = n+t+2$ for $j \in [1, n]$,

- $c(\triangle_{ij}) = \triangle_D^{c(y_{ij})}$, where $D \subset A$, such that $t+1 \in D$ and $\triangle_D^{c(y_{ij})} \setminus \triangle_D^{c(y_{iq})} \neq \emptyset$ for $j \neq q$, $j, q \in [1, n]$.

For $i = 3(n+2)+1$ use the following coloring:

1. $c(x_i) = c(x_{i-2})$,
2. $c(\triangle_{ij}) = \triangle^j_{A_{j+1}}$, for $j \in [1, n]$.

Note that all values of $c(a)$ are calculated on modulo $n+2$.

Now, we will show that c is a locating coloring on G. Let Π be the partition of $V(G)$ induced by the above coloring c. Let $u, v \in V(G)$ with $c(u) = c(v)$. We must show that $c_\Pi(u) \neq c_\Pi(v)$. If $c(N(u)) \neq c(N(v))$, then clearly $c_\Pi(u) \neq c_\Pi(v)$. If $c(N(u)) = c(N(v))$, then the following three cases should be considered.

1. If $u = x_i$ and $v = x_j$ for $i \neq j$, then by the Definition of c, $c(N(u)) = c(N(v))$ is only satisfied when $i = 3n+2$ and $j = 3(n+2)+1$. Hence, the color codes $c_\Pi(u) \neq c_\Pi(v)$ since they differ in the $(n+2)^{th}$-ordinate.
2. If $u = x_i$ and $v = y_{pq}$ for $i \neq p$, then by the Definition of c, $c(N(u)) = c(N(v))$ holds if one of the following conditions is satisfied.
   (i) $n = 2$, $i = 3t+1$ and $p = 3t+2$, for $t = 0, 1, 2, 3$. By Table 1, $c_\Pi(u) \neq c_\Pi(v)$ since they differ in the first ordinate for $t = 0$, the third ordinate for $t = 1$, the second ordinate for $t = 2$, and the fourth ordinate for $t = 4$.
   (ii) $n = 2$, $i = 3t+2$ and $p = 3(n+2)+1$, for some p and $t = 0, 2$. By Table 1, $c_\Pi(u) \neq c_\Pi(v)$ since for $t = 0$ they differ in the first ordinate and for $t = 2$ they differ in the second ordinate.
   (iii) $n > 2$, $i = 3t+2$ and $p = 3(n+2)+1$ for some q and $0 \leq t \leq n+1$. By the Definition of c, $c_\Pi(u) \neq c_\Pi(v)$ since they differ in the $(t+1)^{th}$-ordinate.
3. If $u = y_{ij}$ and $v = y_{pq}$ for some i, j, p, q with $i \neq p$, then by the Definition of c, the color codes of these two vertices differ in the w^{th}-ordinate where $w \notin c(N(u)) \cup \{c(u)\}$.

Thus, c is a locating $(n+2)$-coloring of $Lb(m, n)$. Hence $\chi_L(Lb(m, n)) \leq n+2$. □

Lemma 2.3. *If $\chi_L(Lb(m, n)) = n+2$, for $m, n \geq 2$ then every subgraph $Lb(2, n)$ of $Lb(m, n)$ must be colored by $n+2$ colors.*

Proof. It follows immediately from Theorem 2.2. □

Corollary 2.4. *If $\chi_L(Lb(m, n)) = n+2$ for $m, n \geq 2$ then the value of any ordinate in the color code of any vertex of $Lb(m, n)$ is an integer $\in [0, 5]$.*

Proof. This is a direct consequence of Lemma 2.3. □

Theorem 2.5. *If* $m > 3(n+2)+1$ *then* $\chi_L(Lb(m,n)) = n+3$.

Proof. First, we show that $\chi_L(Lb(m,n)) \geq n+3$. Suppose $\chi_L(Lb(m,n)) = n+2$. Let $A = [1, n+2]$ and for any i define $A_i = A \backslash \{i\}$. For any i, j, all vertices in $\triangle_{ij}$ must be colored by $n+1$ colors. Since $\chi_L(Lb(m,n)) = n+2$ and $|c(\triangle_{ij})| = n+1$, then there exists a vertex a in color class C_a such that $c(a) \notin c(\triangle_{ij})$. Since $\chi_L(Lb(m,n)) = n+2$, by Corollary 2.4 $d(y_{ij}, C_a) \leq 4$ for any i, j. Now, consider one vertex y_{ij} and fix a color on it. If $d(y_{ij}, C_a) = 1$ then there is exactly one possible color code for vertex y_{ij} . If $d(y_{ij}, C_a) = r$, where $r = 2, 3,$ or 4, then there are exactly $n+1$ possible color codes of vertex y_{ij}. Therefore, in total, there are $3(n+1)+1$ possible color codes of vertex y_{ij}. Since there are $n+2$ possibilities for the color of y_{ij}, then there are exactly $(n+2)(3(n+1)+1)$ possibilities of the color codes of all vertices y_{ij} . However, we will show that not all these possibilities can be used. To do that, consider a vertex y_{ij} with $c(y_{ij}) = u$ for some i, j and let a be a vertex such that $c(a) \notin c(\triangle_{ij})$, say $c(a) = w$. Then, we have the following two cases.

Case 1. The color code of y_{ij} contains 4 in the w^{th}-ordinate.

Then, there is no color code containing 3 in the w^{th}-ordinate for other vertices y_{pq}, where $c(y_{pq}) = c(x_i)$ and $p \neq i$. This case also implies that there is no color code containing 4 in the w^{th}-ordinate for other vertices y_{st}, where $c(y_{st}) = c(x_i)$ and $s \neq i \neq p$. Since there are $n+2$ possibilities for ordinate w , then there are $2(n+2)$ possibilities for this case. Hence, Case 1 removes $2(n+2)$ possible color codes for vertices y_{pq} , $p \neq i$.

Case 2. The color code of y_{ij} consists of (one 0 and the others are 1) or (one 0, one 2, and the others are 1).

If $m > 3(n+2)+1$, then it requires at least $n+3$ such color codes of backbone vertices x_i, that is one color code consists of one 0 and the others are 1 and $n+2$ color codes consists of one 0, one 2, and the others are 1. Otherwise, there are at least two backbone vertices with the same color codes. Hence, Case 3 removes $n+3$ possible color codes for vertices y_{pq} , $p \neq i$.

Based on these two above cases, there are at least $3n+7$ possibilities of color codes that cannot be used. Hence, there are at most $(n+2)(3(n+1)+1) - (3n+7) = 3n^2 + 7n + 1$ possible color codes for all vertices y_{ij} in $Lb(m,n)$. Since $Lb(m,n)$ have mn vertices y_{ij} and $3n^2 + 7n + 1 < mn$ for $m > 3(n+2)+1$, then there are at least two vertices having the same color code. Thus $\chi_L(Lb(m,n)) \geq n+3$.

Now, we show that $\chi_L(Lb(m,n)) \leq n+3$. Let $B = [1, n+3]$ and $B_i = B \backslash \{i\}$ for any i. Define $c : V(Lb(m,n)) \to \{1, 2, \ldots, n+3\}$ as follows:

- $c(x_i) = \begin{cases} 1 & \text{if } i \text{ is odd,} \\ 2 & \text{if } i \text{ is even.} \end{cases}$

- $c(\triangle_{1j}) = \triangle_D^2$, where $D \subset B_{n+2}$ such that $c(\triangle_{1j}) \setminus c(\triangle_{1q}) \neq \emptyset$ for every $j \neq q$, $j, q \in [1, n]$.
- if i is even, then $c(\triangle_{ij}^1) = \triangle_D^1$, where $D \subset B_{n+3}$ such that $c(\triangle_{ij}) \setminus c(\triangle_{iq}) \neq \emptyset$ for every $j \neq q$, $j, q \in [1, n]$.
- if i is odd, $i \geq 3$, then $c(\triangle_{ij}) = \triangle_D^2$, where $D \subset B_{n+3}$ such that $c(\triangle_{ij}) \setminus c(\triangle_{iq}) \neq \emptyset$ for every $j \neq q$, $j, q \in [1, n]$.

It is clear that there are pairs of vertices in $Lb(m, n)$ in which their color codes have the same value in the $(n+3)^{th}$-ordinate, namely:

1. x_i and y_{pq} for some $p = i - 1$ or $p = i + 1$, or
2. x_i and z_{pqr} where $p = i - 2$, or
3. y_{ij} and y_{pq} where $i = p$, or
4. z_{ijk} and z_{pqr} where $i = p$.

Now, we have to show that these vertices have distinct color codes by considering the following cases.

1. If $c(x_i) = c(y_{pq})$ for $p = i - 1$ or $p = i + 1$, then $c_\Pi(y_{pq})$ contains at least n entries with value 1 but $c_\Pi(x_i)$ contains exactly one entries with value 1. Thus, $c_\Pi(x_i) \neq c_\Pi(y_{pq})$.
2. If $c(x_i) = c(z_{pqr})$ where $p = i - 2$, then $c_\Pi(x_i)$ contains at least n entries with value 2 but $c_\Pi(z_{pqr})$ contains exactly $n-1$ entries with value 2. Thus, $c_\Pi(x_i) \neq c_\Pi(z_{pqr})$.
3. If $c(y_{ij}) = c(y_{pq})$, $i = p$ and $j \neq q$ then by Definition of c, $c(\triangle_{ij}) \setminus c(\triangle_{iq}) \neq \emptyset$. Then, there is some $a \in c(\triangle_{ij}^{c(x_i)})$, but $a \notin c(\triangle_{iq}^{c(x_i)})$. Hence $c_\Pi(y_{ij}) \neq c_\Pi(y_{pq})$.
4. If $c(z_{ijk}) = c(z_{pqr})$, $i = p$ and $j \neq q$ then by Case 3, $c_\Pi(z_{ijk}) \neq c_\Pi(z_{pqr})$.

Thus, c is a locating $(n+3)$-coloring of $Lb(m, n)$, and so $\chi_L(Lb(m, n)) \leq n + 3$. Therefore, $\chi_L(Lb(m, n)) = n + 3$. □

Ackowledgement

This research was supported by the Directorate General of Higher Education (DGHE), Ministry of Education and Culture, by Research Grant "International Research Collaboration and Scientific Publication" 082/SP2H/PL/Dit.Litabmas/V/2013.

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