

THE (Δ, D) AND (Δ, N) PROBLEMS FOR NEW AMSTERDAM AND MANHATTAN DIGRAPHS

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To Joan Gimbert, in memoriam.

Abstract

We give a quasi-complete solution of the (Δ, N) problem for two well-known families of digraphs used as good models for large interconnection networks. In our study we also relate both families, the New Amsterdam and Manhattan digraphs, with the double-step graphs (or circulant graphs with degree two).

Keywords: (Δ, D) problem, (Δ, N) problem, New Amsterdam digraph, Manhattan digraph.

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1. Introduction

In this paper we concentrate on two families of digraphs which are Cayley digraphs of the plane crystallographic groups. Namely, we consider the (Δ, D) and (Δ, N) problems for the so-called New Amsterdam and Manhattan digraphs. The first problem consists of maximizing the number N of vertices giving the (maximum degree) Δ and the diameter D , whereas the second one (somehow dual of the first) consists of minimizing the diameter for a fixed degree and number of vertices. Our study is based on known results about the same problems of another family, the double-step graphs (also called circulants or Cayley graphs of Abelian groups). In fact, the (Δ, D) problem was already solved by Morillo, Fiol and Fàbrega in [8] for the New Amsterdam digraphs with odd diameter, and also for the Manhattan digraphs with even diameter. Although a solution for the other diameters was also claimed in the same paper, the digraphs proposed were not vertex transitive and, as a consequence, the eccentricity from the odd vertices was not the correct one. Here we show that, for such values of the diameter, the number of vertices is much smaller than the theoretical upper (Moore-like) bounds. For a comprehensive survey on Moore graphs and the (Δ, D) problem, see Miller and Sirán [7].

2. Double-step graphs, New Amsterdam and Manhattan digraphs

In this section we define the different families of digraphs considered and recall the corresponding theoretical (Moore-like) upper bounds for their number of vertices.

2.1. Double-step graphs

A *double-step graph* $G(N; \pm a, \pm b)$ has set of vertices \mathbb{Z}_N (the integers modulo N) and every vertex i is adjacent to the vertices $i \pm a$ and $i \pm b$ (arithmetic is always mod N), for some different integers a, b called *steps* such that $\gcd(N, a, b) = 1$. For more details, see Yebra, Fiol, Morillo and Alegre [9]. Then, as is readily seen in that paper, the maximum number N_{DS} of vertices of a double-step graph with diameter k is upper bounded by the Moore-like bound

$$N_{DS} \leq M_{DS}(2, k) = 1 + \sum_{n=1}^k 4n = 2k^2 + 2k + 1. \tag{1}$$

2.2 New Amsterdam digraphs

Let N be an even integer and let $\alpha, \beta, \gamma, \delta$ be some odd integers ($\alpha \neq \beta$) as before called *steps* satisfying

$$\alpha + \beta + \gamma + \delta \equiv 0 \pmod{N}. \tag{2}$$

Then, a *New Amsterdam digraph* $NA(N; \alpha, \beta, \gamma, \delta)$ is a bipartite digraph with set of vertices $V = \mathbb{Z}_N$, $V = V_0 \cup V_1$, $V_0 = \{0, 2, \dots, N - 2\}$, $V_1 = \{1, 3, \dots, N - 1\}$, and where each vertex $i \in V_0$ is adjacent to the vertices $i + \alpha, i + \beta \in V_1$, and every vertex $j \in V_1$ is adjacent to the vertices $j + \gamma, j + \delta \in V_0$. See Figure 1 (left) for the plane local pattern followed by the vertices.

Since the digraph is regular and bipartite, if it has diameter k , its maximum number N_k of vertices is twice the number of vertices in V_0 when k is odd, or in V_1 if k is even, at distance at most $k - 1$ from vertex 0. This leads to the following Moore-like bounds (see Morillo, Fiol, and Fàbrega [8]):

$$N_{NA} \leq M_{NA}(2, k) = \begin{cases} 2 \left(1 + \sum_{m=1}^n 4m \right) = k^2 + 1 & \text{for odd } k = 2n + 1, & (3) \\ 2 \sum_{m=1}^n (4m - 2) = k^2 & \text{for even } k = 2n. & (4) \end{cases}$$

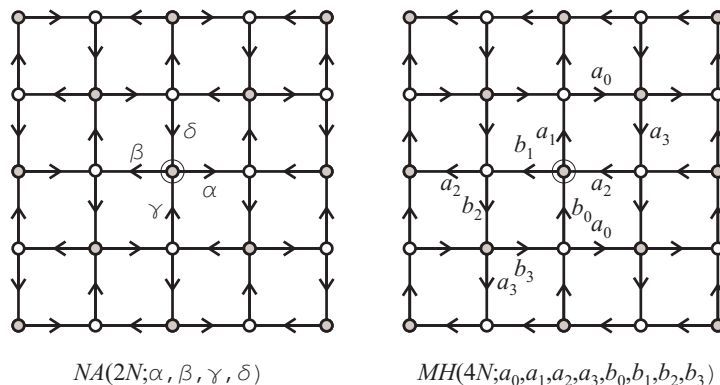


Figure 1: A New Amsterdam digraph (left) and a Manhattan digraph (right). In both cases, the even vertices are grey and the odd ones are white.

2.3. Manhattan digraphs

Manhattan digraphs were introduced independently (in slightly different forms and contexts) by Morillo, Fiol, and Fàbrega [8] and Maxemchuk [6]. They are called in this way because locally resemble the topology of the avenues and streets of Manhattan (or *l'Exampel* in downtown Barcelona); see Figure 1 (right). More precisely, given an integer N multiple of 4, the *Manhattan digraph* $MH = MH(N; a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3)$ has set of vertices $V = \bigcup_{j=0}^3 V_j$, where $V_j = \{i \in \mathbb{Z}_N : i \equiv j \pmod{4}\}$. Moreover, each vertex $i \in V_j$ is adjacent to the vertices $i + a_j, i + b_j$, where the steps $a_0, b_0, \dots, a_3, b_3$ satisfy $a_j \equiv 3 \pmod{4}, b_j \equiv 1 \pmod{4}$, for $j = 0, 1, 2, 3$, and

$$a_0 + a_2 \equiv -(a_1 + a_3) \equiv b_0 + b_2 \equiv -(b_1 + b_3) \pmod{N}. \tag{5}$$

The number of vertices for such a digraphs are upper bounded by the following Moore-like bounds (see again Morillo, Fiol, and Fàbrega [8]):

$$N_{MH} \leq M_{MH}(2, k) = \begin{cases} 2(k-1)^2 & \text{for odd } k = 2n+1, \\ 2[(k-1)^2 + 1] & \text{for even } k = 2n. \end{cases} \tag{6}$$

$$\tag{7}$$

3. New Amsterdam and Manhattan digraphs associated to double-step graphs

In this section we show that every double-step graph has associated both a New Amsterdam digraph and a Manhattan digraph. In each step of this process, while the number of vertices is doubled, the diameter only increases (at most) by one. Moreover, the relationships between the steps characterizing the digraphs are easily devised by superimposing the corresponding plane patterns.

3.1. From a double-step graph to a New Amsterdam digraph

Given a double-step graph $G(N; \pm a, \pm b)$, we can obtain a New Amsterdam digraph $NA(N_{NA}; \alpha, \beta, \gamma, \delta)$, where $N_{NA} = 2N$ and the steps $\alpha, \beta, \gamma, \delta$ satisfy the following conditions:

- (i) $\alpha, \beta, \gamma, \delta$ are odd integers,
- (ii) $\alpha + \gamma \equiv -\beta - \delta \equiv 2a \pmod{N_{NA}}$,
- (iii) $\beta + \gamma \equiv -\alpha - \delta \equiv 2b \pmod{N_{NA}}$,

where (ii), (iii) are made clear in Figure 2. There are different possible solutions of these equations. For instance, we can take

$$\alpha = -1, \quad \beta = 2(b - a) - 1, \quad \gamma = 2a + 1, \quad \delta = -2b + 1, \quad (1)$$

and we are lead to the following result.

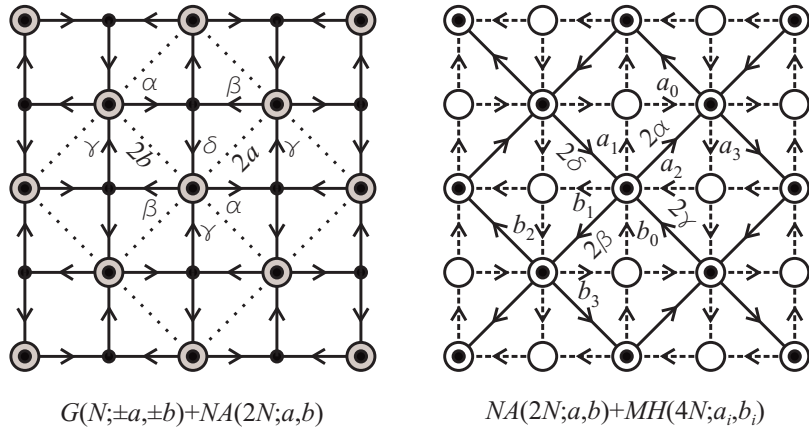


Figure 2: Left: A New Amsterdam digraph $NA(2N; a, b)$ superimposed on a double-step graph $G(N; \pm a, \pm b)$. Right: A Manhattan digraph $MH(4N; a_i, b_i)$, $0 \leq i \leq 3$, superimposed on a New Amsterdam digraph $NA(2N; \alpha, \beta, \gamma, \delta)$. The double-step digraph has grey vertices and its edges are dotted lines, the New Amsterdam digraph has black vertices and continuous arcs, and the Manhattan digraph has white vertices and discontinuous arcs.

Theorem 3.1. For any double-step graph $G = G(N; \pm a, \pm b)$ with diameter $D = k$, there exists a New Amsterdam digraph $NA = NA(N_{NA}; \alpha, \beta, \gamma, \delta)$ on $N_{NA} = 2N$ vertices, steps $\alpha = -1, \beta = 2(b - a) - 1, \gamma = 2a + 1, \delta = -2b + 1$, and diameter D_{NA} satisfying

$$2k \leq D_{NA} \leq 2k + 1. \quad (2)$$

Proof. Let us first prove that there is a path of length at most $2k + 1$ between any two given vertices i, j of a New Amsterdam digraph NA . By symmetry, it is enough to check the cases $i \in \{0, 1\}$. Then we must distinguish the cases when j is even or odd.

- (a) $0 \rightarrow 2j$: Since $\gcd(a, b, N) = 1$ and G has diameter k , for any $j \in \mathbb{Z}_N$ there exist integers m, n such that $|m| + |n| \leq k$ and

$$ma + nb \equiv j \pmod{N}. \quad (3)$$

So, multiplying by 2 and using Eqs. (ii) and (iii), we have that

$$m\alpha + m\gamma + n\beta + n\gamma \equiv 2j \pmod{N_{NA}} \quad (4)$$

and, hence, $\text{dist}_{NA}(0, 2j) \leq 2|m| + 2|n| \leq 2k$.

- (b) $0 \rightarrow 2j - 1$: Since in at most $2k$ steps we reach every even vertex from 0, by using an additional step $\alpha = -1$, Eq. (4) yields

$$(m + 1)\alpha + m\gamma + n\beta + n\gamma \equiv 2j - 1 \pmod{N_{NA}} \quad (5)$$

and we reach every odd vertex in at most $2k + 1$ steps, $\text{dist}_{NA}(0, 2j - 1) \leq 2|m| + 1 + 2|n| \leq 2k + 1$.

- (c) $1 \rightarrow 2j + 1$: By using again Eq. (4), we have

$$1 + m\alpha + m\gamma + n\beta + n\gamma \equiv 2j + 1 \pmod{N_{NA}} \quad (6)$$

and, hence, $\text{dist}_{NA}(1, 2j + 1) \leq 2k$.

- (d) $1 \rightarrow 2j$: Let us consider the value $2j - \gamma = 2(j - a) - 1$. By the same reasons as before, there exist integers m', n' such that $|m'| + |n'| \leq k$ and

$$m'a + n'b \equiv j - a - 1 \pmod{N}. \quad (7)$$

and, hence, using again Eqs. (ii) and (iii),

$$1 + m'\alpha + m'\gamma + n'\beta + (n' + 1)\gamma \equiv 2j \pmod{N_{NA}}$$

and, so, $\text{dist}_{NA}(1, 2j) \leq 2k + 1$.

Finally to prove that $D_{NA} \geq 2k$ it suffices to take the vertex i such that $\text{dist}(0, i) = k$. \square

3.2. From a New Amsterdam digraph to a Manhattan digraph

Similarly, from a New Amsterdam digraph $NA(N_{NA}; \alpha, \beta, \gamma, \delta)$ we can obtain the Manhattan digraph $MH(N_{MH}; a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3)$, with $N_{MH} = 2N_{NA}$ and steps $a_0, b_0, \dots, a_3, b_3$ satisfying:

- (i) a_i, b_i are odd integers for $i = 0, 1, 2, 3$.
- (ii) $a_0 + a_2 \equiv -(a_1 + a_3) \equiv b_0 + b_2 \equiv -(b_1 + b_3) \pmod{N_{MH}}$.
- (iii) $a_0 + a_1 \equiv 2\alpha \pmod{N_{MH}}$; $b_1 + b_2 \equiv 2\beta \pmod{N_{MH}}$;
 $b_3 - a_1 \equiv 2\delta \pmod{N_{MH}}$; $b_0 - a_0 \equiv 2\gamma \pmod{N_{MH}}$.

For understanding where Eqs. (ii) and (iii) come from, see Figure 2 (right). Again, there are different possible solutions to these equations. For instance, we can take:

$$\begin{aligned} a_0 = 1, & & a_1 = 2\alpha - 1, & & a_2 = 1, & & a_3 = -2\alpha - 1, \\ b_0 = 2\gamma + 1, & & b_1 = 2\beta + 2\gamma - 1, & & b_2 = -2\gamma + 1, & & b_3 = -2\beta - 2\gamma - 1. \end{aligned} \quad (8)$$

3.3. From a double-step graph to a Manhattan digraph

As a consequence of the above results (1), and (8), the following equalities give the steps of a Manhattan digraph from the steps of its corresponding double-step graph:

$$\begin{aligned} a_0 = 1, & & a_1 = -3, & & a_2 = 1, & & a_3 = 1, \\ b_0 = 4a + 3, & & b_1 = 4b - 1, & & b_2 = -4a - 1, & & b_3 = -4b - 1. \end{aligned} \quad (9)$$

As a consequence, we get the following theorem.

Theorem 3.2. *For any double-step graph $G = G(N; \pm a, \pm b)$ with diameter $D = k$, there exists a Manhattan digraph $MH = MH(N_{MH}; a_0, b_0, \dots, a_3, b_3)$, where $N_{MH} = 4N$ vertices, steps given by (9), and diameter D_{MH} satisfying*

$$2k + 1 \leq D_{MH} \leq 2k + 2. \quad (10)$$

Proof. As was shown by Dalfó, Comellas, and Fiol [3], every Manhattan digraph can be seen as the line digraph of a New Amsterdam digraph. Then, the result is a direct consequence of Theorem 3.1 and the properties of line digraphs. Namely, if G is a δ -regular digraph (different from a directed cycle) with N vertices and diameter D , then its line digraph $L(G)$ has δN vertices and diameter $D + 1$. See more details about the line digraph technique in Fiol, Yebra and Alegre [5]. \square

4. Dense New Amsterdam and Manhattan digraphs

Now we are ready to give results about the (Δ, N) problem for both families considered. We begin recalling some basic known results concerning double-step graphs.

4.1. A basic pair of steps

As it was shown by Bermond, Iliades and Peyrat [2] (see also Beivide, Herrada, Balcázar and Arruabarrena [1]), the pair of steps

$$a = k, \quad b = k + 1,$$

solves both the (Δ, D) and (Δ, N) problems for the family of double-step graphs. Indeed, we have the following result.

Theorem 4.1. [2, 1] *For any number N of vertices satisfying*

$$M_{DS}(2, k - 1) = 2(k - 1)^2 + 2(k - 1) + 1 < N \leq M_{DS}(2, k) = 2k^2 + 2k + 1$$

the double-step graph $G(N; \pm k, \pm[k + 1])$ has diameter $D = k$.

Notice that, for such numbers N of vertices, k is the smallest possible value of the diameter D that solves the (Δ, N) problem for these graphs. Moreover, for diameter $D = k$, the number $N = 2k^2 + 2k + 1$ is the maximum possible number of vertices, so solving the corresponding (Δ, D) problem.

4.2. The (Δ, N) problem for New Amsterdam digraphs

From the double-step graphs described in Theorem 4.1, we solve the (Δ, N) problem for the New Amsterdam digraphs for all, but one, (even) values of N .

Theorem 4.2. *From $a = k$ and $b = k + 1$, for $k \geq 1$, we get a New Amsterdam digraph with number N_{NA} of vertices and steps*

$$\beta = -\alpha = 1, \quad \gamma = -\delta = 2k + 1, \tag{1}$$

such that

- (a) *If $4k^2 + 4 \leq N_{NA} \leq 4k^2 + 4k + 2$, then $D_{NA} = 2k + 1$.*
- (b) *If $N_{NA} = 4k^2 + 4k + 4$, then $D_{NA} = 2k + 2$.*
- (c) *If $4k^2 + 4k + 8 \leq N_{NA} \leq 4(k + 1)^2 + 2$, then $D_{NA} = 2k + 3$.*

Proof. (a) When N_{NA} is within the range of (a), Theorems 4.1 and 3.1 yield that the diameter is $D_{NA} \leq 2k + 1$ but, from the Moore bounds (3) and (4), D_{NA} is smaller than $2k + 1$. Thus, $D_{NA} = 2k + 1$. The same reasoning applies for the case $N_{NA} = 4(k + 1)^2 + 2$ in (c), giving diameter $D_{NA} = 2k + 3$. In fact if, in this case, we change $k + 1$ for k , we get the diameter $D_{NA} = 2k + 1$ for the order $N_{NA} = 4k^2 + 2$. This is the lower missing value in (a) to cover all the range $(2k)^2 < N_{NA} \leq (2k + 1)^2 + 1$ with the minimum possible diameter.

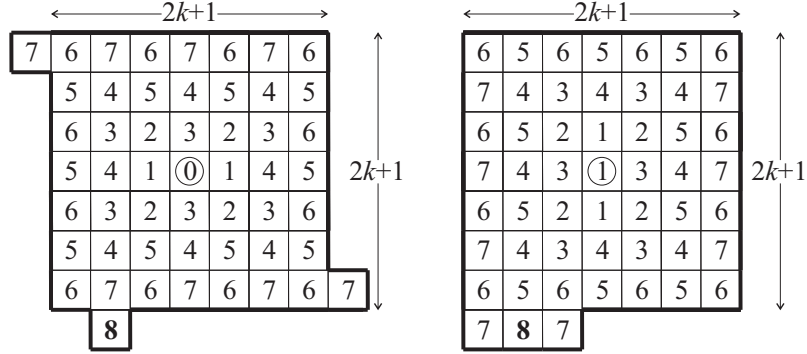


Figure 3: Optimal tiles for case (b): $N_{NA} = 4k^2 + 4k + 4$ when $k = 3$. In bold, there are the distance of the unique farthest vertex from 0 or 1.

(b) When $N_{NA} = 4k^2 + 4k + 4$, we have the tiles centered at 0 or at 1 of Fig. 3, which gives diameter $D_{NA} = 2k + 2$, and their corresponding tessellations. Moreover, the distribution of the 0's yields the equations with solutions $a = k$ and $b = k + 1$, as claimed (see Morillo, Fiol, and Fàbrega [8]).

(c) Although, according to (3) and (4), for the range $4k^2 + 4k + 8 \leq N_{NA} \leq 4(k + 1)^2$ the minimum theoretical diameter is $D_{NA} = 2k + 2$, a detailed study of the corresponding tiles and tessellations show that this is not attainable and, then, $D_{NA} = 2k + 3$ (a value guaranteed again by Theorems 4.1 and 3.1). \square

Note that the only missing value in the above interval is $N_{NA} = 4k^2 + 4k + 6$. In this case, computer programming shows that, to obtain the minimum diameter $D_{NA} = 2k + 2$, we need to use other steps different from $a = k$ and $b = k + 1$.

In terms of the diameter $D = D_{NA}$ of the New Amsterdam digraph, we have the following results:

- For every odd diameter $D = D_{NA} \geq 3$, there is a New Amsterdam digraph for any order $N = N_{NA}$ satisfying

$$(D - 1)^2 - 2D + 10 \leq N \leq D^2 + 1, \tag{2}$$

to be compared with the theoretical optimal values which, according to the Moore bounds (3) and (4), would be

$$(D - 1)^2 + 2 \leq N \leq D^2 + 1. \tag{3}$$

- For every even diameter $D = D_{NA} \geq 2$, there is a New Amsterdam digraph for any order $N = N_{NA}$ satisfying

$$D^2 - 2D + 4 \leq N \leq D^2 - 2D + 6, \tag{4}$$

to be compared with the theoretical optimal values which, according to (3) and (4), would be

$$(D - 1)^2 + 3 \leq N \leq D^2. \tag{5}$$

Notice that, according to the upper bound in (2), the value $N_{NA} = 4k^2 + 4k + 2$ for odd diameter $D_{NA} = 2k + 1$ solves the (Δ, D) problem (see also Morillo, Fiol and Fàbrega [8]). In fact, for $k = 1$ the New Amsterdam digraph $NA(10, -1, 1, 3, 3)$ is a bipartite Moore digraph with degree 2 and diameter 3. That is, its order attains the Moore bound for a (general) bipartite digraph with such parameters.

4.3. The (Δ, N) problem for Manhattan digraphs

From the above New Amsterdam digraphs, we also can solve the (Δ, N) problem for the Manhattan digraphs. The results are based on the already mentioned fact that, if G is a New Amsterdam digraph on N_{NA} vertices and diameter k , then its line digraph $L(G)$ is a Manhattan digraph on $N_{MH} = 2N_{NA}$ vertices and diameter $D = k + 1$.

Theorem 4.3. *From the steps $\beta = -\alpha = 1, \gamma = -\delta = 2k + 1$ of a New Amsterdam digraph, we get a Manhattan digraph with number $N_{MH} = 2N_{NA} = 4N$ of vertices and steps*

$$\begin{aligned} a_0 &= 1, & a_1 &= -3, & a_2 &= 1, & a_3 &= 1, \\ b_0 &= 4k + 3, & b_1 &= 4k + 3, & b_2 &= -4k - 1, & b_3 &= -4k - 5, \end{aligned} \tag{6}$$

such that

- If $8k^2 + 8 \leq N_{MH} \leq 8k^2 + 8k + 4$, then $D_{MH} = 2k + 2$.
- If $N_{MH} = 8k^2 + 8k + 8$, then $D_{MH} = 2k + 3$.
- If $8k^2 + 8k + 16 \leq N_{MH} \leq 8(k + 1)^2 + 4$, then $D_{MH} = 2k + 4$.

Now the only missing value of N_{MH} is $N_{MH} = 8k^2 + 8k + 12$ where, according to the program outputs for the New Amsterdam digraphs, with some other steps different from $\beta = -\alpha = 1$ and $\gamma = -\delta = 2k + 1$, we obtain the diameter $D_{MH} = 2k + 3$.

In terms of the diameter $D = D_{MH}$ of the Manhattan digraph, we now have the following results:

- For every even diameter $D = D_{MH} \geq 4$, there is a Manhattan digraph for any order $N = N_{MH}$ satisfying

$$2[(D - 2)^2 - 2(D - 1) + 10] \leq N \leq 2[(D - 1)^2 + 1], \tag{7}$$

to be compared with the theoretical optimal values which, according to the Moore bounds (6) and (7), would be

$$2[(D - 2)^2 + 3] \leq N \leq 2[(D - 1)^2 + 1]. \tag{8}$$

- For every odd diameter $D = D_{MH} \geq 5$, there is a Manhattan digraph for any order $N = N_{MH}$ satisfying

$$2[(D-1)^2 - 2(D-1) + 4] \leq N \leq 2[(D-1)^2 - 2(D-1) + 6], \quad (9)$$

to be compared with the theoretical optimal values which, according to (6) and (7), would be

$$2[(D-2)^2 + 3] \leq N \leq 2(D-1)^2. \quad (10)$$

Notice that, according to the upper bound in (7), the value $N_{MH} = 8k^2 + 8k + 4$ for even diameter $D_{MH} = 2k$ solves the (Δ, D) problem (see also Morillo, Fiol and Fàbrega [8]). In particular, for $k = 1$ the Manhattan digraph MH with 20 vertices has diameter 4, whereas its line digraph $L(MH)$, with 40 vertices and diameter 5, turns out to be a quasi-Moore bipartite digraph (that is, it has only two vertices less than the unattainable Moore-bound). For more details about almost Moore bipartite digraphs, see Fiol and Gimbert [4].

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