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Decomposition of complete graphs into small generalized prisms

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Abstract

A prism is the Cartesian product $C_m \Box P_2$. In other words, it is a graph consisting of two cycles of the same length whose corresponding vertices are joined by additional edges forming a matching. The problem of decomposition of complete graphs into prisms with 12 or 16 vertices was completely solved in [12]. In this paper we completely characterize the complete graphs that are decomposable into 3-regular bipartite graphs of order 12 or 16 that are a simple modification of prisms.

Keywords: graph decomposition, prism, 3-regular bipartite graph, graceful labeling, α-labeling.

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1. Introduction

We use standard terminology and notation of graph theory. All graphs in this paper are simple, finite and undirected.

A graph H has a G-*decomposition* if there are subgraphs $G_1, G_2, \ldots, G_s$ of H, all isomorphic to G, such that each edge of H belongs to exactly one G_i.

For many years, one of the most popular problems in graph decompositions has been the problem of decompositions into 2-regular graphs, that is, into cycles and unions of cycles ([4, 5, 6]). Investigation of analogous problems for 3-regular graphs is a natural next step in this field of research. A. Kotzig [15] and D. Bryant, S. El-Zanati, and R. Gardner [7] completely solved the problem of decomposition of complete graphs into cubes Q_3. P. Adams, D. Bryant, and B. Maenhaut [3] completely solved the problem of factorization of complete graphs into unions of cubes Q_3. P. Adams, D. Bryant, and A. Khodkar characterized all cubic graphs factorizing K_{10} [2], and P. Adams, H. Ardal, J. Manuch, V. Hoa, M. Rosenfeld, and L. Stacho characterized all cubic graphs that factorize K_{16} [1]. Recall that a *prism* is a graph of the form $C_m \Box P_2$. The authors of this paper [12] completely characterized the complete graphs that are decomposable into prisms with 12 or 16 vertices.

We continue our effort in this paper and study decompositions of complete graphs into small 3-regular graphs that arise from prisms by a slight modification. As in [9, 10, 11] we generalize prisms and let the $(0, j)$*-prism* (pronounced "oh-jay prism") of order $2n$ for j even be the graph with two vertex disjoint cycles $R_n^i = v_0^i, \ldots, v_{n-1}^i$ for $i \in \{1, 2\}$ of length n called *rims* and edges $v_0^1 v_0^2, v_2^1 v_2^2, v_4^1 v_4^2, \ldots$ and $v_1^1 v_{j+1}^2$, $v_3^1 v_{3+j}^2, v_5^1 v_{5+j}^2, \ldots$ called *spokes* of *type* 0 and *type* j, respectively (see Fig. 1). It is easy to observe that an $(0, j)$-prism is a 3-regular graph and is isomorphic to an $(0, -j)$-prism, $(j, 0)$-prism and $(-j, 0)$-prism. We can therefore always assume that $0 \le j \le \frac{n}{2}$. In our terminology the usual prism is an $(0, 0)$-prism. We will denote an $(0, j)$-prism with $n = 2m$ vertices and $3m$ edges by $Pr_n(0, j)$.

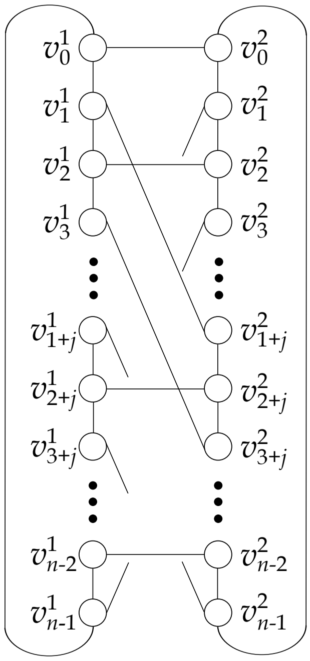

Figure 1: $(0, j)$-prism $Pr_n(0, j)$.

A labeling of a graph G with m edges is an injection ρ from $V(G)$ into set $S \subset \{0, 1, \ldots, 2m\}$. A. Rosa ([16]) introduced several types of graph labelings as tools for decompositions of complete graphs. The length of an edge xy is $l(x, y) = \min\{|\rho(x) - \rho(y)|, |\rho(x) - \rho(y)|\}$ where the subtraction is performed modulo $2m + 1$. If the set of all lengths of the n edges is equal to $\{1, 2, \ldots, m\}$ and $S \subset \{0, 1, 2, \ldots, 2m\}$, then ρ is a *rosy labeling* (called ρ-valuation by A. Rosa [16]). If $S \subset \{0, 1, 2, \ldots, m\}$ instead then ρ is a *graceful labeling* (called β-valuation by A. Rosa [16]).

A graceful labeling ρ is an α*-labeling* if there exists a number α such that for every edge $xy \in E(G)$ with $\rho(x) < \rho(y)$ it holds that $\rho(x) \le \alpha < \rho(y)$.

It is obvious that G must be bipartite to allow an α-labeling. Labelings are important tools in graph decomposition, as follows from two results by A. Rosa.

Theorem 1.1. [16] *If a graph G with m edges has a rosy labeling, then there is a G-decomposition of K_{2m+1} into $2m+1$ copies of G.*

Theorem 1.2. [16] *If a bipartite graph G with m edges has an α-labeling, then there is a G-decomposition of K_{2mx+1} for any positive integer x.*

In [14] S.I. El-Zanati, N. Punnim and C. Vanden Eynden defined a ρ^+*-labeling*. A labeling of a bipartite graph G with bipartition X, Y is called a ρ^+-labeling if it is a rosy labeling with the additional property that for every edge $xy \in E(G)$ with $x \in X, y \in Y$ it holds that $\rho^+(x) < \rho^+(y)$.

The difference between these labelings is that while in an α-labeling we require all vertices in X to have labels smaller than every vertex in Y, in a ρ^+-labeling we require that all neighbors of each given vertex $y \in Y$ have their labels smaller than $\rho^+(y)$. Moreover, we can use labels from the set $\{0, 1, \ldots, 2m\}$ while in an α-labeling only from the set $\{0, 1, \ldots, m\}$.

Theorem 1.3. [14] *If a bipartite graph G with m edges has an ρ^+-labeling, then there is a G-decomposition of K_{2mx+1} for any positive integer x.*

A simple modification of α-labeling has been also introduced independently by many authors for decomposition of complete bipartite graphs. A bipartite graph G with m edges and a bipartition $X \cup Y$ has a *bipartite labeling* f^* if there are bijections f_X and f_Y from X and Y, respectively, to $\{0, 1, \ldots, m-1\}$ such that the set of all edge lengths is equal to $\{0, 1, \ldots, m-1\}$. Here the edge length of and edge xy is defined as $f_Y(y) - f_X(x)$ and the subtraction is performed in Z_m. The following analogue of the above theorems was proved many times.

Theorem 1.4. *If a bipartite graph G with m edges has a bipartite labeling, then $K_{m,m}$ has a G-decomposition.*

When G has an α-labeling, we can modify it to obtain a bipartite labeling f^* as follows. Define f^* as $f_X(x) = f(x)$ for $x \in X$ and $f_Y(y) = f(y) - 1$ for $y \in Y$. Obviously, f^* is a bipartite labeling. It is easy to observe that the following corollary holds.

Corollary 1.5. *If a bipartite graph G with m edges has an α-labeling, then $K_{mk,mk}$ has a G-decomposition for every positive integer k.*

The following was proved in [8].

Theorem 1.6. [8] *Every* $Pr_n(0,2)$ *has a* ρ^+ *-labeling.*

By Theorems 1.3 and 1.6 we immediately have the following.

Theorem 1.7. [8] *Every* $Pr_n(0,2)$ *decomposes the complete graph* K_{6nx+1} *for any positive integer* x.

For small prisms it was shown in [12].

Theorem 1.8. [12] *A complete graph* K_n *is* $Pr_{12}(0,0)$ *-decomposable only if and only if* $n \geq 28$ *and* $n \equiv 1$ *or* $n \equiv 28 \pmod{36}$.

Theorem 1.9. [12] *A complete graph* K_n *is* $Pr_{16}(0,0)$ *-decomposable only if and only if* $n \geq 16$ *and* $n \equiv 1$ *or* $16 \pmod{48}$.

In this paper we generalize the above Theorems 1.8 and 1.9 for $Pr_{12}(0,j)$ and $Pr_{16}(0,j)$.

2. Decomposition into $Pr_{12}(0,j)$

We start by proving that a complete graph K_n is always $Pr_{12}(0,j)$-decomposable when the necessary conditions are satisfied. As $Pr_{12}(0,j)$ is 3-regular, it follows that $n-1$ must be divisible by 3. Since $Pr_{12}(0,j)$ has 18 edges, we must have $n(n-1)/2 = 18k$. Therefore, we have the following necessary condition.

Observation 2.1. *If the complete graph* K_n *is* $Pr_{12}(0,j)$ *-decomposable for some* j, *then* $n \equiv 1$ *or* $28 \pmod{36}$.

Notice that there exist only two types of $Pr_{12}(0,j)$, namely $Pr_{12}(0,0)$ and $Pr_{12}(0,2)$. Hence, by Theorems 1.7 and 1.8 the following holds.

Theorem 2.2. *For any feasible* j, *the complete graph* K_n *is* $Pr_{12}(0,j)$ *-decomposable for every* $n \equiv 1 \pmod{36}$.

To solve the case of $n \equiv 28 \pmod{36}$, we start with an easy observation.

Observation 2.3. *The complete graph* K_{36m+28} *for any* $m \geq 0$ *can be decomposed into* K_{28}, m *copies of* K_{37}, m *copies of* $K_{27,36}$ *and* $m(m-1)/2$ *copies of* $K_{36,36}$.

Thus we need to find $Pr_{12}(0,2)$-decompositions of the four graphs in the previous observation.

Observation 2.4. *The complete bipartite graph* $K_{36,36}$ *has a* $Pr_{12}(0,2)$ *-decomposition.*

Proof. Let us recall that $Pr_{12}(0,2)$ has a ρ^+-labeling by Theorem 1.6. Let $f : V(Pr_{12}(0,2)) \to \{0,1,\ldots,18\}$ be a ρ^+-labeling of $Pr_{12}(0,2)$. Define a new labeling f^* as $f^*(x) = f(x)$ for the vertices in the "lower" partite set and $f^*(x) = f(x) - 1$ for the vertices in the "upper" partite set. Then f^* is a bipartite labeling of $Pr_{12}(0,2)$, and $K_{24,24}$ is $Pr_{12}(0,2)$-decomposable by Corollary 1.5. □

Lemma 2.5. *The complete graph K_{28} has a $Pr_{12}(0,2)$-decomposition.*

Proof. Let $V(K_{28}) = Z_7 \times \{0,1,2,3\}$. Let us construct three base copies of $Pr_{12}(0,2)$ (starters). The first one consists of two cycles: $(0_0,2_0,0_1,5_0,4_1,2_2)$ and $(1_0,4_0,1_1,3_1,0_2,1_2)$. The second starter has cycles $(0_0,3_1,2_2,1_0,0_2,2_3)$ and $(0_1,4_1,4_2,0_3,3_0,1_3)$. Cycles in the third starter are $(0_0,3_3,5_2,5_3,4_3,6_3)$ and $(1_3,6_1,1_2,0_3,3_2,2_1)$. In every starter, the spokes are edges between corresponding vertices x_i and y_i of two cycles for $i = 0,2,4$ and between vertices x_i and y_{i+2} for $i = 1,3,5$. To get 18 remaining copies of $Pr_{12}(0,2)$, for every starter we apply the mapping $i \mapsto i+1$ for labels of vertices while indices are fixed. □

Lemma 2.6. *The complete bipartite graph $K_{27,36}$ is $Pr_{12}(0,2)$-decomposable.*

Proof. It is sufficient to show that $K_{9,6}$ is $Pr_{12}(0,2)$-decomposable. Let the partite sets of $K_{9,6}$ be $X = \{x_0,x_1,\ldots,x_8\}$ and $Y = \{y_0,y_1,\ldots,y_5\}$. We construct the first copy of $Pr_{12}(0,2)$ as follows. Both cycles consist of edges $\{x_i,y_i\}$ and $\{x_i,y_{i+2}\}$, the former for $i = 0,2,4$ and the latter for $i = 1,3,5$. The spokes are $\{x_j,y_{j-1}\}$, $j = 0,1,2,3,4,5$. Notice that x_j has neighbors y_j, y_{j+2} and y_{j+5} in Y, for $j = 0,1,2,3,4,5$. To get the second copy of $Pr_{12}(0,2)$ we apply the mapping $i \mapsto i+3$ for labels of vertices in X, the mapping $i \mapsto i+1 \pmod 3$ for y_0,y_1,y_2 and the mapping $i \mapsto i-1 \pmod 3 + 3$ for y_3,y_4,y_5. Thus x_k gets neighbors y_{k+1}, y_{k+3} and y_{k+4} for $k = 3,4,5$, and y_{k-6}, $y_{k-2 \ (\mathrm{mod}\ 3)+3}$ and $y_{k-1 \ (\mathrm{mod}\ 3)+3}$ for $k = 6,7,8$. Similarly, the third copy is obtained from the first after applying the mapping $i \mapsto i+6$ for labels of vertices in X, the mapping $i \mapsto i-1 \pmod 3$ for y_0,y_1,y_2 and the mapping $i \mapsto i+1 \pmod 3 + 3$ for y_3,y_4,y_5. Now every x_l has neighbors y_{l+1}, y_{l+3} and y_{l+4} for $l = 0,1,2$ and $y_{l-5 \ (\mathrm{mod}\ 3)}$, $y_{l-4 \ (\mathrm{mod}\ 3)}$ and y_{l-3} for $l = 6,7,8$. □

Now the case of $n \equiv 28 \pmod{36}$ is complete.

Theorem 2.7. *The complete graph K_n is $Pr_{12}(0,2)$-decomposable for every $n \equiv 28 \pmod{36}$.*

Proof. The result follows immediately from Observations 2.3 and 2.4 and Lemma 2.5 and Lemma 2.6. □

Combining Observation 2.1 and Theorems 1.8, 2.2 and 2.7, we obtain the complete characterization.

Theorem 2.8. *For any feasible* j, *the complete graph* K_n *is* $Pr_{12}(0,j)$ *-decomposable if and only if* $n \geq 28$ *and* $n \equiv 1$ *or* $n \equiv 28 \pmod{36}$.

3. Decomposition into $Pr_{16}(0,j)$

In this section we will show that a complete graph K_n is $Pr_{16}(0,j)$-decomposable whenever the necessary conditions are satisfied. Since $Pr_{16}(0,j)$ is 3-regular, we must have $n-1$ divisible by 3. Since $Pr_{16}(0,j)$ has 24 edges, $n(n-1)/2$ must be divisible by 24. Therefore, the following holds.

Observation 3.1. *If the complete graph* K_n *is* $Pr_{12}(0,j)$ *-decomposable for some* j, *then* $n \equiv 1$ *or* $16 \pmod{48}$.

Notice that there exist only three types of $Pr_{16}(0,j)$, namely $Pr_{16}(0,0)$, $Pr_{16}(0,2)$ and $Pr_{16}(0,4)$.

Observation 3.2. *The generalized prism* $Pr_{16}(0,4)$ *has a* ρ^+ *-labeling.*

A ρ^+-labeling of $Pr_{16}(0,4)$ is shown in Figure 2.

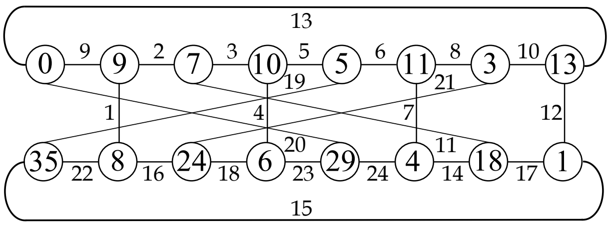

Figure 2: ρ^+-labeling of $Pr_{16}(0,4)$.

Therefore, using Observation 3.2 and Theorems 1.7 and 1.9 the following holds.

Observation 3.3. *For any feasible* j, *the complete graph* K_n *is* $Pr_{16}(0,j)$ *-decomposable for every* $n \equiv 1 \pmod{48}$.

To settle the remaining case of $n \equiv 16 \pmod{48}$, we first notice that K_{16} allows a $Pr_{16}(0,j)$-factorization for any j, as proved by [1].

Theorem 3.4. K_{16} *allows a* $Pr_{16}(0,j)$ *-factorization for any* j.

Let G and H be two graphs where $\{x^1, x^2, \ldots, x^p\}$ is the vertex set of G. Based upon the graph G, an isomorphic copy H^j of H replaces every vertex x^j, for $j = 1, 2, \ldots, p$, in such a way that a vertex in H^j is adjacent to a vertex in H^i if and only if $x^j x^i$ was an edge in G. Let $G[H]$ denote the resulting graph. The graph $G[H]$ is called the *composition* of graphs G and H.

Now we make an easy observation.

Observation 3.5. *Let $N = 3m + 1$. Then K_N can be decomposed into the complete m-partite graph $K_{3,3,\dots,3}$ and m copies of K_4.*

If $m = 1$, then the graph $K_{3,3,\dots,3}$ does not appear in the decomposition. We can blow up K_N into $K_N[K_{16}] = K_{16N}$ and set $n = 16N$. Then the following obviously holds.

Observation 3.6. *Let $n = 48m + 16$. Then K_n can be decomposed into the complete m-partite graph $K_{48,48,\dots,48}$, m copies of the complete 4-partite graph $K_{16,16,16,16}$ and $3m + 1$ copies of K_{16}.*

Now we present a $Pr_{16}(0,2)$-decomposition and $Pr_{16}(0,4)$-decomposition of $K_{48,48,\dots,48}$. Obviously, $K_{48,48}$ can be decomposed into four copies of $K_{24,24}$. Then $K_{48,48,\dots,48}$ can be decomposed into $2m(m-1)$ copies of $K_{24,24}$.

Let $g : V(Pr_{16}(0,2)) \to \{0,1,\dots,24\}$ ($g : V(Pr_{16}(0,4)) \to \{0,1,\dots,24\}$) be a ρ^+-labeling of $Pr_{16}(0,2)$ (or $Pr_{16}(0,4)$, respectively). Define a new labeling g^* as $g^*(x) = g(x)$ for the "lower" vertices and $g^*(x) = g(x) - 1$ for the "upper" vertices. Then again g^* is a bipartite labeling of $Pr_{16}(0,2)$ ($Pr_{16}(0,4)$, respectively), and $K_{24,24}$ is $Pr_{16}(0,2)$-decomposable (or $Pr_{16}(0,4)$-decomposable, respectively). Therefore, we obtain the following observation.

Observation 3.7. *The complete m-partite graph $K_{48,48,\dots,48}$ can be decomposed into $96m(m-1)$ copies of $Pr_{16}(0,2)$ or $Pr_{16}(0,4)$.*

It is well-known (see, e.g., [13]) that $K_{4,4,4,4}$ can be decomposed into 16 copies of K_4. Since $K_{16,16,16,16} = K_{4,4,4,4}[\overline{K}_4]$, it can be decomposed into 16 copies of $K_{4,4,4,4} = K_4[\overline{K}_4]$.

Proposition 3.8. *$K_{4,4,4,4}$ can be decomposed into four copies of $Pr_{16}(0,2)$ or $Pr_{16}(0,4)$.*

Proof. Denote the partite sets by X_0, X_1, X_2, X_3 with $X_k = \{x_{k,0}, x_{k,1}, x_{k,2}, x_{k,3}\}$. Let $p = \frac{j}{2}$. Construct the first copy of $Pr_{16}(0,j)$ for $j = 2,4$ as follows:

One cycle consists of edges $x_{0,i}x_{1,i}$ and $x_{0,i}x_{1,i+1}$, the other one of $x_{2,i}x_{3,i}$ and $x_{2,i}x_{3,i+1}$ for $i = 0,1,2,3$. The spokes are $x_{0,i}x_{2,i}$ and $x_{1,i}x_{3,i+p}$ for $i = 0,1,2,3$. So we have edges of lengths 0 and 1 between the sets X_0 and X_1 and also between X_2 and X_3. Between X_0 and X_2 there are spokes of length 0 and between X_1 and X_3 we have spokes of length p.

Now we rotate the vertices of the cycles within X_1 and X_2 by two, obtaining the edge lengths 2 and 3 between the sets X_0 and X_1 and also between X_2 and X_3. At the same time, we obtain spokes of length 2 between X_0 and X_2 and of length $(2 + p) \pmod 4$ between X_1 and X_3.

To be more precise, the first cycle consists of edges $x_{0,i}x_{1,i+2}$ and $x_{0,i}x_{1,i+3}$ and the other one of $x_{2,i}x_{3,i+2}$ and $x_{2,i}x_{3,i+3}$ for $i = 0,1,2,3$. The spokes are $x_{0,i}x_{2,i+2}$ and $x_{1,i}x_{3,i+p+2}$ for $i = 0,1,2,3$.

The two remaining factors are constructed with the cycles between sets X_0 and X_3 and between X_1 and X_2. First, we construct a cycle containing edges $x_{0,i}x_{3,i}$ and $x_{0,i}x_{3,i+1}$, and another cycle with edges $x_{1,i}x_{2,i}$ and $x_{1,i}x_{2,i-1}$ for $i = 0,1,2,3$. The spokes are $x_{0,i}x_{2,i+1}$ and $x_{1,i}x_{3,i-p-1}$ for $i = 0,1,2,3$. So we have edges of lengths 0 and 1 between the sets X_0 and X_3 and edges of length 3 and 0 between X_1 and X_2. Between X_0 and X_2 the spokes have length 1 and between X_1 and X_3 length $(3-p) \pmod 4$.

The last factor has again the cycles between X_0 and X_3 and between X_1 and X_2. One has edges $x_{0,i}x_{3,i+2}$ and $x_{0,i}x_{3,i+3}$, the other one $x_{1,i}x_{2,i+1}$ and $x_{1,i}x_{2,i+2}$ for $i = 0,1,2,3$. The spokes are $x_{0,i}x_{2,i+3}$ and $x_{1,i}x_{3,i+1-p}$ for $i = 0,1,2,3$. So we have edges of lengths 2 and 3 between the sets X_0 and X_3 and edges of length 1 and 2 between X_1 and X_3. Between X_0 and X_2 the spokes have length 3 and between X_1 and X_3 length $(1-p) \pmod 4$. □

Now the case of $n \equiv 16 \pmod{49}$ is settled completely.

Theorem 3.9. *For any feasible j, the complete graph K_n is $Pr_{16}(0,j)$-decomposable for every $n \equiv 16 \pmod{48}$.*

Proof. The case of $j = 0$ holds by Theorem 1.9. Hence, we can assume that $j = 2,4$.

Let $n = 48m + 16$. By Observation 3.6, K_n can be decomposed into copies of the m-partite graph $K_{48,48,\ldots,48}$, 4-partite graph $K_{4,4,4,4}$ and complete graph K_{16}.

$Pr_{16}(0,2)$- and $Pr_{16}(0,4)$-decompositions of $K_{48,48,\ldots,48}$ exist by Observation 3.7, of $K_{4,4,4,4}$ by Proposition 3.8 and of K_{16} by Theorem 3.4. This completes the proof. □

The main result of this section follows from Observation 3.1 and Theorems 3.3 and

Theorem 3.10. *For any feasible j, the complete graph K_n is $Pr_{16}(0,j)$-decomposable if and only if $n \geq 16$ and $n \equiv 1$ or $16 \pmod{48}$.*

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