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On the metric dimension of convex polytopes*

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Abstract

Metric dimension is a generalization of affine dimension to arbitrary metric spaces (provided a resolving set exists).

Let $\mathcal{F}$ be a family of connected graphs $G_n : \mathcal{F} = (G_n)_{n\geq 1}$ depending on n as follows: the order $|V(G)| = \varphi(n)$ and $\lim\limits_{n\to\infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $dim(G_n) \leq C$ for every $n \geq 1$ then we shall say that $\mathcal{F}$ has bounded metric dimension. If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on n), $\mathcal{F}$ is called a family with constant metric dimension.

In this paper, we study the properties of of some classes of convex polytopes having pendent edges with respect to their metric dimension.

Keywords: Metric dimension, basis, resolving set, plane graph, convex polytope, pendant.

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1. Notation and preliminary results

If G is a connected graph, the *distance* $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), ..., d(v,w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving set* or

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locating set for G [1]. A resolving set of minimum cardinality is called a *metric basis* for G and this cardinality is the *metric dimension* of G, denoted by $dim(G)$. The concepts of resolving set and metric basis have previously appeared in the literature (see [1-4, 6-18]).

For a given ordered set of vertices $W = \{w_1, w_2, ..., w_k\}$ of a graph G, the *ith* component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G)\backslash W$. A useful property in finding $dim(G)$ is the following lemma [17]:

Lemma 1.1. *Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap W \neq \emptyset$.*

By denoting $G + H$ the join of G and H a *wheel* W_n is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a fan is $f_n = K_1 + P_n$ for $n \geq 1$ and $Jahangir\ graph$ $J_{2n}, (n \geq 2)$ (also known as *gear graph*) is obtained from a *wheel* W_{2n} by alternately deleting n spokes. Buczkowski *et al.* [1] determined the dimension of *wheel* W_n, Caceres *et al.* [3] the dimension of a fan f_n and Tomescu and Javaid [18] the dimension of $Jahangir\ graph$ J_{2n}.

Theorem 1.2. [1, 3, 18] *Let W_n be a wheel of order $n \geq 3$, f_n be fan of order $n \geq 1$ and J_{2n} be a Jahangir graph. Then*

(i) For $n \geq 7$, $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$;

(ii) For $n \geq 7$, $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$;

(iii) For $n \geq 4$, $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$.

The metric dimension of all these *plane graphs* depends upon the number of vertices in the graph.

On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim(\mathrm{G})$ is finite and does not depend upon the choice of G in $\mathcal{G}$. In [4] Chartrand et al. proved that a graph has metric dimension 1 if and only if it is a *path*, hence paths on n vertices constitute a family of graphs with constant metric dimension. A nice property of graphs with metric dimension 2 is the following result of Khuller et al. [13].

Theorem 1.3. [13] *Let G be a graph with metric dimension 2 and let $\{u,v\} \subset V(G)$ be a metric basis in G. Then the following are true:*

(a) There is a unique shortest path between u and v.

(b) The degree of each u and v is at most 3.

It is shown in [6] that some families of plane graphs generated by convex polytopes constitute the families of plane graphs with constant metric dimension. Note that the problem of determining whether $dim(G) < k$ is an NP-complete problem [5].

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. If G' is a graph obtained by adding a pendent edge to a nontrivial connected graph G, then it is easy to verify that

$$dim(G) \leq dim(G') \leq dim(G) + 1$$

A *helm* H_n, $n \geq 3$ is a graph obtained from a wheel W_n by attaching a pendant vertex to each rim vertex. Javaid [11] proved that $dim(H_n) = dim(W_n)$. In this paper, we extend this study by considering some classes of convex polytopes with pendant edges. It is natural to ask for the characterization of classes of convex polytopes G' obtained from convex polytope G by attaching a pendant edge at each vertex of the outer cycle of G such that $dim(G') = dim(G)$.

2. The plane graph S_n^p

The graph of convex polytope S_n defind in [7] consisting of $2n$ 3-sided faces, $2n$ 4-sided faces and a pair of n-sided faces, and is obtained by the combination of the graph of convex polytope R_n [6] and the graph of a prism D_n.

The plane graph S_n^p (p from pendant) (Figure 1) is obtained from a graph of convex polytope S_n by attaching a pendant edge at each vertex of outer cycle of S_n. We have

$$V(S_n^p) = V(S_n) \cup \{e_i : 1 \leq i \leq n\}$$

and

$$E(S_n^p) = E(S_n) \cup \{d_ie_i : 1 \leq i \leq n\}.$$

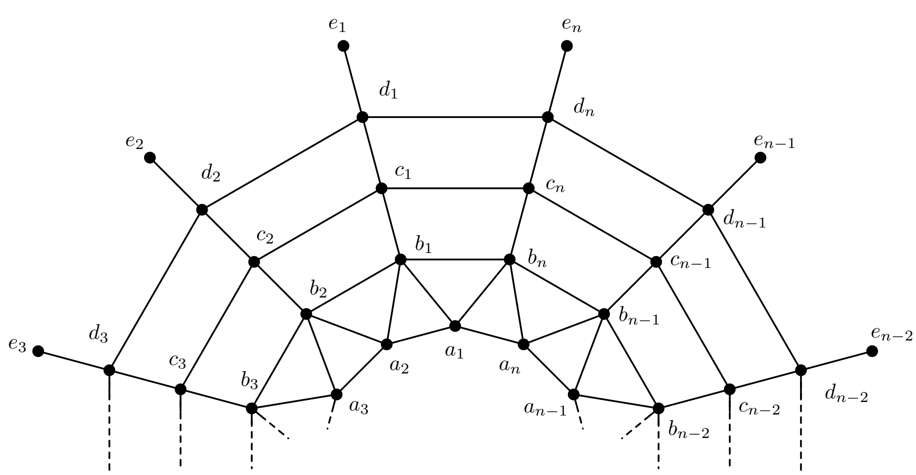

Figure 1: The plane graph S_n^p

For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$, the inner cycle; cycle induced by $\{b_i : 1 \leq i \leq n\}$, the interior cycle; cycle induced by $\{c_i : 1 \leq i \leq n\}$, the exterior cycle, the cycle induced by $\{d_i : 1 \leq i \leq n\}$, the outer cycle and the set of vertices $\{e_i : 1 \leq i \leq n\}$, the pendant vertices.

The metric dimension of graph of convex polytope S_n has been studied in [7] and it has been proved that the graph of convex polytope S_n has constant metric dimension 3. In the next theorem, we prove that the metric dimension of plane graph S_n^p is the same as the graph of convex polytope S_n. Note that the choice of an appropriate basis of vertices (also referred to as landmarks in [13]) is core of the problem.

Theorem 2.1. *Let S_n^p be the plane graph defined above; then $dim(S_n^p) = 3$ for every $n \geq 6$.*

Proof. We will prove the above equality by double inequalities. We consider the two cases.

Case(i) When n is even.

In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(S_n^p)$, we show that W is a resolving set for S_n^p in this case. For this we give representations for any vertex of $V(S_n^p) \backslash W$ with respect to W.

Representations for the vertices of inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations for the vertices of interior cycle are

$$r(b_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \leq i \leq k; \\ (k, k, 1), & i = k+1; \\ (2k-i+1, 2k-i+2, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations for the vertices of exterior cycle are

$$r(c_i|W) = (1, 1, 1) + r(b_i|W)$$

Representations for the vertices of outer cycle are

$$r(d_i|W) = (1, 1, 1) + r(c_i|W) = (2, 2, 2) + r(b_i|W)$$

Representations for the pendant vertices are

$$r(e_i|W) = (1, 1, 1) + r(d_i|W) = (2, 2, 2) + r(c_i|W) = (3, 3, 3) + r(b_i|W).$$

We note that there are no two vertices having the same representations implying that $dim(S_n^p) \leq 3$.

On the other hand, we show that $dim(S_n^p) \geq 3$ by proving that there is no resolving set W such that $|W| = 2$. Suppose on contrary that $dim(S_n^p) = 2$. Then by Theorem 1.3, the degree of basis vertices can be at most 3. But except the vertices e_i, all other vertices of S_n^p have degree 4 or 5. So we have the only possibility to be discussed when both vertices belong to the set of pendant vertices. Without loss of generality we suppose that one resolving vertex is e_1. Suppose that the second resolving vertex is e_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(d_n|\{e_1, e_t\}) = r(c_1|\{e_1, e_t\}) = (2, t+1)$ and when $t = k+1$, we have $r(d_2|\{e_1, e_t\}) = r(d_n|\{e_1, e_t\}) = (2, t-1)$, a contradiction.

Hence, from above it follows that there is no resolving set with two vertices for $V(S_n^p)$ implying that $dim(S_n^p) = 3$ in this case.

Case(ii) When n is odd.

In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Again we show that $W = \{a_1, a_2, a_{k+1}\} \subset V(S_n^p)$ is a resolving set for S_n^p in this case. For this we give representations of any vertex of $V(S_n^p)\backslash W$ with respect to W.

Representations for the vertices of inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (k, k, 1), & i = k+2 \\ (2k-i+3, 2k-i+4, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations for the vertices of interior cycle are

$$r(b_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \leq i \leq k; \\ (k+1, k, 1), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations for the vertices of exterior cycle are

$$r(c_i|W) = (1, 1, 1) + r(b_i|W)$$

Representations for the vertices of outer cycle are

$$r(d_i|W) = (1, 1, 1) + r(c_i|W) = (2, 2, 2) + r(b_i|W)$$

Representations for the pendant vertices are

$$r(e_i|W) = (1, 1, 1) + r(d_i|W) = (2, 2, 2) + r(c_i|W) = (3, 3, 3) + r(b_i|W).$$

Again we see that there are no two vertices having the same representations which implies that $dim(S_n^p) \leq 3$.

On the other hand, suppose that $dim(S_n^p) = 2$, then there are the same possibilities as in Case(i) and contradiction can be deduced analogously. This implies that $dim(S_n^p) = 3$ in this case, which completes the proof. $\square$

3. The plane graph T_n^p

The graph of convex polytope T_n defined in [7] consists of $4n$ 3-sided faces, n 4-sided faces and a pair of n-sided faces, and is obtained by the combination of the graph of convex polytope R_n [6] and the graph of an antiprism A_n [11]. The plane graph T_n^p (p from pendant) (Figure 2) is obtained from a graph of convex polytope T_n by attaching a pendant edge at each vertex of outer cycle of T_n. We have

$$V(T_n^p) = V(T_n) \cup \{e_i : 1 \leq i \leq n\}$$

and

$$E(T_n^p) = E(T_n) \cup \{d_i e_i : 1 \leq i \leq n\}.$$

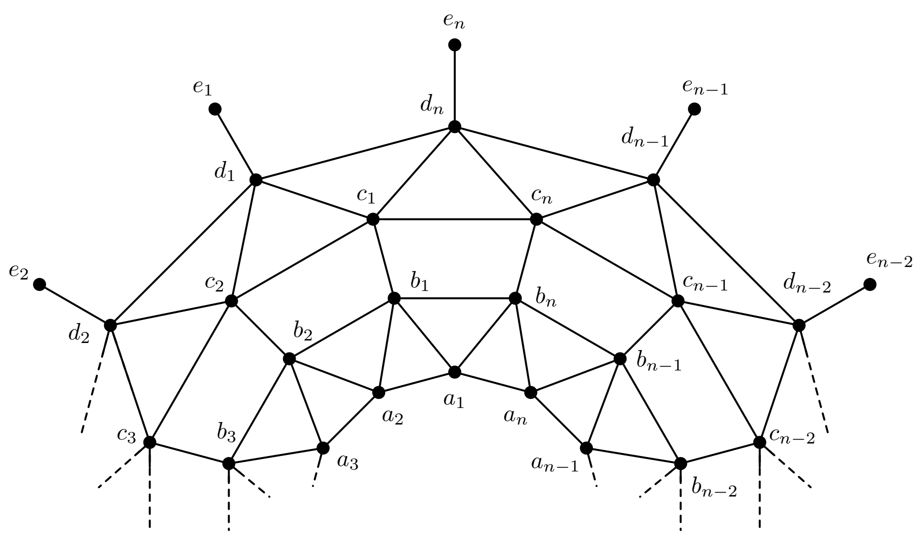

Figure 2: The plane graph T_n^p

For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$, the inner cycle; cycle induced by $\{b_i : 1 \leq i \leq n\}$, the interior cycle; cycle induced by $\{c_i : 1 \leq i \leq n\}$, the exterior cycle, the cycle induced by $\{d_i : 1 \leq i \leq n\}$, the outer cycle and the set of vertices $\{e_i : 1 \leq i \leq n\}$, the pendant vertices.

The metric dimension of graph of convex polytope T_n has been studied in [7] and where it was shown that the graph of convex polytope S_n has constant metric dimension. In the next theorem, we prove that the metric dimension of plane graph T_n^p is the same as the graph of convex polytope T_n. Again, the choice of an appropriate basis of vertices (also referred to as landmarks in [13]) is core of the problem.

Theorem 3.1. *Let T_n^p be the plane graph defined above; then $dim(T_n^p) = 3$ for every $n \geq 6$.*

Proof. We will prove the above equality by double inequalities. We consider the two cases.

Case(i) When n is even.

In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(T_n^p)$, we show that W is a resolving set for T_n^p in this case. For this we give representations for any vertex of $V(T_n^p)\backslash W$ with respect to W.

Representations for the vertices of inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations for the vertices of interior cycle are

$$r(b_i|W) = \begin{cases} (1,1,k), & i = 1; \\ (i, i-1, k-i+1), & 2 \leq i \leq k; \\ (k,k,1), & i = k+1; \\ (2k-i+1, 2k-i+2, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations for the vertices of exterior cycle are

$$r(c_i|W) = (1,1,1) + r(b_i|W)$$

Representations for the vertices of outer cycle are

$$r(d_i|W) = \begin{cases} (3,3,k+1), & i = 1; \\ (i+2, i+1, k-i+2), & 2 \leq i \leq k-1; \\ (k+2, k+1, 3), & i = k; \\ (k+1, k+2, 3), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+2), & k+2 \leq i \leq 2k-1; \\ (3,3,k+2), & i = 2k. \end{cases}$$

Representations for the pendant vertices are

$$r(e_i|W) = (1,1,1) + r(d_i|W).$$

We note that there are no two vertices having the same representations implying that $dim(T_n^p) \leq 3$.

On the other hand, we show that $dim(T_n^p) \geq 3$. The proof follows the same lines as in Theorem 2.1. Hence, from above it follows that there is no resolving set with two vertices for $V(T_n^p)$ implying that $dim(T_n^p) = 3$ in this case.

Case(ii) When n is odd.

In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(T_n^p)$, we show that W is a resolving set for T_n in this case. For this we give representations of any vertex for $V(T_n^p)\backslash W$ with respect to W.

Representations for the vertices of inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (k, k, 1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \le i \le 2k+1. \end{cases}$$

Representations for the vertices of interior cycle are

$$r(b_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \le i \le k; \\ (k+1, k+1, 1), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \le i \le 2k+1. \end{cases}$$

Representations for the vertices of exterior cycle are

$$r(c_i|W) = (1, 1, 1) + r(b_i|W)$$

Representations for the vertices of outer cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+1), & i = 1; \\ (i+2, i+1, k-i+2), & 2 \le i \le k-1; \\ (k+2, k+1, 3), & i = k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \le i \le 2k; \\ (3, 3, k+2), & i = 2k+1. \end{cases}$$

Representations for the pendant vertices are

$$r(c_i|W) = (1, 1, 1) + r(c_i|W).$$

Again we see that there are no two vertices having the same representations which implies that $dim(T_n^p) \le 3$ in this case.

On the other hand, suppose that $dim(T_n^p) = 2$, then there are the same subcases as in Case(i) and contradiction can be obtained analogously. This implies that $dim(T_n^p) = 3$ in this case, which completes the proof. □

4. The plane graph U_n^p

The graph of convex polytope U_n defined in [7] consists of n 4-sided faces, $2n$ 5-sided faces and a pair of n-sided faces, and is obtained as a combination of the graph of convex polytope $\mathbb{D}_n$ [6] and graph of a prism D_n.

The plane graph U_n^p (p from pendant) (Figure 3) is obtained from a graph of convex polytope U_n by attaching a pendant edge at each vertex of outer cycle of U_n. We have

$$V(U_n^p) = V(U_n) \cup \{f_i : 1 \le i \le n\}$$

and

$$E(U_n^p) = E(U_n) \cup \{e_i f_i : 1 \leq i \leq n\}.$$

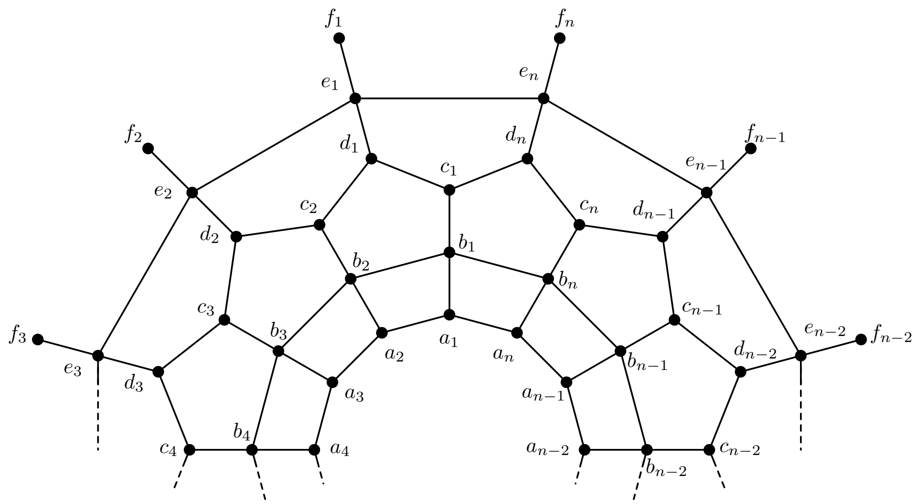

Figure 3: The plane graph U_n^p

For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$, the inner cycle; cycle induced by $\{b_i : 1 \leq i \leq n\}$, the interior cycle; cycle induced by $\{c_i : 1 \leq i \leq n\} \cup \{d_i : 1 \leq i \leq n\}$, the exterior cycle, the cycle induced by $\{e_i : 1 \leq i \leq n\}$, the outer cycle and the set of vertices $\{f_i : 1 \leq i \leq n\}$, the pendant vertices.

The metric dimension of graph of convex polytope U_n has been studied in [7] and where it was shown that the graph of convex polytope U_n does not depend upon its order and its size. In the next theorem, we prove that the metric dimension of plane graph U_n^p is the same as the graph of convex polytope U_n. Once again, the choice of an appropriate basis of vertices (also referred to as landmarks in [13]) is core of the problem.

Theorem 4.1. *Let U_n^p be the plane graph defined above; then $dim(U_n^p) = 3$ for every $n \geq 6$.*

Proof. We will prove the above equality by double inequalities. We consider the two cases.

Case(i) When n is even.

In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(U_n^p)$, we show that W is a resolving set for U_n^p in this case. For this we give representations for any vertex of $V(U_n^p)\backslash W$ with respect to W.

Representations for the vertices of inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \le i \le 2k. \end{cases}$$

Representations for the vertices of interior cycle are

$$r(b_i|W) = (1,1,1) + r(a_i|W)$$

Representations for the vertices of exterior cycle are

$$r(c_i|W) = (1,1,1) + r(b_i|W) = (2,2,2) + r(a_i|W)$$

and

$$r(d_i|W) = \begin{cases} (3, 3, k+3), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \le i \le k; \\ (k+3, k+3, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \le i \le 2k. \end{cases}$$

Representations for the vertices of outer cycle are

$$r(e_i|W) = (1,1,1) + r(d_i|W)$$

Representations for the vertices of outer cycle are

$$r(f_i|W) = (1,1,1) + r(e_i|W) = (2,2,2) + r(d_i|W).$$

We note that there are no two vertices having the same representations implying that $dim(U_n^p) \le 3$.

On the other hand, we show that $dim(U_n^p) \ge 3$. The proof follows the same lines as in Theorem 2.1 and Theorem 3.1.

Hence, from above it follows that there is no resolving set with two vertices for $V(U_n^p)$ implying that $dim(U_n^p) = 3$ in this case.

Case(ii) When n is odd.

In this case, we can write $n = 2k+1$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(U_n^p)$, we show that W is a resolving set for U_n^p in this case. For this we give representations for any vertex of $V(U_n^p) \backslash W$ with respect to W.

Representations for the vertices of inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (k, k, 1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \le i \le 2k+1. \end{cases}$$

Representations for the vertices of interior cycle are

$$r(b_i|W) = (1,1,1) + r(a_i|W)$$

Representations for the vertices of exterior cycle are

$$r(c_i|W) = (1,1,1) + r(b_i|W) = (2,2,2) + r(a_i|W)$$

and

$$r(d_i|W) = \begin{cases} (3,3,k+2), & i=1; \\ (i+2,i+1,k-i+3), & 2 \le i \le k; \\ (k+3,k+2,3), & i=k+1; \\ (k+2,k+1,4), & i=k+2; \\ (k+1,k,5), & i=k+2; \\ (2k-i+4,2k-i+5,i-k+2), & k+3 \le i \le 2k+1. \end{cases}$$

Representations for the vertices of outer cycle are

$$r(e_i|W) = (1,1,1) + r(d_i|W)$$

Representations for the pendant vertices are

$$r(f_i|W) = (1,1,1) + r(e_i|W) = (2,2,2) + r(d_i|W).$$

Again we see that there are no two vertices having the same representations which implies that $dim(U_n^p) \le 3$ in this case.

On the other hand, suppose that $dim(U_n^p) = 2$, then there are the same subcases as in Case(i) and contradiction can be obtained analogously. This implies that $dim(U_n^p) = 3$ in this case, which completes the proof. □

5. Concluding remarks

In this paper, we have studied the metric dimension of some classes of plane graphs which are obtained from some graph of convex polytopes by attaching a pendant edge at each vertex of the outer cycle of these convex polytopes. We prove that the metric dimension of these classes of plane graphs is finite and does not depend upon the number of vertices in these graphs and only three vertices appropriately chosen suffice to resolve all the vertices of these classes of plane graphs. It is natural to ask for the characterization of classes of convex polytopes with pendant edges G' obtained from convex polytope G by attaching a pendant edge at each vertex of outer cycle of these graphs G such that $dim(G') = dim(G)$. We propose the following conjecture.

Conjecture 5.1. *Let G' be the plane graph obtained from rotationally-symmetric graph of convex polytope G by attaching a pendant edge at each vertex of the out cycle of G. If G has constant metric dimension, then G' will always have constant metric dimension.*

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