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The locating-chromatic number for a subdivision of a wheel on one cycle edge

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Abstract

Let G be a connected graph. Let c be a k-coloring on G which induces a partition Π of $V(G)$ into color classes $R_1, R_2, \ldots, R_k$, where $R_i = \{v \in V(G)|c(v) = i\}$ for $1 \leq i \leq k$. The color code $c_\Pi(v)$ of vertex v is the ordered k-tuple $(d(v,R_1), d(v,R_2), ..., d(v,R_k))$, where $d(v,R_i) = \min\{d(v,x)|x \in R_i\}$ for $1 \leq i \leq k$. The coloring c is called a locating k-coloring on G if the color codes of distinct vertices are distinct. The locating-chromatic number of G, denoted by $\chi_L(G)$, is the smallest k such that G possess a locating k-coloring.

Recently, Behtoei presented the locating-chromatic number of a wheel. Furthermore, Purwasih and Baskoro (2012) gave the locating-chromatic number of the subdivision of a wheel on one of its spoke edges. In this paper, we determine the locating-chromatic number of the subdivision of a wheel on one of its cycle edges.

Keywords: locating-chromatic number, subdivision, wheel.

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1. Introduction

Let $G = (V, E)$ be a connected graph without loops and multiple edges. Let c be a k-coloring on G which induces a partition Π of $V(G)$ into *color classes* $R_1, R_2, \ldots, R_k$, where $R_i = \{v \in V(G)|c(v) = i\}$ for $i \in [1, k]$. The *color code* $c_\Pi(v)$ of vertex v is the ordered k-tuple $(d(v,R_1), d(v,R_2), ..., d(v,R_k))$, where $d(v,R_i) = \min\{d(v,x)|x \in R_i\}$ for $i \in [1, k]$. The coloring c is called a *locating k-coloring* on G if the color codes of distinct vertices are distinct. The *locating-chromatic number* of G, denoted by $\chi_L(G)$, is the smallest k such that G possess a locating k-coloring.

This concept was introduced and studied by Chartrand et al. [7]. They established the bounds for the locating-chromatic number k of a connected graph G in terms of its order $n \geq 3$ and diameter $d \geq 2$, namely $\log_{d+1} n \leq k \leq n - d + 2$ and $n \leq kd^{k-1} - 1$. They showed that for a connected graph G of order $n \geq 3$, $\chi_L(G) = n$ if and only if G is a complete multipartite graph. They also showed that for any integers a, b with $2 \leq a \leq b$, there exists a connected graph with the chromatic number a and the locating-chromatic number b. They also determined the locating-chromatic number of some well-known classes of graphs such as paths, cycles, and complete graphs.

Furthermore, Chartrand et al. [7] also studied the locating-chromatic number on trees. They showed that for $n \geq 5$, there exists a tree on n vertices having locating-chromatic number k if and only if $k \in \{3, 4, \ldots, n-2, n\}$. Recently, Asmiati et al. gave the locating-chromatic number of special classes of trees, i.e., an amalgamation of stars [1] and a firecracker graph [2].

For a certain locating-chromatic number, Chartrand et al. [8] characterized all graphs of order n with locating-chromatic number $n-1$. Recently, Asmiati and Baskoro [3] determined all graphs on n vertices containing a cycle with locating-chromatic number 3.

The locating-chromatic numbers of graphs obtained by some graph operations are also interesting to be studied. Baskoro and Purwasih [4] studied the locating-chromatic number for the corona product of graphs. They gave the upper bound for the locating-chromatic number of the corona product of two connected graphs G and H with $diam(H) \geq 2$. They also gave the exact value of the locating-chromatic number for the corona product of some well-known graphs. For results on the locating-chromatic number of the Cartesian product of graphs, please refer to [6].

In this paper, we determine the locating-chromatic number for the graphs obtained by a subdivision operation on a given graph G. In particular, we consider graph G as a wheel.

2. Wheels

Recently, Behtoei [5] derived the locating-chromatic number of the join of graphs. He defined a new parameter. Let f be a proper k-coloring on a graph G. The coloring f is called a *neighbor locating coloring* if for each pair of distinct vertices u and v with $f(u) = f(v)$ implies that $f(N_G(u)) \neq f(N_G(v))$. The *neighbor locating-chromatic number* of G, denoted by $\chi_{L2}(G)$, is the smallest k such that G possess a neighbor locating k-coloring. Behtoei then used the neighbor locating coloring to determine the locating-chromatic number for the join of graphs. One of his results is on the neighbor locating-chromatic number of a path, as follows.

Theorem 2.1. [5] *For a positive integer* $n \geq 2$, $\chi_{L2}(P_n) = m$, *where* $m = min\{k | k \in N, n \leq \frac{1}{2}(k^3 - k^2)\}$. *Particularly, there exists a neighbor locating* m*-coloring* f *on path* $P_n = u_1 u_2 \ldots u_n$ *such that* $f(u_{n-1}) = 2$ *and* $f(u_n) = 1$. *For* $n \geq 9$, $f(u_{n-2}) = m$. *Moreover, for* $n \geq 9$ *and* $n \neq \frac{1}{2}(m^3 - m^2) - 1$, $f(u_1) = 2$ *and* $f(u_2) = 1$.

By using Theorem 2.1, Behtoei derived the locating-chromatic number of wheels, as follows.

Theorem 2.2. [5] *For* $n \geq 3$, *let* $W_n = K_1 + C_n$ *and* $m = \min\{k \in N \mid n \leq \frac{1}{2}(k^3 - k^2)\}$.

Then,

$$\chi_L(W_n) = \begin{cases} 1+\chi_L(C_n) & \text{if } 3 \le n < 9, \\ m+1 & \text{if } n \ne \frac{1}{2}(m^3-m^2)-1 \text{ and } n \ge 9, \\ m+2 & \text{if } n = \frac{1}{2}(m^3-m^2)-1 \text{ and } n \ge 9. \end{cases}$$

Furthermore, by using Theorem 2.1, we can construct a neighbor locating coloring on a wheel. For $n \ge 3$, let $W_n = K_1 + C_n$, where $K_1 = \{v\}$ and $C_n = u_1u_2 \dots u_nu_1$. Let $m = \min\{k|k \in N, n \le \frac{1}{2}(k^3-k^2)\}$. For simplicity, we represent any coloring f on W_n by a sequence $[f(v); f(u_1), f(u_2), \dots, f(u_n)]$.

For any integer $3 \le n \le 8$ define a coloring f on W_n such that $[4; 3, 2, 1]$, $[5; 4, 3, 2, 1]$, $[4; 2, 1, 3, 2, 1]$, $[5; 4, 2, 3, 1, 2, 1]$, $[4; 2, 1, 3, 2, 3, 2, 1]$, and $[5; 4, 2, 3, 1, 3, 1, 2, 1]$.

For the case $n \ge 9$ and $n \ne \frac{1}{2}(m^3-m^2)-1$, by Theorem 2.1, there exists a neighbor locating m-coloring f on path $P_n = u_1u_2 \dots u_n$ such that $f(u_{n-1}) = 2, f(u_n) = 1, f(u_{n-2}) = m, f(u_1) = 2$, and $f(u_2) = 1$. Since a W_n can be constructed from a P_n by connecting u_1 with u_n and u_i with v for every $1 \le i \le n$, then $V(W_n) = V(P_n) \cup \{v\}$ and $E(W_n) = E(P_n) \cup \{u_nu_1\} \cup \{u_iv|1 \le i \le n\}$. Therefore, we can define a coloring f' on W_n such that $f'(u_i) = f(u_i)$ and $f'(v) = m+1$. Since $f'(u_1) = f(u_1) \ne f(u_n) = f'(u_n)$ and $f'(v) = m+1 \ne f'(u_i)$ for every $1 \le i \le n$, then f' is a proper coloring on W_n. For each $1 \le i \le n$, we have $f'(N_{W_n}(u_i)) = f(N_{P_n}(u_i)) \cup \{m+1\}$. Therefore, f' is a neighbor locating $(m+1)$-coloring on W_n where $f'(u_{n-1}) = 2, f'(u_n) = 1, f'(u_{n-2}) = m, f'(u_1) = 2, f'(u_2) = 1$, and $f'(v) = m+1$.

Now assume that $n = \frac{1}{2}(m^3-m^2)-1$. Consequently, $n \ge 9$ and $n-1 \ne \frac{1}{2}(m^3-m^2)-1$. Hence, by Theorem 2.1, there exists a neighbor locating m-coloring f on path $P_{n-1} = u_1u_2 \dots u_{n-1}$ such that $f(u_{n-2}) = 2$, $f(u_{n-1}) = 1$, $f(u_{n-3}) = m$, $f(u_1) = 2$, and $f(u_2) = 1$. Now, define a coloring f' on W_n such that $f'(u_i) = f(u_i)$ for $1 \le i \le n-1, f'(u_n) = m+1$, and $f'(v) = m+2$. Note that $m+1 \in f'(N_{W_n}(u_1)) \cap f'(N_{W_n}(u_{n-1}))$, $f'(u_1) \ne f'(u_{n-1})$, and $f'(N_{W_n}(u_i)) = f(N_{P_{n-1}}(u_i)) \cup \{m+2\}$ for each $2 \le i \le n-2$. Thus, f' is a neighbor locating $(m+2)$-coloring on W_n. By some permutation of colors in coloring f', we can define a new neighbor locating coloring f'' on W_n such that $f'' = [f'(v); f'(u_n), f'(u_1), f'(u_2), \dots, f'(u_{n-2}), f'(u_{n-1})]$. So, we have a neighbor locating $(m+2)$-coloring on W_n with $f''(u_{n-1}) = 2, f''(u_n) = 1, f''(u_{n-2}) = m, f''(u_1) = m+1, f''(u_2) = 2, f''(u_3) = 1$, and $f''(v) = m+2$.

Therefore we have showed the following lemma.

Lemma 2.3. *For $n \ge 3$, let $m = min\{k|k \in N, n \le \frac{1}{2}(k^3-k^2)\}$ and $W_n = K_1 + C_n$, where $K_1 = \{v\}$ and $C_n = u_1u_2 \dots u_nu_1$. Let $\chi_L(W_n) = t$, then there exists a neighbor locating* t*-coloring f on W_n such that $f(u_{n-1}) = 2$ and $f(u_n) = 1$. For $n \ge 9$, $f(u_{n-2}) = m$ and $f(v) = t$. Moreover, $f(u_1) = 2$ and $f(u_2) = 1$, when $n \ne \frac{1}{2}(m^3-m^2)-1$; $f(u_1) = m+1$, $f(u_2) = 2$, and $f(u_3) = 1$, when $n = \frac{1}{2}(m^3-m^2)-1$.*

3. Subdivision of wheels

Let $G = (V, E)$ and $e \in E$. Let $S(G(e, k))$ be a graph obtained by a subdivision of graph G on edge e in $k \geq 1$ times. In 2012, Purwasih and Baskoro [9] gave the locating-chromatic number of $S(G(e, k))$ if G is a wheel and e is a spoke edge. They derived the locating-chromatic number of $H_1 = S(W_n(e, k))$ as in the following theorems.

Theorem 3.1. *Let $H_1 = S(W_n(e, k))$ for a spoke edge e, $n \geq 3$, and $k \geq 1$. Then, $\chi_L(W_{n-1}) - 1 \leq \chi_L(H_1) \leq \chi_L(W_{n-1})$.*

Theorem 3.2. *Let $H_1 = S(W_n(e, k))$ for $n \geq 3$ and $k \neq 2$. Let v, u_i, w_i are the center vertex, cycle and subdivision vertices, respectively. Let $e = vu_1$ be a subdivision edge where $u_1 \sim u_2$, and $u_1 \sim u_n$. Let Π be an ordered partition of $V(W_n)$ induced by a minimum locating coloring c on W_n. Then, $\chi_L(H_1) = \chi_L(W_n) - 1$ if $\chi_L(W_n) \geq 5$ and one of the following conditions is satisfied:*

i. Vertex u_1 belongs to a singleton color class in Π and $c(u_2) = c(u_n)$.

ii. Vertex u_1 belongs to a singleton color class in Π, $c(u_2) \neq c(u_n)$, and u_2 (or u_n) with any other vertex is not only distinguished by u_1.

In this paper, we determine the locating-chromatic number of $S(W_n(e, k))$ if e is a cycle edge of W_n, $n \geq 3$ and $k \geq 1$. From now on, let $H_2 = S(W_n(e, k))$ with the vertex-set and edge-set

$V(H_2) = \{v, u_1, u_2, \cdots, u_n\} \cup \{w_1, w_2, \cdots, w_k\}$,

$E(H_2) = \{vu_i | 1 \leq i \leq n\} \cup \{u_2u_3, u_3u_4, \;\cdots, u_nu_1\} \cup \{u_1w_1, w_1w_2, \cdots, w_{k-1}w_k, w_ku_2\}$,
respectively. Note that, in this case, we have that $e = u_1u_2$ and w_i are the subdivision vertices.

Lemma 3.3. *Let c be a minimum locating coloring on H_2. If $c(u_1) \neq c(u_2)$, then for any vertex w_i there exists a vertex $x \neq w_i$ such that $c(w_i) = c(x)$.*

Proof. For a contradiction, let c be a minimum locating coloring on H_2 such that $\{c(w_i)\}$ is a singleton color class for some i. From the definition of H_2, we know that $d(u_s, w_i) = d(u_t, w_i)$ for all $4 \leq s, t \leq n - 1$. It implies that u_s and u_t can not be resolved by w_i. Furthermore, since $c(u_1) \neq c(u_2)$, then all vertices $u_3, u_n, w_j \neq w_i$ can be resolved by u_1 or u_2. Therefore, the coloring c is not minimum, a contradiction. □

In the following theorem, we give lower and upper bounds on the locating-chromatic number of H_2.

Theorem 3.4. $\chi_L(W_n) - 1 \leq \chi_L(H_2) \leq \chi_L(W_n)$.

Proof. Let $\chi_L(W_n) = t$. Since $n \geq 3$ then $t \geq 4$. Firstly, we will show that $\chi_L(H_2) \geq t-1$. For a contradiction, assume that $\chi_L(H_2) = t-2$. So, there exists a locating coloring $c : V(H_2) \to [1, t-2]$ on H_2. Now consider the following two cases:

Case 1. $c(u_1) \neq c(u_2)$.

By Lemma 3.3, for any vertex w_i there exists a vertex $x \neq w_i$ such that $c(w_i) = c(x)$. Now, construct a coloring c' on the corresponding W_n by removing all vertices w_i on H_2 such that

$$c'(x) = \begin{cases} c(u_i) & \text{if } x = u_i \text{ for } 1 \leq i \leq n, \\ c(v) & \text{if } x = v. \end{cases}$$

Then, c' is a (t-2)-coloring on W_n. Of course c' is not a locating coloring. Now, let Π' be the partition of $V(W_n)$ induced by c'. Thus, there are at most two vertices a and b such that $c_{\Pi'}(a) = c_{\Pi'}(u_1)$ or $c_{\Pi'}(b) = c_{\Pi'}(u_2)$ or both hold. In any case, define a new coloring c'' on W_n such that $c''(a) = t-1$, $c''(b) = t-1$, and $c''(x) = c'(x)$ for any remaining vertices $x \in W_n$. By this definition, c'' is a locating coloring on W_n with $t-1$ colors, a contradiction. So, $\chi_L(H_2) \geq t-1$.

Case 2. $c(u_1) = c(u_2)$.

Construct a coloring c' on the corresponding W_n by removing all vertices w_i on H_2 such that

$$c'(x) = \begin{cases} t-1 & \text{if } x = u_1, \\ c(u_i) & \text{if } x = u_i \text{ for } 2 \leq i \leq n, \\ c(v) & \text{if } x = v. \end{cases}$$

Now, let x and y be two distinct vertices of W_n which satisfy $c'(x) = c'(y)$. Then, $x, y \notin \{v, u_1\}$. Since $c(u_1) = c(u_2)$ then $c(u_n) \neq c(u_2)$ (as well as $c'(u_n) \neq c'(u_2)$). If $x = u_2$ then $y = u_i$ for some $i \in [3, n-1]$. Since $d(x, R_{t-1}) = 1 < 2 = d(y, R_{t-1})$, then $c_{\Pi'}(x) \neq c_{\Pi'}(y)$. If $x = u_i$ and $y = u_j$, $3 \leq i, j \leq n$ and $i \neq j$, then the color codes of x and y are distinct under c'. Therefore, c' is a locating coloring on W_n with at most $t-1$ colors, a contradiction. So, $\chi_L(H_2) \geq t-1$.

Next, we will show that $\chi_L(H_2) \leq \chi_L(W_n) = t$. Let c be a locating t-coloring on W_n and $m = \min\{k \in N \mid n \leq \frac{1}{2}(k^3 - k^2)\}$. For $n \geq 9$, consider the following two cases:

Case 1. $n \neq \frac{1}{2}(m^3 - m^2) - 1$.

By Theorem 2.2, if $n \neq \frac{1}{2}(m^3 - m^2) - 1$, then $t = m + 1$. By Lemma 2.3, there exists a locating t-coloring c of W_n such that $c(u_1) = 2, c(u_2) = 1, c(u_{n-2}) = t-1, c(u_{n-1}) = 2, c(u_n) = 1$, and $c(v) = t$. Now, let H_2' be a subdivision of W_n on edge $u_n u_1$.

Now, construct a t-coloring c' on H_2' as in Figure 1 by defining

$$c'(x) = \begin{cases} c(u_i) & \text{if } x = u_i \text{ for } 1 \leq i \leq n, \\ t & \text{if } x = v \text{ or } x = w_i \text{ for odd } i, \\ 1 & \text{if } x = w_i \text{ for even } i. \end{cases}$$

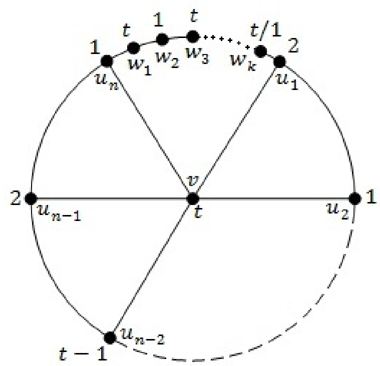

Figure 1: A locating t-coloring on $H_2' = S(W_n(e,k))$ for $n \neq \frac{1}{2}(m^3 - m^2) - 1$.

We will show that c' is a locating coloring on H_2'. Let $\Pi = \{R_1, R_2, \ldots, R_t\}$ be the partition of $V(W_n)$ induced by c and $\Pi' = \{R_1, R_2, \ldots, R_t\}$ be the partition of $V(H_2')$ induced by the coloring c', where the vertices of R_i are colored by i for $1 \leq i \leq t$.

Let $x, y \in V(H_2')$ with $c'(x) = c'(y)$. If $x = u_i$ and $y = u_j$ for $1 \leq i, j \leq n$ and $i \neq j$, then clearly that $c_{\Pi'}(x) = c_{\Pi}(x) \neq c_{\Pi}(y) = c_{\Pi'}(y)$. If $x = u_i$ and $y = w_j$ for $1 \leq i \leq n$ and $1 \leq j \leq k$, then $c_{\Pi'}(x) \neq c_{\Pi'}(y)$ since $d(x, R_{t-1}) = 2 < 3 \leq d(y, R_{t-1})$. If $x = v$ and $y = w_j$ for $1 \leq j \leq k$, then $c_{\Pi'}(x) \neq c_{\Pi'}(y)$ since $d(x, R_{t-1}) = 1 < 3 \leq d(y, R_{t-1})$. It remains to prove that $c_{\Pi'}(x) \neq c_{\Pi'}(y)$ if $x = w_i$ and $y = w_j$ for $1 \leq i, j \leq k$ and $i \neq j$.

By the definition of c', then $c'(u_1) = 2, c'(u_2) = 1, c'(u_{n-2}) = t - 1, c'(u_{n-1}) = 2, c'(u_n) = 1$, and $c'(v) = t$. Note that $d(w_l, R_2) = d(w_l, u_1) = l + 1$ for $l < \lceil \frac{k}{2} \rceil$ and $k - l + 1$ otherwise. Hence, we have

i. If $i, j < \lceil \frac{k}{2} \rceil$, then $d(x, R_2) = i + 1 \neq j + 1 = d(y, R_2)$;

ii. If $i, j \geq \lceil \frac{k}{2} \rceil$, then $d(x, R_2) = k - i + 1 \neq k - j + 1 = d(y, R_2)$;

iii. If $i < \lceil \frac{k}{2} \rceil$ and $j \geq \lceil \frac{k}{2} \rceil$, then $d(x, R_2) = i + 1$ and $d(y, R_2) = k - j + 1$. Note that, $i + 1 = k - j + 1$ if and only if $i = k - j$. Therefore, $d(x, R_2) \neq d(y, R_2)$ if and only if $i \neq k - j$. Note that $d(w_l, R_{t-1}) = d(w_l, u_{n-2}) = l + 2$ for $l < \lceil \frac{k}{2} \rceil$ and $k - l + 3$ otherwise. Therefore, if $i = k - j$, then $d(x, R_{t-1}) = i + 2 = k - j + 2 < k - j + 3 = d(y, R_{t-1})$.

So, $c_{\Pi'}(x) \neq c_{\Pi'}(y)$ if $x = w_i$ and $y = w_j$ for $1 \leq i, j \leq k$ and $i \neq j$.

Since $c_{\Pi'}(x) \neq c_{\Pi'}(y)$ for each pair of distinct vertices $x, y \in V(H_2')$ with $c'(x) = c'(y)$, then c' is a locating coloring on H_2'. By relabeling the vertices of H_2' we can get the graph $H_2 = S(W_n(e,k))$ with the subdivision edge $e = u_1u_2$ and having the same locating coloring c'.

Case 2. $n = \frac{1}{2}(m^3 - m^2) - 1$.

By Theorem 2.2, if $n = \frac{1}{2}(m^3 - m^2) - 1$, then $t = m + 2$. By Lemma 2.3, there exists a locating t-coloring c on W_n such that $c(u_1) = m+1, c(u_2) = 2, c(u_3) = 1, c(u_{n-2}) = t-2, c(u_{n-1}) = 2, c(u_n) = 1$, and $c(v) = t$. Now, let H_2' be a subdivision of W_n on edge $u_{n-1}u_n$.

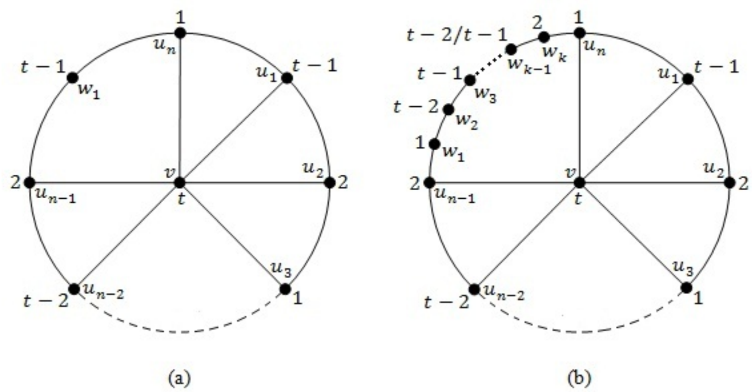

Figure 2: A locating t-coloring on $H_2' = S(W_n(e,k))$ for $n = \frac{1}{2}(m^3 - m^2) - 1$.

For $k = 1$, construct a t-coloring c' on H_2' as in Figure 2 (a) by defining

$$c'(x) = \begin{cases} c(u_i) & \text{if } x = u_i \text{ for } 1 \leq i \leq n, \\ t & \text{if } x = v, \\ t-1 & \text{if } x = w_1. \end{cases}$$

We will show that c' is a locating coloring on H_2'. Let $\Pi = \{R_1, R_2, \ldots, R_t\}$ be the partition of $V(W_n)$ induced by c and $\Pi' = \{R_1, R_2, \ldots, R_t\}$ be the partition of $V(H_2')$ induced by the coloring c', where the vertices of R_i are colored by i for $1 \leq i \leq t$.

Let $x, y \in V(H_2')$ with $c'(x) = c'(y)$. If $x = u_i$ and $y = u_j$ for $1 \leq i, j \leq n-2$ and $i \neq j$, then clearly that $c_{\Pi'}(x) = c_\Pi(x) \neq c_\Pi(y) = c_{\Pi'}(y)$. If $x = u_{n-1}$ and $y = u_j$ for $1 \leq j \leq n-2$, then $c_{\Pi'}(x) \neq c_{\Pi'}(y)$ since $d(x, R_b) \neq d(y, R_b)$ where b is $t-1$ or $t-2$. If $x = u_n$ and $y = u_j$ for $1 \leq j \leq n-2$, then $c_{\Pi'}(x) \neq c_{\Pi'}(y)$ since $d(x, R_{t-1}) = 1 < 2 = d(y, R_{t-1})$. If $x = w_1$ and $y = u_1$, then $c_{\Pi'}(x) \neq c_{\Pi'}(y)$ since $d(x, R_t) = 2 > 1 = d(y, R_t)$. So, c' is a locating coloring on H_2'.

For $k \geq 2$, construct a t-coloring c' on H_2' as in Figure 2 (b) by defining

$$c'(x) = \begin{cases} c(u_i) & \text{if } x = u_i \text{ for } 1 \leq i \leq n, \\ t-2 & \text{if } x = w_i \text{ for even } i \neq k, \\ t-1 & \text{if } x = w_i \text{ for odd } i \notin \{1, k\}, \\ t & \text{if } x = v, \\ 1 & \text{if } x = w_1, \\ 2 & \text{if } x = w_k. \end{cases}$$

We will show that c' is a locating coloring on H_2'. Let $\Pi = \{R_1, R_2, \ldots, R_t\}$ be the partition of $V(W_n)$ induced by c and $\Pi' = \{R_1, R_2, \ldots, R_t\}$ be the partition of $V(H_2')$ induced by the coloring c', where the vertices of R_i are colored by i for $1 \le i \le t$.

Let $x, y \in V(H_2')$ with $c'(x) = c'(y)$. If $x = u_i$ and $y = u_j$ for $1 \le i, j \le n$ and $i \ne j$, then clearly that $c_{\Pi'}(x) = c_\Pi(x) \ne c_\Pi(y) = c_{\Pi'}(y)$. If $x = u_i$ and $y = w_j$ for $1 \le i \le n$ and $1 \le j \le k$ then $c_{\Pi'}(x) \ne c_{\Pi'}(y)$ since $d(x, R_t) = 1 < 2 \le d(y, R_t)$. It remains to prove that $c_{\Pi'}(x) \ne c_{\Pi'}(y)$ if $x = w_i$ and $y = w_j$ for $2 \le i, j \le k-1$ and $i \ne j$.

By the definition of c', we know that $c'(u_1) = t-1, c'(u_2) = 2, c'(u_3) = 1, c'(u_{n-2}) = t-2, c'(u_{n-1}) = 2, c'(u_n) = 1$, and $c'(v) = t$. Note that $d(w_l, R_1) = d(w_l, w_1) = l-1$ for $l < \lceil \frac{k}{2} \rceil$ and $k-l+1$ otherwise. Hence, we have

i. If $i, j < \lceil \frac{k}{2} \rceil$, then $d(x, R_1) = i-1 \ne j-1 = d(y, R_1)$;

ii. If $i, j \ge \lceil \frac{k}{2} \rceil$, then $d(x, R_1) = k-i+1 \ne k-j+1 = d(y, R_1)$;

iii. If $i < \lceil \frac{k}{2} \rceil$ and $j \ge \lceil \frac{k}{2} \rceil$, then $d(x, R_1) = i-1$ and $d(y, R_1) = k-j+1$. Note that, $i-1 = k-j+1$ if and only if $i = k-j+2$. Therefore, $d(x, R_1) \ne d(y, R_1)$ if and only if $i \ne k-j+2$. Note that $d(w_l, R_2) = d(w_l, u_{n-1}) = l$ for $l < \lceil \frac{k}{2} \rceil$ and $k-l$ otherwise. Therefore, if $i = k-j+2$, then $d(x, R_2) = i = k-j+2 > k-j = d(y, R_2)$.

So, $c_{\Pi'}(x) \ne c_{\Pi'}(y)$ if $x = w_i$ and $y = w_j$ for $2 \le i, j \le k-1$ and $i \ne j$.

Since $c_{\Pi'}(x) \ne c_{\Pi'}(y)$ for each pair of distinct vertices $x, y \in V(H_2')$ with $c'(x) = c'(y)$, then c' is a locating coloring on H_2'. By relabeling the vertices of H_2' we can get the graph $H_2 = S(W_n(e,k))$ with the subdivision edge $e = u_1u_2$ and having the same locating coloring c'.

If $3 \le n < 9$, then use coloring f' on W_n defined in the proof of Lemma 2.3. If $n \ne 7$, then choose $u_{n-1}u_n$ as a subdivision edge of H_2'. Otherwise, choose u_nu_1. Then, define a new coloring c' such that $c'(v) = f'(v), c'(u_i) = f'(u_i), c'(w_i) = f'(v)$ if i is odd and $c'(w_i) = f'(u_1)$ if i is even. By considering the color codes of all vertices, it is easy to verify that c' is a locating coloring on H_2'. Again, by relabeling the vertices of H_2' we can get the graph $H_2 = S(W_n(e,k))$ with the subdivision edge $e = u_1u_2$ and having the same locating coloring c'. □

In the following theorem, we will give the graphs which satisfy the lower bound in Theorem 3.4.

Theorem 3.5. *Let $H_2 = S(W_n(e,k))$ for $n \ge 3$ and $k \ge 1$. If $n \in \{4, 6, 8, \frac{1}{2}(m^3 - m^2) - 1\}$ where $m \ge 3$, then $\chi_L(H_2) = \chi_L(W_n) - 1$.*

Proof. By Theorem 2.2, if $n = \frac{1}{2}(m^3 - m^2) - 1$, then $\chi_L(W_n) = m+2$ and $\chi_L(W_{n+1}) = m+1$. Now, let c be a locating $(m+1)$-coloring on W_{n+1} which induces a partition $\Pi = \{R_1, R_2, \ldots, R_{m+1}\}$, where $R_i = \{x \in V(W_{n+1}) | c(x) = i\}$ for each $1 \le i \le m+1$.

Let $V(W_{n+1}) = \{v, u_1, u_2, \ldots, u_{n+1}\}$. Without loss of generality, by Lemma 2.3, we can choose c such that $c(v) = m+1$, $c(u_2) = c(u_{n+1}) = 1$, $c(u_1) = c(u_n) = 2$, and $c(u_{n-1}) = m$. It is obvious that $c(u_3) \neq c(u_n)$. Otherwise, $c_\Pi(u_2) = c_\Pi(u_{n+1})$.

Note that for $k \geq 1$, graph H_2 is isomorphic with a graph obtained by removing an edge vu_1 of W_{n+1} and then subdividing u_1u_2 in $k-1$ times. In this case, vertices u_{n+1} and u_1 on W_{n+1} are u_1 and w_1 on H_2, respectively. Now, construct a new coloring c' on H_2 such that

$$c'(x) = \begin{cases} c(u_i) & \text{if } x = u_i \text{ for } 2 \leq i \leq n, \\ m+1 & \text{if } x = v \text{ or } x = w_i \text{ for even } i \neq k, \\ 1 & \text{if } x = u_1 \text{ or } x = w_i \text{ for odd } i \notin \{1, k\}, \\ 2 & \text{if } x = w_1 \text{ or } w_k. \end{cases}$$

By this definition, we obtain a $(m+1)$-coloring on H_2 which is less one than $\chi_L(W_n) = m+2$. Now, let $\Pi' = \{R_1, R_2, \ldots, R_{m+1}\}$ be an ordered partition of $V(H_2)$ induced by c' where $R_i = \{x \in V(H_2) | c'(x) = i\}$. We will show that c' is a locating coloring on H_2.

Let $x, y \in V(H_2)$ with $c'(x) = c'(y)$. If $x = u_i$ and $y = u_j$ for $1 \leq i, j \leq n$ and $i \neq j$, then clearly that $c_{\Pi'}(x) = c_\Pi(x) \neq c_\Pi(y) = c_{\Pi'}(y)$. If $x = u_i$ and $y = w_j$ for $1 \leq i \leq n$ and $1 \leq j \leq k$, then $c_{\Pi'}(x) \neq c_{\Pi'}(y)$ since $d(y, R_m) \geq 3 > 2 = d(x, R_m)$. If $x = w_i$ and $y = w_j$ for $1 \leq i, j \leq k$ and $i \neq j$, then $c_{\Pi'}(x) \neq c_{\Pi'}(y)$ since $d(y, R_b) \neq d(x, R_b)$ where R_b is class contained u_3 or u_n. Therefore, c' is a locating $(m+1)$-coloring on H_2.

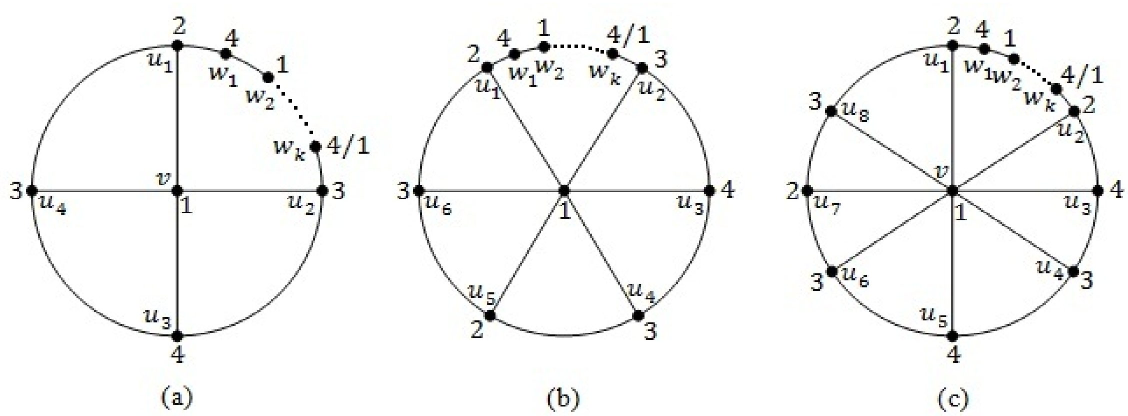

Figure 3: The locating 4-coloring on $H_2 = S(W_n(e,k))$ for (a) $n = 4$, (b) $n = 6$, and (c) $n = 8$.

For $n = 4$, construct a locating coloring as in Figure 3 (a) such that $c'(v) = 1, c'(u_1) = 2, c'(u_2) = 3, c'(u_3) = 4, c'(u_4) = 3$, and $c'(w_i) = 1$ for even $1 \leq i \leq k$ or 4 for odd $1 \leq i \leq k$.

For $n = 6$, construct a locating coloring as in Figure 3 (b) such that $c'(v) = 1, c'(u_1) = 2, c'(u_2) = 3, c'(u_3) = 4, c'(u_4) = 3, c'(u_5) = 2, c'(u_6) = 3$, and $c'(w_i) = 1$ for even

$1 \leq i \leq k$ or 4 for odd $1 \leq i \leq k$.

For $n = 8$, construct a locating coloring as in Figure 3 (c) such that $c'(v) = 1, c'(u_1) = 2, c'(u_2) = 2, c'(u_3) = 4, c'(u_4) = 3, c'(u_5) = 4, c'(u_6) = 3, c'(u_7) = 2, c'(u_8) = 3$, and $c'(w_i) = 1$ for even $1 \leq i \leq k$ or 4 for odd $1 \leq i \leq k$. $\square$

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