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A gap in the achievable radio number line

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Abstract

Let $d(u,v)$ denote the distance between two distinct vertices of a connected graph G, and $\mathrm{diam}(G)$ be the diameter of G. A *radio labeling* c of G is an assignment of positive integers to the vertices of G satisfying $d(u,v) + |c(u) - c(v)| \geq \mathrm{diam}(G) + 1$. The maximum integer in the range of the labeling is its span. The *radio number* of G, $rn(G)$, is the minimum possible span. We show that the path on n vertices, P_n, achieves the maximum possible radio number. We then ask whether any integer in the range $[n, rn(P_n)]$ fails to be the radio number of some connected graph of order n, and answer this question in the affirmative.

Keywords: achievable radio number, radio labeling, multilevel distance labeling.

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1. Introduction

Given a function $f : S \to T$ from a set of objects S to another set T, the first question typically asked is "What is $f(s)$ for some $s \in S$?" An inverse question – asking which elements of T are elements of $f(S)$ – has a rich history within mathematics. For example, the exponents of primitive matrices have been completely characterized ([10, 6, 7, 9, 11]) while the possibilities for exponents of various subgroups of the primitive matrices are still being explored (e.g., [4]). Asking which spectral radii are achievable (e.g., [1]), and investigating the attainable orders of bases of finite cyclic groups (e.g., [5]) are also instances of this inverse question. This paper introduces the analogous question for the radio number of graphs, a function with input consisting of connected simple graphs and output the positive integers.

Question 1.1. *What are the achievable radio numbers of graphs of order n?*

All graphs discussed herein will be connected, simple, and undirected. Let G be such a graph. A *radio labeling* is a function $c : V(G) \to \mathbb{Z}_+$ satisfying the *radio condition* $d(u,v)+|c(u)-c(v)| \geq \text{diam}(G)+1$ for every distinct pair $u, v \in V(G)$. As such, radio labeling, also called multilevel distance labeling, is a type of distance labeling. Distance labelings are assignments of integers to the vertices of the graph so as to satisfy some condition involving the distances between vertices and the differences between corresponding label values. This type of labeling is motivated by considerations arising in assigning frequencies to radio transmitters: the closer the transmitters, the larger the difference necessary in their frequencies so as to avoid interference [3].

For a radio labeling c of a graph G, the *span* of c, denoted *span*(c), is the largest integer in $c\,(V(G))$. Minimizing the span over all possible radio labelings of G gives us the *radio number* of G, $rn(G)$. I.e., $rn(G) = \min\{span(c) \,|\, c \text{ is a radio labeling of } G\}$. A *minimal radio labeling* of G is a radio labeling c of G for which $span(c) = rn(G)$. We will say that $z \in \mathbb{Z}_+$ is an *achievable radio number* when $z = rn(G)$ for some graph G.

A lower bound for achievable radio numbers follows immediately from the radio condition. As $d(u,v) \leq \text{diam}(G)$ for distinct $u, v \in V(G)$, every radio labeling c of G must satisfy $|c(u)-c(v)| \geq 1$. Thus if G is of order n, $rn(G) \geq n$. In fact, this bound is strict, as the radio number of the complete graph on n vertices is n, as evidenced by any labeling that assigns each vertex a distinct integer from $[1, \ldots, n]$.

Examining the radio condition might lead one to conjecture that decreasing the diameter while leaving the order fixed would lead to a smaller radio number. Such a conjecture would be false, as shown in Example 1.2.

Example 1.2. *Consider C_8, the cycle on 8 vertices. Liu and Zhu show that $rn(C_8) = 14$ (Theorem 5, [8]). Let H be the complete graph on 6 vertices $\{v_1, \ldots, v_6\}$ with an additional two vertices $\{w_1, w_2\}$ and additional edges $\{v_6w_1, w_1w_2\}$ (see Figure). We have diam$(H) = 3 < 4 =$ diam(C_8), yet $rn(H) = 17 > rn(C_8)$.*

The graph of order n with maximum diameter is P_n, the path on n vertices. Liu and Zhu also determine the radio number of paths (Theorem 3, [8]):

$$rn(P_n) = \begin{cases} 2k^2+3, & \text{if } n = 2k+1, \\ 2(k^2-k)+2, & \text{if } n = 2k. \end{cases}$$

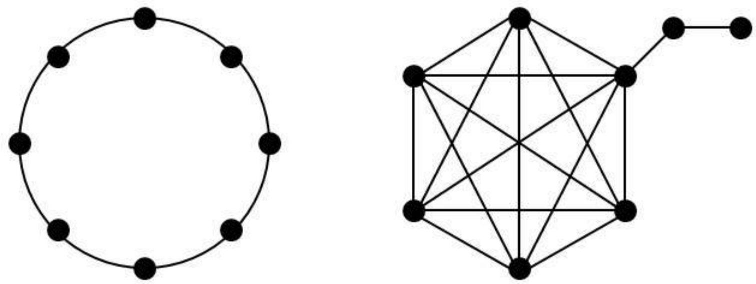

Figure 1: Graphs C_8 and H of Example 1.2

In Section 2 we show that the set of achievable radio numbers of graphs of order n is bounded above by $rn(P_n)$. Thus all achievable radio numbers lie within $[n, rn(P_n)]$, with both bounds strict. Section 3 demonstrates the existence of an unachievable radio number when n is odd.

2. The maximum achievable radio number

In this section we first give an upper bound for the radio number of any graph in terms of its diameter. This bound is sufficient to show that $rn(G) \leq rn(P_n)$ if $diam(G) \leq n-3$ or if $diam(G)$ is odd. In the second part of this section we show that if G has diameter $2k-2$ then its radio number is at most $rn(P_{2k})$ by giving a labeling with that span for all such graphs. The final theorem summarizes these results to show that $rn(G) \leq rn(P_n)$ whenever G has n vertices.

2.1 Upper bounds for radio numbers in terms of diameters

Let G' be a subgraph of a graph G. Chartrand et al give two conditions sufficient to ensure that $rn(G')$ be bounded above by $rn(G)$ (Theorems 2.3 and 2.4, [2]). We require a similar result regarding the radio number of a subgraph G' of G, where G and G' have the same vertex set and equal diameters.

Lemma 2.1. *Suppose G and G' are graphs satisfying $V(G') = V(G)$, $diam(G') = diam(G)$, and $E(G') \subset E(G)$. Then $rn(G') \leq rn(G)$.*

Proof. The hypotheses imply $d_G(u,v) \leq d_{G'}(u,v)$ for every pair of vertices $u, v \in V(G) = V(G')$. Let c be a minimal radio labeling of G (so $span(c) = rn(G)$). We have

$$\begin{aligned} \mathrm{diam}(G') + 1 &= \mathrm{diam}(G) + 1 \\ &\leq d_G(u,v) + |c(u) - c(v)| \\ &\leq d_{G'}(u,v) + |c(u) - c(v)|. \end{aligned}$$

for all distinct $u, v \in V(G')$. This implies that c is also a radio labeling of G'. Therefore $rn(G') \leq span(c) = rn(G)$. □

Our next lemma give an upper bound on $rn(G)$, where G is any graph of order n. (A statement analogous to this lemma is stated without proof as Theorem 3.2 in [2].)

Lemma 2.2. *Let G be a graph with $|V(G)| = n$ and $diam(G) = \delta$. Then*

$$rn(G) \leq \begin{cases} -\frac{\delta^2}{2} + (n-1)\delta + \frac{3}{2}, & \delta \text{ odd} \\ -\frac{\delta^2}{2} + (n-1)\delta + 3, & \delta \text{ even.} \end{cases}$$

Proof. Let c be a minimal radio labeling of $P_{\delta+1}$. Identify the vertices of a subgraph of G isomorphic to $P_{\delta+1}$ with the vertices of $P_{\delta+1}$ and assign the vertices of this subgraph the corresponding values of c. Note that the largest value thus assigned is $rn(P_{\delta+1})$. Assign $rn(P_{\delta+1}) + i\delta$, $i = 1, 2, \ldots, n - \delta - 1$, to the remaining vertices. This labeling of G satisfies the radio condition and has span $rn(P_{\delta+1}) + \delta(n - \delta - 1)$.

We know (from [8]) that

$$rn(P_m) = \begin{cases} \frac{(m-1)^2}{2} + \frac{3}{2}, & m \text{ even} \\ \frac{(m-1)^2}{2} + 3, & m \text{ odd.} \end{cases}$$

Substituting $rn(P_{\delta+1})$ into $rn(P_{\delta+1}) + \delta(n - \delta - 1)$ and simplifying leads to the conclusion. □

2.2 Upper bound for radio numbers when diameter is $2k - 2$

We first introduce a particular graph of order $2k$ and diameter $2k - 2$, and show the radio number of this graph does not exceed the radio number of P_{2k}. This graph is a special case of a class of graphs analyzed in Lemma 2.5.

Lemma 2.3. *Let H be the graph with vertex set $\{v_1, v_2, \ldots v_{2k}\}$ $(k > 1)$ and edge set $\{v_i v_{i+1} \,|\, i = 1, 2 \ldots, 2k-1\} \cup \{v_{k-1}v_{k+1}, v_k v_{k+2}\}$. Then $rn(H) \leq rn(P_{2k})$.*

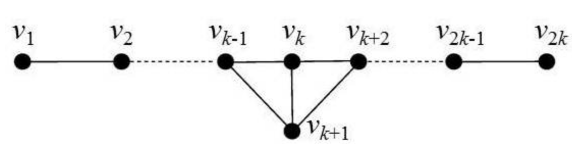

Figure 2: Graph H of Lemma 2.3

Proof. The diameter of H is $2k-2$; a path of length $\operatorname{diam}(H)$ is given by $v_1, v_2, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots v_{2k}$ (Figure 2). We wish to associate a labeling of P_{2k} with a labeling of H. Say $V(P_{2k}) = \{w_1, w_2, \ldots, w_{2k}\}$ and $E(P_{2k}) = \{w_i w_{i+1} \mid i = 1, 2 \ldots, 2k-1\}$. Define the labeling $c = c_P = c_H$ of both P_{2k} and H by

$$c_P(w_i) = c_H(v_i) = \begin{cases} (k-i)(2k-1)+1, & i \in \{1, 2, \ldots, k\} \\ (2k-i)(2k-1)+k+1, & i \in \{k+1, \ldots, 2k\}. \end{cases}$$

In fact, c is a different form of the minimal radio labeling of P_{2k} given in [8]. We claim that c is a radio labeling of H. As $span(c) = rn(P_{2k})$, proving the claim suffices to establish the lemma. Note that $\operatorname{diam}(H) = \operatorname{diam}(P_{2k}) - 1$.

Case 1. Say $\{u,v\} \subseteq \{v_1, v_2, ..., v_k\}$, $\{u,v\} \subseteq \{v_k, v_{k+2}, ..., v_{2k}\}$, or $\{u,v\} = \{v_k, v_{k+1}\}$. Then $d_H(u,v) = d_{P_{2k}}(u,v)$. This gives

$$\begin{aligned} d_H(u,v) + |c_H(u) - c_H(v)| &= d_{P_{2k}}(u,v) + |c_P(u) - c_P(v)| \\ &\geq \operatorname{diam}(P_{2k}) + 1 \\ &> \operatorname{diam}(H) + 1. \end{aligned}$$

Case 2. If $u \in \{v_1, v_2, ..., v_{k-1}\}$ and $v \in \{v_{k+1}, v_{k+2}, ..., v_{2k}\}$, then $d_H(u,v) = d_{P_{2k}}(u,v) - 1$. In this case we have

$$\begin{aligned} d_H(u,v) + |c(u) - c(v)| &= (d_{P_{2k}}(u,v) - 1) + |c(u) - c(v)| \\ &\geq (\operatorname{diam}(P_{2k}) - 1) + 1 \\ &= \operatorname{diam}(H) + 1. \end{aligned}$$

Thus c is a radio labeling of H. We conclude that $rn(H) \leq span(c_H) = span(c_P) = rn(P_{2k})$. □

We wish to show that all graphs of order $2k$ with diameter $2k-2$ have radio number not exceeding $rn(P_{2k})$. To this end, we define a family of graphs that includes the graph H of Lemma 2.3.

Definition 2.4. *Let $k \geq 2$. A triangle graph of order $2k$ has vertex set $\{v_1, v_2, \ldots v_{2k}\}$ and edge set $\{v_i v_{i+1} \mid i = 1, 2 \ldots, 2k-2\} \cup \{v_i v_{2k}, v_{i+1} v_{2k}, v_{i+2} v_{2k}\}$, where i is one of $\{1, 2, \ldots, k-1\}$.*

The triangle graph with $i = k-1$ is graph H of Lemma 2.3, with the vertices named in a different order. (See Figure 3.) The diameter of every triangle graph of order $2k$ is $2k-2$.

Lemma 2.5. *All triangle graphs of order $2k$ have radio numbers not exceeding $rn(P_{2k})$.*

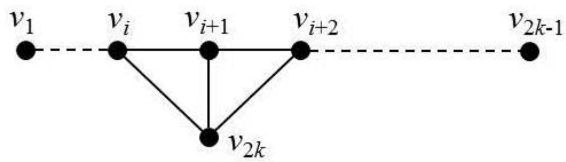

Figure 3: A triangle graph

Proof. Lemma 2.3 provides the desired conclusion when $i = k-1$ (and thus when $k = 2$).

Assume $1 \leq i \leq k-2$ and $k \geq 3$. Refer to the triangle graph under consideration as H'. Observe that the subgraph of H' induced by the vertices $\{v_1, v_2, \dots, v_{2k-1}\}$ is P_{2k-1}.

We provide c, the labeling of P_{2k-1} given in the proof of Theorem 3 in [8], in three steps. First, order the vertices of P_{2k-1} as follows:

$$v_{k-1}, v_{2k-2}, v_1, v_{2k-1}, v_3, v_{k+2}, v_4, v_{k+3}, v_5, v_{k+4}, \dots, v_{k-2}, v_{2k-3}, v_2, v_{k+1}.$$

Next, let x_i be the ith vertex in this list, i.e., $x_1 = v_{k-1}$, $x_2 = v_{2k-2}$, etc. Note that $x_{2k-2} = v_2$ and $x_{2k-1} = v_{k+1}$. Finally, define $c : V(P_{2k-1}) \to \mathbb{Z}_+$ by

$$c(x_i) = \begin{cases} 1, & i = 1 \\ c(x_{i-1}) + 2k - 1 - d(x_{i-1}, x_i), & i = 2, 3, \dots, 2k-1. \end{cases}$$

This gives a radio labeling of P_{2k-1} satisfying $span(c) = rn(P_{2k-1})$ [8].

We now define the labeling $c' : V(H') \to \mathbb{Z}_+$ by

$$c'(v_j) = \begin{cases} c(v_j), & j \in \{1, 2, \dots, 2k-1\} \\ span(c) + 2k - 3, & j = 2k, \end{cases}$$

We claim that c' is a radio labeling of H'. Consider three possibilities for $u, v \in V(H')$ $(u \neq v)$.

1. Say $\{u, v\} \subset \{v_1, \dots, v_{2k-1}\}$. Then the distance and label difference between u and v are the same as they are for the corresponding vertices in P_{2k-1}, so c' satisfies the radio condition on u and v.

2. Suppose $u = x_{2k-1}$ and $v = v_{2k}$. Then $d(u, v) \geq 2$ and as $c'(v) - c'(u) = 2k - 3$, we have $d(u, v) + |c(u) - c(v)| \geq 2k - 1 = diam(H') + 1$.

3. Say $u \in \{x_1, \dots, x_{2k-2}\}$ and $v = v_{2k}$. We have $c'(u) < c'(x_{2k-1}) < c'(v)$, so $c'(v) - c'(u) \geq 2k - 2$. Thus the radio condition is again satisfied.

Note that $span(c) = rn(P_{2k-1}) = 2k^2 - 4k + 5$. As $span(c') = span(c) + 2k - 3$, we have $rn(H') \leq span(c') = rn(P_{2k})$. $\square$

Lemma 2.6. *Let G be a graph of order $2k$ with diameter $2k-2$. Then $rn(G) \leq rn(P_{2k})$.*

Proof. Any graph G of diameter $2k-2$ has a subgraph P that is isomorphic to P_{2k-1}. If the graph is of order $2k$, there is exactly one additional vertex not on this subgraph. This vertex may not be adjacent solely to an endpoint of P, or the diameter of G would be greater than $2k-2$. Also, the vertex in question may not be simultaneously adjacent to two vertices u and v in $V(P)$ for which $d_P(u,v) \geq 2$, or the diameter of G would be less than $2k-2$. Thus G is either a triangle graph or G results from removing one or two edges incident to v_{2k} from a triangle graph in a manner that does not change the diameter. The result follows from Lemma 2.5 and Proposition 2.1. □

2.3 Upper bound for radio numbers

We now have the tools necessary to prove that the radio number of a graph of order n is bounded above by $rn(P_n)$.

Theorem 2.7. *If G is a graph of order n, then $rn(G) \leq rn(P_n)$.*

Proof. Fix n.

Case 1. Let n be odd.

If $\mathrm{diam}(G) = n-1$, then $G = P_n$, so $rn(G) = rn(P_n)$. The bound given for $rn(G)$ in Lemma 2.2 for δ even, a quadratic, is maximized when $\delta = n-1$. So $\max\{rn(G) \,|\, \mathrm{diam}(G) = n-1\} \geq \max\{rn(G) \,|\, \mathrm{diam}(G) < n-1\}$, thus $rn(G) \leq rn(P_n)$ when n is odd.

Case 2. Let n be even.

First examine graphs of diameter $n-3$ or less: using Lemma 2.2 we have

$$\begin{aligned}\max\{rn(G) \,|\, \mathrm{diam}(G) \leq n-3\} &\leq \frac{-(n-3)^2}{2} + (n-1)(n-3) + \frac{3}{2} \\ &= \frac{1}{2}(n-1)^2 - 1 \\ &< rn(P_n).\end{aligned}$$

To finish, consider graphs of diameter $n-2$ and $n-1$. If $\mathrm{diam}(G) = n-2$ then Lemma 2.6 shows $rn(G) \leq rn(P_n)$. Finally, $\mathrm{diam}(G) = n-1$ exactly when $G = P_n$. Thus for any possible diameter of G, $rn(G) \leq rn(P_n)$. □

We may now conclude that all achievable radio numbers of graphs of order n must lie in $[n, rn(P_n)]$, and that these bounds are best possible.

3. An unachievable radio number

Once the range for achievable radio numbers is determined, it is natural to ask whether any integers within that range are not achievable as radio numbers. Indeed, Theorem 3.1 shows there is at least one such integer.

Theorem 3.1. *When n is odd, the integer $rn(P_n) - 1 = \frac{(n-1)^2}{2} + 2$ is an unachievable radio number.*

Proof. Let n be odd. Assume there exists a graph G of order n satisfying $rn(G) = rn(P_n) - 1$. Consider the possible values of $\delta = \text{diam}(G)$.

Case 1. Say δ is odd.

Substituting $rn(G) = rn(P_n) - 1$ into Lemma 2.2 yields

$$\frac{(n-1)^2}{2} + 2 \leq -\frac{\delta^2}{2} + (n-1)\delta + \frac{3}{2},$$

which is equivalent to

$$0 \leq -n^2 + 2n - \delta^2 + 2\delta n - 2\delta - 2.$$

As δ may not exceed $n-1$, we write $\delta = n - a$ for some positive integer a. Substitute and simplify:

$$0 \leq -n^2 + 2n - (n-a)^2 + 2(n-a)n - 2(n-a) - 2$$
$$0 \leq -(a-1)^2 - 1.$$

There are no positive solutions for a, thus no graph G with odd diameter satisfying $rn(G) = rn(P_n) - 1$.

Case 2. Say δ is even.

Again, substitute $rn(G) = rn(P_n) - 1$ into Lemma 2.2, simplify, and substitute $\delta = n - a$:

$$\frac{(n-1)^2}{2} + 2 \leq -\frac{\delta^2}{2} + (n-1)\delta + 3$$
$$0 \leq -n^2 + 2n - \delta^2 + 2\delta n - 2\delta + 1$$
$$0 \leq -n^2 + 2n - (n-a)^2 + 2(n-a)n - 2(n-a) + 1$$
$$0 \leq -(a-1)^2 + 2$$

The only positive integer solutions for a are 1 and 2. That is, we need only consider $\delta \in \{n-1, n-2\}$. As δ is even while n is odd, $\delta \neq n-2$. But $\delta = n-1$ implies $G = P_n$, so $rn(G) = rn(P_n)$. Thus no graph G with even diameter has radio number $rn(P_n) - 1$. □

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