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Analyzing graphs by degrees

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Dedicated to the memory of Dr. B.D. Acharya

Abstract

Let $G = (V, E)$ be a graph and let $N(v) = \{u : uv \in E\}$ be the open neighborhood of a vertex $v \in V$. The *degree* of v, $deg(v) = |N(v)|$, equals the number of vertices u that are adjacent to v. By considering the relationships between $deg(v)$ and the degrees $deg(u)$, for every $u \in N(v)$, we define 10 types of vertices and study some of their properties.

Keywords: strong vertex, very strong vertex, regular vertex, typical vertex, very typical vertex, weak vertex, very weak vertex.

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1. Introduction

Let $G = (V, E)$ be a graph of *order* $n = |V|$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u|uv \in E\}$, while the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. Each vertex in $u \in N(v)$ is called a *neighbor* of v, and $|N(v)|$ is called the *degree* of v, and denoted $deg(v)$. The minimum and maximum degree of a vertex in a graph G are denoted $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively.

The *open neighborhood* of set $S \subseteq V$ of vertices is the set $N(S) = \bigcup_{v \in S} N(v)$. The *subgraph induced by a set* S is defined to be the graph $G[S] = (S, E \cap (S \times S))$. A set $S \subseteq V$ of vertices is called *independent* if no two vertices in S are adjacent.

Let the vertices in $V = \{v_1, v_2, \ldots, v_n\}$ be given in the order their degrees:

$$\delta = deg(v_1) \leq deg(v_2) \leq \ldots \leq deg(v_n) = \Delta.$$

We say that a graph G is *k-regular* if $deg(v) = k \geq 0$ for every vertex $v \in V$, that is, $\delta = deg(v) = \Delta = k$.

This paper is motivated by the notions of strong and weak vertices, introduced by Sampathkumar and Latha [4] and later defined by Kamath and Bhat [3] as follows.

Definition 1.1. *A vertex $u \in V$ in a graph $G = (V, E)$ is called strong if for every neighbor $v \in N(u)$, $deg(u) \geq deg(v)$, and vertex u is called weak if for every neighbor $v \in N(u)$, $deg(u) \leq deg(v)$.*

Notice that according to this definition every vertex in a k-regular graph is both strong and weak. Therefore, we refine and extend this basic definition as follows. Given a graph $G = (V, E)$, for each vertex $u \in V$, determine whether there exists a neighbor $v \in N(u)$ with $deg(u) < deg(v)$, $deg(u) = deg(v)$, or $deg(u) > deg(v)$. There are seven combinations. We can therefore give the following definitions.

Definition 1.2. *A vertex $u \in V$ in a graph $G = (V, E)$ is called:*

1. *very strong (VS) if $deg(u) \geq 2$ and for every vertex $v \in N(u)$, $deg(u) > deg(v)$.*
2. *strong (S) if $deg(u) \geq 2$, and for every vertex $v \in N(u)$, $deg(u) \geq deg(v)$, at least one neighbor $v \in N(u)$ has $deg(u) > deg(v)$, and at least one neighbor $w \in N(u)$ has $deg(u) = deg(w)$.*
3. *regular (R) if $deg(u) \geq 0$ and for every vertex $v \in N(u)$, $deg(u) = deg(v)$. Note: we assume that isolated vertices, those for which $deg(v) = 0$, are regular vertices.*
4. *very typical (VT) if $deg(u) \geq 2$ and for every vertex $v \in N(u)$, $deg(u) \neq deg(v)$, at least one neighbor $v \in N(u)$ has $deg(u) < deg(v)$, and at least one neighbor $w \in N(u)$ has $deg(u) > deg(w)$.*
5. *typical (T) if $deg(u) \geq 3$ and there are three distinct vertices $v, w, x \in N(u)$, $deg(v) < deg(u) = deg(x) < deg(w)$.*
6. *weak (W) if $deg(u) \geq 2$ and for every vertex $v \in N(u)$, $deg(u) \leq deg(v)$, at least one neighbor $v \in N(u)$ has $deg(u) < deg(v)$, and at least one neighbor $w \in N(u)$ has $deg(u) = deg(w)$.*
7. *very weak (VW) if $deg(u) \geq 1$ and for every vertex $v \in N(u)$, $deg(u) < deg(v)$.*

We can also make the following three definitions.

Definition 1.3. *A vertex u in a graph $G = (V, E)$ is called:*

1. *above average (AVG+) if $deg(u) > (\Sigma_{v \in N(u)} deg(v))/deg(u)$.*
2. *average (AVG) if $deg(u) = (\Sigma_{v \in N(u)} deg(v))/deg(u)$.*
3. *below average (AVG-) if $deg(u) < (\Sigma_{v \in N(u)} deg(v))/deg(u)$.*

In Figure 1 we illustrate these definitions with a tree having vertices of all seven types; we call such a graph *pantypical*.

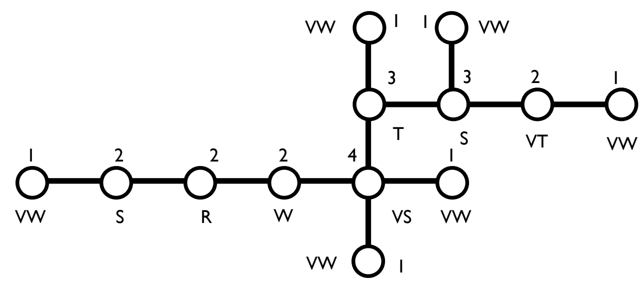

Figure 1: A Pantypical Tree

Let VS, S, R, VT, T, W, and VW denote the sets of vertices that are very strong, strong, regular, very typical, typical, weak and very weak, respectively. Because every vertex in a connected graph G of order $n \geq 2$ is of exactly one of these types, $\pi = \{VS, S, R, VT, T, W, VW\}$ is a partition of V. An immediate consequence of these definitions is the following result.

Proposition 1.4. *For any graph G, the sets VS and VW are independent sets.*

Notice that the definitions of above average, average and below average vertices are given in terms of the average of the degrees of the vertices in the open neighborhood of a vertex, rather than the average of the degrees of the vertices in the closed neighborhood of a vertex. It can be observed that the determination of a vertex being above average, average or below average will be the same using either open or closed neighborhoods, provided one changes the denominator to $deg(u) + 1$ when using closed neighborhoods.

We pose the following interesting open problem: for the random graph G, what is the expected number of vertices of each of the seven types $\{VS, S, R, VT, T, W, VW\}$?

2. Preliminary observations

Proposition 2.1. *Every connected graph G of order $n \geq 2$ is either k-regular, in which case all vertices are regular, or is not k-regular and has at least one vertex that is strong or very strong and at least one vertex that is weak or very weak.*

Proof. Assume that a connected graph G is not regular. Then there must exist at least two vertices r and s with $deg(r) = \delta < deg(s) = \Delta$. Let u be any vertex of minimum degree, $deg(u) = \delta$, and let z be any vertex of maximum degree, $deg(z) = \Delta$. Since G is a connected graph, there is a path $P(u, z)$ from u to z. It follows that since $deg(u) < deg(z)$, there must be two adjacent vertices v and w on this path with $deg(v) = \delta(G)$ and $deg(v) < deg(w)$. Therefore, vertex v must be either very weak or weak. Similarly, there must be two adjacent vertices x and y on this path with $deg(y) = \Delta(G)$ and $deg(x) < deg(y)$. Therefore, vertex y must be either strong or very strong. □

Corollary 2.2. *Every graph G of order $n \geq 2$, connected or not, contains at least one vertex u that is either very strong, strong or regular, and at least one vertex v, $v \neq u$, that is either very weak, weak or regular.*

Corollary 2.3. *Every connected graph that is not k-regular contains a vertex of maximum degree that is either strong or very strong, and contains a vertex of minimum degree that is either weak or very weak.*

Since the following results are easily proved, we omit the proofs.

Proposition 2.4. *In every graph G, (i) every (very) strong vertex is above average, (ii) every (very) weak vertex is below average, and (iii) every regular vertex is average.*

Proposition 2.5. *In any connected graph of order $n \geq 3$, every leaf is very weak and below average.*

Note that it is quite possible for a graph G to have regular vertices of different degrees.

Proposition 2.6. *If a graph G is k-regular, then all of its vertices are average, but not conversely. If a graph G is connected then G is k-regular if and only if every vertex is regular.*

Note that in a graph consisting of two connected components, one of which is k-regular and the other is j-regular, where $k \neq j$, every vertex is average.

Proposition 2.7. *In a connected graph G, not every vertex can be above average and not every vertex can be below average. Furthermore, every graph of order $n \geq 2$ that is not k-regular contains at least one above average vertex and at least one below average vertex.*

3. Extremal results

In this section, we consider the general question: for a given type of vertex, how many vertices of this type can there be in a graph G of order n?

3.1 The maximum number of very strong vertices

The complete bipartite graph $K_{r,r+1}$, of order $n = 2r+1$, contains r very strong vertices. Thus, $|VS(K_{r,r+1})| = \lfloor \frac{n-1}{2} \rfloor$. We prove that this is best possible.

Theorem 3.1. *Let G be any connected graph of order n. Then $|VS(G)| \leq \lfloor \frac{n-1}{2} \rfloor$.*

In order to prove this result we will need the following lemma. Let $I \subseteq VS$ be an arbitrary subset of very strong vertices in a graph $G = (V, E)$. Let $G_I = (I \cup N(I), E \cap (I \times N(I)))$ be the graph having vertex set $I \cup N(I)$ and all edges $uv \in E$ with $u \in I$ and $v \in N(I)$. Note that since I is a set of very strong vertices, it is an independent set. Thus, $I \cap N(I) = \emptyset$, and the graph G_I is a bipartite graph.

Lemma 3.2. *Let $G = (V, E)$ be a connected graph, let VS be the set of very strong vertices in G and let $I \subseteq VS$. Then if G_I is a complete bipartite graph of the form $K_{r,r+1}$, then $G = G_I = K_{r,r+1}$.*

Proof. Let G, VS, $I \subseteq VS$ and G_I be as defined and assume that $G_I = K_{r,r+1}$. Assume, to the contrary, that $G \neq G_I$. This can happen in two ways: either $V(G) \neq V(G_I)$ or $V(G) = V(G_I)$ but $E(G) \neq E(G_I)$.

Assume first that $V(G) \neq V(G_I)$. Since G is connected, and since $V(G_I) = I \cup N(I)$, there must be a vertex $w \in V - V(G_I)$ which is adjacent to some vertex $v \in N(I)$; w can't be adjacent to a vertex in I, else it would be in $N(I)$. But this is a contradiction, since for every $u \in I$ and $v \in N(I)$, $deg(u) = r + 1 > deg(v) = r$. But if v is also adjacent to a vertex $w \notin V(G_I)$, then $deg(v) \geq deg(u)$ in G and therefore u is not a very strong vertex in G.

Assume, therefore, that $V(G) = V(G_I)$ but $E(G) \neq E(G_I)$. Since I is an independent set, this means that there must be two vertices in $N(I)$, say v and w, that are adjacent. But this too is a contradiction, since it implies that $deg(v) \geq deg(u)$ for all vertices $u \in I$, and therefore u is not a very strong vertex in G. □

Theorem 3.3. *Let $G = (V, E)$ be a connected graph of odd order $n = 2r + 1$, and let $\emptyset \neq I \subseteq VS(G)$. Then*

1. *$|I| < |N(I)|$.*
2. *If $|I| = |N(I)| - 1$, then $G = G_I = K_{r,r+1}$.*

Proof. Assume that $|I| = 1$, that is, $I = \{x\}$, where x is a very strong vertex. This means that $deg(x) = |N(x)| \geq 2$, and therefore $|I| < |N(I)|$. Additionally, if $1 = |I| = |N(I)| - 1$, then $|N(I)| = 2$ and $G_I = K_{1,2}$ and, by Lemma 3.2, $G = G_I$. Thus, the theorem is true for $|I| = 1$.

Assume that the theorem is not true for some $I \subseteq VS$ with $|I| \geq 2$. Let I be a smallest cardinality counterexample. Therefore, we may assume that the theorem fails for I but holds for every proper subset of I.

Assume that $|I| \geq |N(I)|$, let $x \in I$ and consider the set $I' = I - \{x\}$. Since the theorem is true for I', we know that $|I'| < |N(I')|$, or $|N(I')| \geq |I'| + 1 = |I|$, and therefore,

$$|I| \geq |N(I)| \geq |N(I')| \geq |I|.$$

Therefore, $|I| = |N(I)| = |N(I')|$. Thus, in reducing I to I', we do not reduce the size of the neighborhood $N(I)$. But we also know that

$$|I'| = |I| - 1 = |N(I')| - 1.$$

Since the theorem is true for I', we know that $G_{I'}$ is a complete bipartite graph of the form $K_{r,r+1}$ and $G = G_{I'} = K_{r,r+1}$, which contradicts our assumption that x is a vertex not in $G_{I'}$.

Therefore, the set I' must satisfy Condition 1 but fail to satisfy Condition 2. Assume therefore that $|I| = |N(I)| - 1$, but $G \neq G_I = K_{r,r+1}$. Assume that $|I| = k \geq 2$ and $|N(I)| = k + 1$.

As in Figure 2, arrange the vertices of I and $N(I)$ by their degrees, from the smallest up to the largest in two vertical columns. Let the degrees of the vertices in I be $\delta_1 \leq \delta_2 \leq \ldots \leq \delta_k$, and let the vertices in $N(I)$ be $d_1 \leq d_2 \leq \ldots \leq d_k \leq d_{k+1}$. We will show that the degree sequence $\{d_i\}$ imposes some minimal conditions on the degree sequence $\{\delta_i\}$, namely that

$$\delta_i \geq d_{i+2} + 1, \text{ for } 1 \leq i \leq k - 1, \text{ and } \delta_k = d_{k+1} + 1.$$

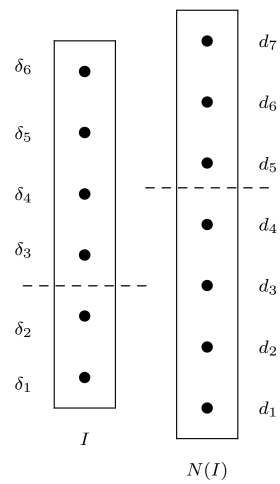

Figure 2: Arrangement by Degrees

For $i \geq 2$, let D_i denote the i vertices of $N(I)$ that have degree labels d_1 through d_i. Let

$$I_i = \{x \in I : N(x) \subseteq D_i\}.$$

Thus, a very strong vertex x is in I_i only if all of its neighbors are among the vertices of D_i. Since D_i has i elements and since I_i is a proper subset of I, by Conditions 1 and 2 of the theorem, the number of vertices in I_i cannot be more than $i-2$. Therefore, the number of elements in $I - I_i$ must be at the very least $k-(i-2) = k+2-i$.

In other words, at least $k+2-i$ vertices are adjacent to a vertex with degree at least d_{i+1}. Any such vertex, being very strong, must have degree at least $d_{i+1}+1$. This means that the top $k+2-i$ vertices in the I column must each have degree at least $d_{i+1}+1$. In particular, counting down from the top of the I column, we have $\delta_{i-1} \geq d_{i+1}+1$. Notice that when $i = k$ we obtain $\delta_{k-1} \geq d_{k+1}+1$ and $\delta_k \geq d_{k+1}+1$.

Consider therefore the difference between the sum of the degrees of the vertices in I and the sum of the degrees of the vertices in $N(I)$.

$$(\delta_1 + \delta_2 + \ldots + \delta_k) - (d_1 + d_2 + \ldots + d_{k-1} + d_k + d_{k+1})$$

$$\geq (d_3+1) + (d_4+1) + \ldots + (d_{k+1}+1) + (d_{k+1}+1) - d_1 - d_2 - \ldots d_{k-1} - d_k - d_{k+1}$$

$$\geq (k - d_2) + (d_{k+1} - d_1).$$

We now claim that the assumption that G_I is not a complete bipartite graph implies that $k - d_2 + d_{k_1} - d_1 > 0$, which contradicts the fact the the sum of the degrees of the vertices in I must equal the sum of the degrees of the vertices in $N(I)$. Note that $d_{k+1} - d_1 \geq 0$ and $k - d_2 \geq 0$. There are two possibilities. If $d_{k+1} - d_1 > 0$ then we proved the claim. If, on the other hand, $d_{k+1} - d_1 = 0$, then all of the vertices in $N(I)$ must have the same degree, and because we have assumed that G_I is not a complete bipartite graph, this common degree must be strictly less than k, the maximum possible degree. Hence, $k - d_2 > 0$, and the claim is proved. Thus, the assumption that Condition 1 holds, but Condition 2 fails, leads to a contradiction, and the theorem is proved. □

3.2 The maximum number of strong (and above average) vertices

Recall from the definition of a strong vertex that if a vertex v is strong, then it must have at least one neighbor w with $deg(v) > deg(w)$ (in which case we say that w is a *weak neighbor of* v), and at least one neighbor u with $deg(v) = deg(u)$.

Theorem 3.4. *If a graph $G = (V, E)$ has a strong vertex, then it must have at least two vertices that are not strong.*

Proof. Assume that graph $G = (V, E)$ contains a strong vertex $v \in V$. Let w be a weak neighbor of v, i.e. $deg(v) > deg(w)$, and let u be a neighbor of v with $deg(v) = deg(u)$. Clearly, vertex w is not strong. If u is not a strong vertex, then the theorem in proved, so assume that vertex u is a strong vertex, and assume further that all neighbors of v not equal to w are strong vertices, else the theorem is proved. Let this set be denoted $S(v)$. Since all vertices in $S(v)$ are strong, each one of them has a weak neighbor. If any of these weak neighbors is not equal to w then the theorem is proved. Therefore, assume that each vertex in $S(v)$ has exactly one weak neighbor and that vertex is w.

But this is a contradiction, since in this case $deg(w) \geq deg(v)$. Therefore, if a graph G has a strong vertex v, then it must have at least two vertices that are not strong. □

Corollary 3.5. *The maximum number of strong vertices in a connected graph G of order n is $n - 2$.*

There are many ways to construct a graph having $n - 2$ strong vertices, the smallest of which is the path P_4. In this case the middle two vertices, of degree two, are both strong, while the two leaves are very weak. Another example is provided by a complete graph K_n, from which a single edge uv is deleted. This creates two very weak vertices, u and v, while the other $n - 2$ vertices are strong. A final example consists of a complete graph K_{2n} of order $2n$ for any $n \geq 2$. To this complete graph add two nonadjacent vertices x and y, and join x by an edge to half of the vertices of the vertices in the K_{2n}, and join y to the other half of the vertices in the K_{2n}.

Corollary 3.6. *The maximum number of above average vertices in a connected graph G of order n is $n - 2$.*

3.3 The maximum number of regular (and average) vertices

The following result is an immediate consequence of the definition of regular vertices.

Proposition 3.7. *The maximum number of regular vertices in a connected graph G of order n is n.*

Corollary 3.8. *The maximum number of average vertices in a connected graph G of order n is n.*

3.4 The maximum number of very typical and typical vertices

Define a graph G to be *very typical (typical)* if every vertex is very typical (typical); otherwise it is *avtypical (atypical)*. We also define a graph to be *k-avtypical (k-atypical* if it has at most k vertices that are not very typical (typical) vertices.

Proposition 3.9. *No (finite) graph is very typical, 1-avtypical, typical or 1-atypical.*

Proof. No vertex of minimum degree can be very typical or typical and no vertex of maximum degree can be very typical or typical. □

Theorem 3.10. *There exist infinitely many 2-avtypical graphs.*

Proof. Let G denote the complete multipartite graph with partite sets of orders: $1, 3, 5, 7, 9, \ldots, 2k+1$. To this graph add one additional vertex x that is adjacent to every vertex in the partite set of order $2k+1$. It can be seen that the resulting graph has $n-2$ very typical vertices, and two vertices that are not very typical, namely the vertex x, having minimum degree, and the single vertex in the partite set of order 1, having maximum degree. □

Corollary 3.11. *The maximum number of very typical vertices in a connected graph G of order n is $n-2$.*

Theorem 3.12. *There exist infinitely many 2-atypical graphs.*

Proof. Let G denote the complete multipartite graph with partite sets of orders: $1, 4, 6, 8, 10, \ldots, 2k$. To this graph add one additional vertex x that is adjacent to every vertex in the partite set of order $2k$. In addition, add the edges of a perfect matching to every partite set of even order.

It can be seen that the resulting graph has $n-2$ typical vertices, and two vertices that are not typical, namely the vertex x and the single vertex in the partite set of order 1, call it y. The vertices in every partite set have neighbors of larger degree, the same degree (in the same partite set) and smaller degree. The vertex x has minimum degree, while the vertex y has maximum degree in the resulting graph. □

3.5 The maximum number of weak (and below average) vertices

Proposition 3.13. *The maximum number of weak vertices in a connected graph G of order n is $n-1$.*

Proof. The wheel graph $W_n = C_{n-1} + \{w\}$ has $n-1$ weak vertices (namely all the vertices of C_{n-1}). Clearly this is best possible, as not all vertices in a connected graph can be weak. □

Corollary 3.14. *The maximum number of below average vertices in a connected graph G of order n is $n-1$.*

3.6 The maximum number of very weak vertices

Proposition 3.15. *The maximum number of very weak vertices in a connected graph G of order n is $n-1$.*

Proof. Every very weak vertex u in a graph G must have at least one neighbor v whose degree is strictly greater than the degree of u. Such a vertex v therefore cannot itself be very weak. Thus, the maximum number of very weak vertices in a graph G of order n is $n-1$, and this is achieved by the star graph $K_{1,n}$. We have previously observed that the set of very weak vertices in a graph G must be an independent set. Therefore, this graph $K_{1,n}$ is unique. □

The following table summarizes the maximum number of vertices of a given type in a graph G of order n. We leave as open problems the determination of five of these maximum numbers in trees. In the table, we indicate with a question mark our conjectured maximum numbers for these types of vertices. We provide the following result for trees.

Theorem 3.16. *The maximum number of very strong vertices in a tree T of order $n \geq 4$ is $\lfloor (n-1)/3 \rfloor$ and this bound is sharp.*

Proof. Let $T = (V, E)$ be a tree of order $n \geq 4$, and let T_r denote the tree obtained by rooting T at any leaf $r \in V$. Let $VS \subset V$ be the set of very strong vertices in T. By Proposition 2.5, since we know that every leaf is very weak, we know that $r \notin VS$, so consider the remaining $(n-1)$ vertices in T.

Let $v \in VS$, let u be the parent (immediate ancestor) of v, and let w be a child (immediate descendant) of v. Since, by Proposition 3.2, VS is an independent set, we know that $u, w \notin VS$, and since v is a very strong vertex, we know that $deg(v) > deg(u)$ and $deg(v) > deg(w)$.

Case 1. $u = r$.

If $deg(u) = 1$, then since $n \geq 4$, vertex v must have at least two children. If, however, $deg(u) \geq 2$, then as well $deg(v) > deg(u) \geq 2$, and so vertex v has at least two children.

Case 2. $u \neq r$.

In this case $deg(u) \geq 2$, and again it follows that vertex v has at least two children.

Thus, in all cases, every vertex $v \in VS$ has at least two children in T_r, neither of which is in VS. Therefore, $|VS| \leq \lfloor (n-1)/3 \rfloor$.

The bound $|VS| \leq \lfloor (n-1)/3 \rfloor$ is achieved by a tree T consisting of a path $v_1, v_2, \ldots, v_{2n+1}$ together with leaves $u_2, u_4, \ldots, u_{2n}$ adjacent to vertices $v_2, v_4, \ldots, v_{2n}$, respectively. □

Vertex Type	**Maximum**	**Maximum in Trees**
1. Very strong	$\lfloor (n-1)/2 \rfloor$	$\lfloor (n-1)/3 \rfloor$
2. Strong (and Above Average)	$n-2$	$(n-2)/2$?
3. Regular (and Average)	n	$n-4, P_n$?
4. Very typical	$n-2$	$(n-1)/2$, $S(K_{1,n})$?
5. Typical	$n-2$	$2(n-1)/11$?
6. Weak (and Below Average)	$n-1$	$(n-4)/2$?
7. Very Weak	$n-1$	$n-1, K_{1,n}$

4. Subgraphs by degrees

Given a graph $G = (V, E)$ of order n, let $G[i]$ denote the subgraph induced by the set of vertices u having $deg(u) = i$, for $0 \leq i \leq n-1$. We say that $G[i]$ is a *degree graph*. This raises the question: is every graph G a degree graph of some graph H?

Theorem 4.1. *Every connected graph G of order $n \geq 2$ is a degree graph of some graph H.*

Proof. Assume that we are given an arbitrary connected graph G of order $n \geq 2$, and assume that $\Delta = k$ for some positive integer $1 \leq k \leq n-1$. To each vertex $v \in V$, attach $\Delta - deg(v)$ leaves. Call the resulting graph H. It follows by this construction that every vertex in the graph G becomes a vertex of degree Δ in the graph H, while every other vertex in H is a leaf, of degree 1. Therefore, $G[k] \simeq G$. □

Consider the family of degree graphs $\{G[0], G[1], G[2], \ldots, G[n-1]\}$. What can you say about this family of graphs? For example, since every graph must have at least two vertices of the same degree [1], it follows that for every graph G there must exist at least one integer i such that $G[i] = \emptyset$.

Proposition 4.2. *If G is a k-regular graph, then $G[k] \simeq G$, and for all j, $j \neq k$, $G[j] \simeq \emptyset$.*

A set $S \subset V$ of vertices is called *k-dependent* if $\Delta(G[S]) \leq k$.

Proposition 4.3. *In any graph G, for every k, $1 \leq k \leq \Delta(G)$, the vertices $V(k)$ in the subgraph $G[k]$ form a k-dependent set.*

Thus, for example, the vertices in $G[0]$ form an independent set, that is, a set of isolated vertices; the vertices in $G[1]$ form a 1-dependent set, that is, a collection of isolated vertices and K_2's; and the vertices in $G[2]$ form a 2-dependent set, that is a collection of isolates, K_2's, paths, and cycles.

Consider the general question: how much information about a graph G is contained in the degree graphs, $\{G[0], G[1], G[2], \ldots, G[n-1]\}$? In general, some of these degree graphs can be empty. Thus, let $d_1 \leq d_2 \leq \ldots d_n$ be the degree sequence of a graph G. Note that this information can be obtained from the information in the degree graphs $\{G[0], G[1], G[2], \ldots, G[n-1]\}$. How much more information about G can be inferred from the degree graphs than from the degree sequence?

For example, it is well known that two non-isomorphic graphs can have the same degree sequence. The trees in Figure 3 show that two non-isomorphic trees can have the same degree sequence and the same degree graphs.

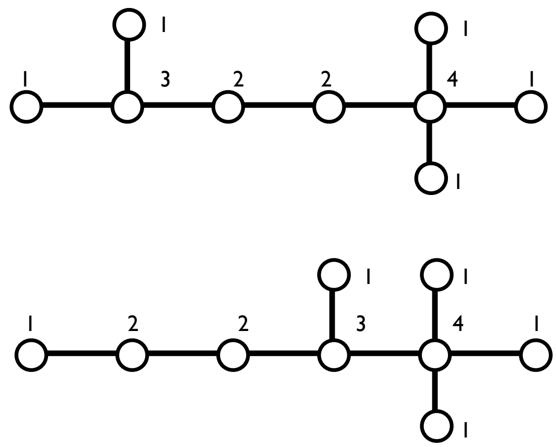

Figure 3: Non-isomorphic trees with the same degree graphs

5. Subgraphs by types

A similar study of subgraphs can be conducted using the previously mentioned seven types of vertices. Therefore, consider the seven induced subgraphs: $G[VS]$, $G[S]$, $G[R]$, $G[VT]$, $G[T]$, $G[W]$, and $G[VW]$. Let us refer to these as *type graphs*. Let us say that a graph G is a *VS-graph, S-graph, R-graph, VT-graph, T-graph, W-graph, or VW-graph* if there is a graph H, such that $H[VS] \simeq G$, $H[S] \simeq G$, $H[R] \simeq G$, $H[VT] \simeq G$, $H[T] \simeq G$, $H[W] \simeq G$, or $H[VW] \simeq G$, respectively. Thus, we are led to ask: given any graph G, is it a type graph of a given kind?

Since, for any graph G, the sets $VS(G)$ and $VW(G)$ are always independent sets, we have the following result.

Proposition 5.1. *The only graphs G that are VS-graphs or VW-graphs are those of the form $\overline{K_n}$.*

The construction used in the proof of Theorem 4.1 can be modified to produce the following results.

Theorem 5.2. *For every connected graph G of order $n \geq 2$, there exists a graph H such that $H[R] \simeq G$.*

Proof. Assume that we are given an arbitrary connected graph G of order $n \geq 2$, and assume that $\Delta(G) = k$ for some positive integer $1 \leq k \leq n-1$. To each vertex $v \in V$, attach $\Delta(G) - deg(v)$ leaves. Call the resulting graph H. It follows by this construction that every vertex in the graph G becomes a vertex of degree $\Delta(G)$ in the graph H, while every other vertex in H is a leaf, of degree 1. To each leaf x added in this construction, attach $k-1$ leaves, making $deg(x) = k$. Denote the resulting graph by H'. In H' all vertices v originally in G have degree k and every neighbor of v in H' also has degree k. Every vertex added to G to construct H' is either a leaf, and is very weak, or is a vertex of degree k that is adjacent to a leaf, and is strong. Therefore, $H'[R] \simeq G$. □

Theorem 5.3. *For every connected graph G of order $n \geq 2$, there exists a graph H such that $H[S] \simeq G$.*

Proof. Similar to the construction used in the proof of Theorem 4.1, assume that we are given an arbitrary connected graph G of order $n \geq 2$, and assume that $\Delta(G) = k$ for some positive integer $1 \leq k \leq n-1$. To each vertex $v \in V$, attach $\Delta(G) - deg(v) + 1$ leaves. Call the resulting graph H. It follows by this construction that every vertex in the graph G becomes a vertex of degree $\Delta(G) + 1$ in the graph H, while every other vertex in H is a leaf, of degree 1. Therefore, every vertex v in G becomes a strong vertex in H, while all leaves in H are very weak. Therefore, $H[S] \simeq G$. □

Theorem 5.4. *For every connected graph G of order $n \geq 2$, there exists a graph H such that $H[W] \simeq G$.*

Proof. Let H be the graph constructed in the proof of Theorem 5.3, where $\Delta(G) = k$. To every leaf in H attach $k+2$ leaves. Denote the resulting graph by H". In H" every vertex v in G has $deg(v) = k+1$. Every neighbor of v in G also has degree $k+1$. All other neighbors of v in H" have degree $k+3$. Therefore, all vertices in G are weak in H". The vertices of degree $k+3$ in H" are very strong, while all leaves in H" are very weak. Therefore, H"$[W] \simeq G$. □

Theorem 5.5. *For every connected graph G of order $n \geq 2$, there exists a graph H such that $H[T] \simeq G$.*

Proof. Let H" be the graph constructed in the proof of Theorem 5.4, in which all vertices in the original graph G have degree $k+1$ and are weak. To every vertex in G, attach a single leaf in H", and denote the resulting graph H^*. This gives every vertex in G a very weak neighbor in H^*, and thereby, every vertex in G has degree $k+2$ in the new graph

H^* and is typical. All leaves in H^* are very weak, and all vertices of degree $k+3$ are very strong. Therefore, $H^*[T] \simeq G$. □

Theorem 5.6. *For every connected graph G of order $n \geq 2$, there exists a graph H such that $H[VT] \simeq G$.*

Proof. Assume that we are given an arbitrary connected graph G of order $n \geq 2$, and assume that $\Delta(G) = k$ for some positive integer $1 \leq k \leq n-1$. Let the vertices $v_1, v_2, \ldots, v_n$ be labelled in ascending order by their degrees, so that $d_1 \leq d_2 \leq \ldots \leq d_n$. To every vertex v_i, $1 \leq i \leq n$, attach $k+i$ leaves, and then to exactly one of these attached leaves, attach $2n$ leaves. Let H denote the resulting graph. Notice that no two vertices in G have the same degree in H, since for $i < j$, $deg(v_i) = d_i + k + i < deg(v_j) = d_j + k + j$. Notice also that the vertices adjacent to v_i in H, other than those to which it is adjacent in G, are either leaves, or one vertex of degree $2n+1$ and $deg(v_i) < 2n+1$. Therefore, each vertex v_i in G is a very typical vertex in H. All other vertices in H are either leaves or are very strong. Therefore, $H[VT] \simeq G$. □

Thus, we have shown that every connected graph G of order $n \geq 2$ can be an S-graph, an R-graph, a VT-graph, a T-graph or a W-graph in some graph H. However, the only graphs that can be a VS-graph or a VW-graph are those of the form $\overline{K_n}$.

6. Coloring graphs by degrees

As we noted earlier, the vertices of any graph G are partitioned into seven (or fewer) sets according to the partition $\pi = \{VS, S, R, VT, T, W, VW\}$. If each of these seven sets is an independent set, then π is a proper coloring of G with seven or fewer colors. This leads us to the following, somewhat unexpected result.

Theorem 6.1. *If the chromatic number of a graph G satisfies $\chi(G) \geq 8$, then G must have two adjacent vertices of at least one of the five types, S, R, VT, T, or W.*

Proof. Let G be a graph for which $\chi(G) \geq 8$. Partition the vertices according to the seven types, $\pi = \{VS, S, R, VT, T, W, VW\}$. Since $\chi(G) \geq 8$, at least one of these seven sets cannot be an independent set, else we have colored G with 7 or fewer colors. But the sets VS and VW are always independent sets. Thus, at least one of the five sets S, R, VT, T, or W must not be an independent set, and therefore there must be two adjacent vertices of one of these five types. □

Corollary 6.2. *Let G be a graph having chromatic number $\chi(G) \geq 7$. Then the subgraph $G[S, R, VT, T, W]$ induced by the vertices of types S, R, VT, T, and W is non-planar.*

Proof. If the induced subgraph $G[S, R, VT, T, W]$ is planar, then by the Four Color Theorem, it has chromatic number at most four. Therefore, $\chi(G) \leq 6$, since the two sets VS and VW are independent, a contradiction. □

Corollary 6.3. *If the vertices of a graph G have only k distinct non-zero degrees, and the chromatic number of G is greater than k, then G must have two adjacent vertices of the same degree.*

Corollary 6.4. *If a graph G has only k of the seven types of vertices, and a chromatic number greater than k, then it must contain two adjacent vertices of the same type.*

7. Open problems

1. For the random graph G of order n, what is the expected number of vertices of each of the: seven types: VS, R, S, VW, VT, W, T, and each of the three types: AVG+, AVG, AVG-?

2. Define a graph G to be *above average* if more than half of its vertices are above average. What can you say about above average graphs?

3. Define a graph G to be *pantypical* if it has vertices of all seven types: VS, S, R, VT, T, W, VW. What can you say about pantypical graphs? In particular, what is the probability that the random graph is pantypical? Does every pantypical graph have at least two very weak vertices?

4. What is the smallest order of a pantypical graph? The pantypical tree in Figure 1 has order $n = 13$. It can be shown that this is an example of a pantypical tree of smallest order. The graph in Figure 4 is a panytypical graph of order $n = 9$ that was found by a computer search. We also generated 1.5 million graphs of order $n = 8$ but failed to find a pantypical graph of order $n = 8$.

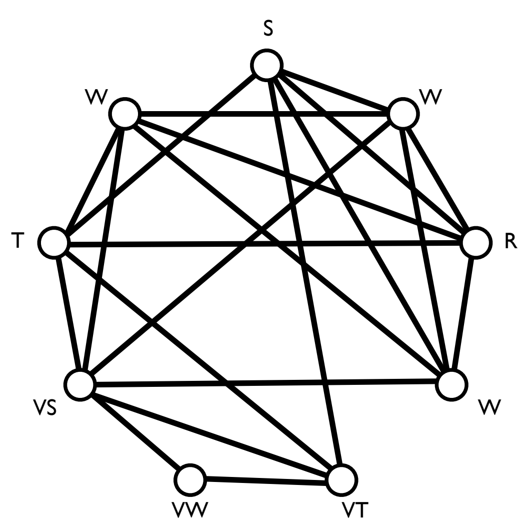

Figure 4: A Pantypical Graph

5. What is the smallest order of a graph having $n-2$ very typical vertices or $n-2$ typical vertices?

6. What are the extremal values for the maximum number of vertices of each of the seven types in a tree T?

7. Can you characterize the class of graphs G, or trees T, having no two adjacent vertices of the same type?

8. How many non-isomorphic degree graphs can a graph of order n have?

9. How many non-isomorphic degree graphs can a tree of order n have?

10. How many non-empty, isomorphic degree graphs can a tree of order n have?

11. Given a set of degree graphs $\{G[0], G[1], G[2], \ldots, G[k]\}$, where $G[i] = \emptyset$ is possible, does there exist a graph G whose degree graphs are $\{G[0], G[1], G[2], \ldots, G[k]\}$? In other words, is the given set of degree graphs *graphical*?

12. Given a set of degree graph forests $\{F[1], F[2], \ldots, F[k]\}$, where $F[i] = \emptyset$ is possible, does there exist a tree T whose degree graph forests are $\{F[1], F[2], \ldots, F[k]\}$?

13. How much information about a graph G can be inferred from its degree graphs:

$$\{G[0], G[1], G[2], \ldots, G[n-1]\}?$$

14. How much information about a graph G can be inferred from its type graphs:

$$\{G[VS], G[R], G[S], G[VW], G[VT], G[W], G[T]\}?$$

    For example, we know that there are no edges between vertices in $G[VS]$, or between vertices in $G[VW]$, there is no edge between a vertex in $G[VS]$ and either $G[R]$ or $G[S]$, and there is no edge between a vertex in $G[VW]$ and either $G[W]$ or $G[R]$.

15. Given $\{G[VS], G[R], G[S], G[VW], G[VT], G[W], G[T]\}$, can you tell if G can be a tree?

16. Can the degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$ of a graph G be determined from its seven type graphs: $G[VS], G[S], G[R], G[VT], G[T], G[W], G[VW]$?

17. The following six types of vertices can be defined. What properties do these types of vertices have?

    (a) *globally above average (GAVG+)* if $deg(u) > (\Sigma_{v \in V} deg(v))/n = 2m/n$.
    (b) *globally average(GAVG)* if $deg(u) = 2m/n$.
    (c) *globally below average (GAVG-)* if $deg(u) < 2m/n$.

(d) *a median vertex (MED)* if the number of its neighbors with a smaller degree equals the number of its neighbors with a larger degree.

(e) *a major vertex (MAJ)* if the number of its neighbors with a smaller degree is greater than the number of its neighbors with a larger degree.

(f) *a minor vertex (MIN)* if the number of its neighbors with a larger degree is greater than the number of its neighbors with a smaller degree.

Added in proof. In a recent paper [2] sharp upper bounds on the maximum numbers of several types of vertices that can appear in a tree have been determined.

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