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Degree/diameter problem for trees and pseudotrees

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Abstract

The *degree/diameter problem* asks: Given natural numbers d and k what is the order (that is, the maximum number of vertices) $n_{d,k}$ that can be contained in a graph of maximum degree d and diameter at most k? The degree/diameter problem is wide open for most values of d and k. A general upper bound exists; it is called the *Moore bound*. Graphs whose order attains the *Moore bound* are called *Moore graphs*. Since the degree/diameter problem is considered to be very difficult in general, it is worthwhile to consider it for special classes of graphs. In this paper we consider the degree/diameter problem on trees, special types of trees such as Cayley trees, caterpillars, lobsters, banana trees and firecracker trees, as well as for tree-like structures such as pseudotrees. We obtain new $n_{d,k}$ values and provide corresponding constructions.

Keywords: degree/diameter problem, trees, pseudotrees.

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1. Introduction

Graphs arise in many areas of mathematics and computer science having applications in many other fields as well. Extremal graph theory problems usually ask for the maximum or minimum size or order of a graph having certain characteristics. Such questions are often quite natural in the construction of networks or circuits.

In this paper we consider the degree/diameter problem which asks: What is the maximum number of vertices $n_{d,k}$ that can be contained in a graph of maximum degree d and diameter at most k? Moore[14] initially posed the problem and introduced the upper bound called the *Moore bound*. Graphs that attain the *Moore bound* are called *Moore graphs.*

For most values of d and k, the degree/diameter problem is open; we only have general upper and lower bounds for the values of $n_{d,k}$. In an effort to tighten the gaps between the upper and lower bounds, research activities related to the degree/diameter problem fall into two main streams. On the one hand there are proofs of non-existence of graphs of order equal to the current best upper bounds, thereby improving (lowering) the upper bounds[2, 11, 16]. On the other hand, there is a great deal of activity in the constructions of large graphs, furnishing better lower bounds on $n_{d,k}$[6, 10, 18].

The study of Moore graphs was initiated by Hoffman and Singleton[14], their pioneering paper was devoted to Moore graphs of diameter 2 and 3. In the case of diameter 2, they proved that Moore graphs exist for $d \in \{2, 3, 7\}$ and possibly 57 but for no other degrees, and that for the first three values of d the graphs are unique. For diameter 3 they showed that the unique Moore graph is the heptagon (for $d = 2$). The proofs exploited eigenvalues and eigenvectors of the adjacency matrix (and its principal submatrices) of graphs.

It turns out that no Moore graphs exist for $d \geq 3$ and $k \geq 3$. This was shown by Damerell [9] by way of an application of his theory of distance-regular graphs to the classification of Moore graphs. An independent proof of this result was also given by Bannai and Ito [1].

The main results concerning Moore graphs can be summed up as follows. Moore graphs for diameter $k = 1$ and maximum degree $d \geq 1$ are the complete graphs K_{d+1}. For diameter $k = 2$, Moore graphs are the cycle C_5 for degree $d = 2$, the Petersen graph for degree $= 3$ and the Hoffman-Singleton graph for degree $d = 7$. The existence or otherwise of a Moore graph of degree 57 and diameter 2 is still unknown. Finally, for diameter $k \geq 3$ and degree $d = 2$, Moore graphs are the cycles on $2k + 1$ vertices C_{2k+1}.

Since the general degree/diameter problem is difficult, research has been also concentrated on various related problems. These include studies of the degree/diameter problem for special types of graphs such as Cayley [8, 21], planar [12, 13, 20], bipartite [3, 7], directed [4, 5, 15] and toroidal [19] graphs (for a general survey of the degree/diameter problem see [17]). Several other areas of research in graph theory turn out to be related or inspired by the theory of Moore graphs; examples include cages, antipodal graphs, Moore geometries and Moore groups. Recall that a $(k; g)$-*cage* is a graph of degree k and girth

g, with the minimum possible number of vertices. Connections between cages and Moore graphs are explained in a survey paper on cages by Wong [22].

In this paper we consider the degree/diameter problem on trees, special types of trees such as Cayley trees, caterpillars, lobsters, banana trees and firecracker trees, as well as for tree-like structures such as pseudotrees, giving the extremal numbers and constructions.

The rest of this paper is structured as follows. In Section 2 we present the basic definitions used throughout the paper and we define the problems. In Section 3 we consider the degree/diameter problem for special types of graphs. Finally, we give some future proposals and a brief conclusion in Section 4.

2. Definitions and problems

Throughout this paper we consider an undirected graph $G(V, E)$, where V is the set of *vertices*, also called *nodes*, and E is the set of *edges*. The *complement* graph $\overline{G}(V, \overline{E})$ of G has the same vertices as G but edges that appear in G do not appear in $\overline{G}$ and edges that do not appear in G appear in $\overline{G}$. The *order* of a graph is the number of its vertices. The *size* of a graph is the number of its edges. A *path* $P_n = P_n(V, E)$ is a graph with $V = \{x_1, x_2, \ldots, x_n\}$ and $E = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$. The end vertices of P_n are x_1 and x_n and the *length* of P_n is equal to $n-1$. The *diameter* of a graph is the length of a longest shortest path between any two vertices of the graph. The *eccentricity* $\epsilon(v)$ of a vertex v in a connected graph G is the maximum distance between v and any other vertex u of G. For a disconnected graph, all vertices are defined to have infinite eccentricity. Note that the maximum eccentricity over all vertices of a graph is the diameter of the graph. The minimum eccentricity over all the vertices of the graph is called the *radius* of the graph.

A *cycle* $C_n = C_n(V, E)(n \geq 3)$ is a graph with $V = \{x_1, x_2, \ldots, x_n\}$ and $E = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$. The *length* of C_n is equal to n. A *cycle* is called *odd/even* if its length is *odd/even*. The *girth* $g = g(G)$ of a graph G is the length of its shortest cycle. A graph containing no cycles is called an *acyclic* graph. The *degree* of a vertex $v \in G$ is denoted by $d(v)$ and is equal to the number of vertices to which v is connected by an edge. A *regular* graph is a graph in which all the vertices have the same degree. A graph is *planar* if it can be drawn in a plane without its edges crossing. A *face* is a region surrounded by a cycle in a planar embedding of a graph without any path crossing the cycle. A *tree* T_n is a maximal acyclic graph on n vertices. A *forest* is a disconnected acyclic graph. A *rooted* tree has a distinguished node which is called the *root*. In such a tree, each of the nodes that is one edge away from a given node (its *parent*) and at distance to the root one more than its parent is called a *child*. Nodes having the same parent node are called *siblings*. The *height* of a tree T_n, denoted by *Height(T_n)*, is defined as the maximum length of a path from the root of T_n to a leaf of T_n.

A *Cayley tree* is a tree in which each non-leaf vertex has the same degree. A *caterpillar* is a tree in which every vertex is on a central path or only one edge away from the central path (in other words, the removal of its endpoints leaves a path). A *lobster* is a tree having the property that the removal of its leaves leaves a caterpillar. A *star* S_n of order n, is a tree on n nodes with one node having degree $n-1$ and the other $n-1$ nodes having degree 1. A (n,k)-*banana tree* is a graph obtained by connecting one leaf of each of n copies of a k-star graph with a single root vertex which does not belong to any of the stars. A (n,k)-*firecracker* is a graph obtained by the concatenation of n S_k by linking one leaf from each to a path. A *pseudotree*, also called a *unicyclic graph*, is a connected graph with exactly one cycle. A *complete* graph on n vertices, denoted by K_n, is a graph in which all n vertices are adjacent to each other.

In this paper we consider the following problems.

Problem 2.1. *Given natural numbers $d \geq 1$ and $k \geq 1$ find the largest possible number of vertices $n_{d,k}$ in a tree of maximum degree d and diameter at most k. Within this problem we consider also the specific cases of Cayley trees, caterpillars, lobsters, banana trees and fireckracker trees.*

Problem 2.2. *Given natural numbers $d \geq 2$ and $k \geq 1$ find the largest possible number of vertices $n_{d,k}$ in a pseudotree of maximum degree d and diameter at most k.*

3. Degree/diameter problem on trees

In this section we investigate the degree/diameter problem on trees, and in particular on some special types of trees such as Cayley trees, caterpillars, lobsters, banana trees and firecracker trees, as well as for tree-like structures such as pseudotrees, giving the extremal numbers and constructions. Unlike the general case when the Moore bound is usually not achieved, in the case of banana trees, caterpillars and lobsters, we can always construct the corresponding Moore graph as well as for trees, Cayley trees and pseudotrees in the case that k is even.

We attack the degree/diameter problem on a tree by considering a maximal shortest path and maximizing the order of hanging subtrees on it. Before going further we introduce the following helpful notation:

$F_{d,h}$ is the tree where all non-leaf nodes have degree d and all its leaves are at height h from the root.
$F'_{d,h}$ is the tree where all non-leaf nodes have degree d, except its root which has degree $d-1$, and all its leaves are at height h from the root.
$F''_{d,h}$ is the tree where all non-leaf nodes have degree d, except its root which has degree $d-2$, and all its leaves are at height h from the root.

Theorem 3.1. *For a tree of maximum degree d and diameter at most k, we have*

$$n_{d,k} = \begin{cases} 2, & d = 1 \ or \ k = 1 \\ k+1, & d = 2 \ and \ k \geq 2 \\ \frac{2(d-1)^{\frac{k+1}{2}}-2}{d-2}, & k \ odd, \ k \geq 3 \ and \ d \geq 3 \\ \frac{d(d-1)^{\frac{k}{2}}-2}{d-2}, & k \ even, \ k \geq 2 \ and \ d \geq 3 \end{cases}$$

Proof. For $d = 1$ or $k = 1$ we have $n_{1,k} = 2$ and $n_{d,1} = 2$, achieved by P_2. For $d = 2$ and $k \geq 2$ we have $n_{2,k} = k+1$, achieved by P_{k+1}. For $d \geq 3$ and $k \geq 2$, we consider a maximal shortest path of length r. We then number its vertices from 0 to r. No subgraph hanging from any one of these vertices has common vertices with a subgraph hanging from another vertex as then we would have a cycle.
It is easy to see that these hanging subgraphs are also trees. An end vertex has no hanging tree attached as that would mean a larger maximal shortest path. The tree hanging from each non-end vertex i has height at most $min(i, r-i)$ (consider distance of the leaves from the end vertices 0 and r).

Therefore, there is at most one $F''_{d,min(i,r-i)}$ hanging from the vertex i. Summing the vertices gives the required upper bound:

$\sum_{i=0}^{r} |F''_{d,min(i,r-i)}|$

Clearly the above is maximized when $r = k$.

For k odd, $n_{d,k} = 2\sum_{i=0}^{\frac{k-1}{2}} (d-1)^i = 2\frac{(d-1)^{\frac{k+1}{2}}-1}{(d-1)-1} = \frac{2(d-1)^{\frac{k+1}{2}}-2}{d-2}$.

For k even, $n_{d,k} = |F''_{d,\frac{k}{2}}| + 2\sum_{i=0}^{\frac{k}{2}-1} |F''_{d,i}| = (d-1)^{\frac{k}{2}} + 2\sum_{i=0}^{\frac{k}{2}-1}(d-1)^i = (d-1)^{\frac{k}{2}} + 2\frac{(d-1)^{\frac{k}{2}}-1}{(d-1)-1} = \frac{d(d-1)^{\frac{k}{2}}-2}{d-2}$. □

Observe that the extremal case for any general tree is a Cayley tree and therefore the upper bound for Cayley trees is the same as for general trees; it is in fact the (unique) solution of the degree/diameter problem for Cayley graphs. However, this is not the case for other types of trees. The next theorems deal with the degree/diameter problem for caterpillars, lobsters, banana trees and firecracker trees.

The extremal cases for caterpillars and lobsters are found by considering the limitations of the tree as in Theorem 3.1 and then applying the limitations for the specific tree. We give the extremal cases without proof.

Theorem 3.2. *For a caterpillar tree:* $n_{d,k} = 2 + (k-1)(d-1)$

Theorem 3.3. *For a lobster tree:*

$$n_{d,k} = \begin{cases} 2, & k = 1 \ or \ d = 1 \\ d+1, & k = 2 \ and \ d \geq 2 \\ (k-3)d^2 + (8-2k)d + k - 3 & k \geq 3 \ and \ d \geq 2 \end{cases}$$

The extremal cases for banana and firecracker trees are found by considering the different type of constructions of those trees. Again we give the extremal cases without proof.

Theorem 3.4. *For a firecracker tree:*

$$n_{d,k} = \begin{cases} 2, & k = 1 \text{ or } d = 1 \\ d+1, & (k \in \{2,3\} \text{ and } d \geq 3) \text{ or } (k = 4 \text{ and } d \geq 5) \\ k+1, & d = 2 \text{ and } k \geq 2 \\ 2k-2, & k = 4 \text{ and } d \in \{3,4\} \\ (k-3)(d+1), & k \geq 5 \text{ and } d \geq 3 \end{cases}$$

Theorem 3.5. *For a banana tree:*

$$n_{d,k} = \begin{cases} 2, & k = 1 \text{ or } d = 1 \\ d+1, & k = 2 \\ d+2, & k = 3 \\ 2d+1, & k = \{4,5\} \\ 1+d+d^2, & k \geq 6 \end{cases}$$

Although a pseudotree is very close to being a tree, it appears that a deeper combinatorial analysis is needed to find the extremal cases of the degree/diameter problem on a pseudotree. To attack the problem we consider cases on the maximum height of subtrees hanging from the cycle of the pseudotree, thus deriving our results.

The following upper bound will be very helpful as it will allow us to disregard constructions with subtrees of small maximum height h on their cycle.

Lemma 3.6. *A pseudotree of maximum degree d, diameter at least k, having subtrees of maximum height h on its cycle and at least one subtree achieving that height has at most $2(k-2h)(d-1)^h + \frac{d(d-1)^h-2}{d-2}$ vertices, where $h \leq \lfloor \frac{k}{2} \rfloor$, $d \geq 3$ and $k \geq 2$.*

Proof. Clearly all subtrees hanging from the cycle of the pseudotree must be of form $F''_{d,q}$, where $0 \leq q \leq h$. On the right side of the subtree of height h there can be at most $k-2h$ subtrees of max height h, then the height of the subtrees gradually decreases as $h-1, h-2, \ldots, 1, 0$. Same on the left of the central subtree of height h. One can observe that

$$\begin{aligned} n &= 2(k-2h)|F''_{d,h}| + |F_{d,h}| \\ &= 2(k-2h)(d-1)^{h-1} + \frac{d(d-1)^h - 2}{d-2}. \end{aligned}$$

□

The next lemma gives the extremal case for k even, $k \geq 2$, and $d \geq 4$ using the upper bound of Lemma 3.6.

Lemma 3.7. *When k is even, a pseudotree of maximum degree d and diameter at most k has at most the same number of nodes as the extremal tree of maximum degree d and diameter at most k, where $k \geq 2$ and $d \geq 4$ or $k \geq 4$ and $d = 3$.*

Proof. Hanging subtrees are of the form $F''_{d,h}$ except possibly one with height $\lceil \frac{k}{2} \rceil$. Clearly, on the right side of the subtree of height $\frac{k}{2}$ the height of the subtrees gradually decreases as $\frac{k}{2} - 1, \frac{k}{2} - 2, \ldots, 1, 0$. Same on the left side of the subtree of height $\frac{k}{2}$. Observe that $n = F_{d,\frac{k}{2}}$.

Subtracting from that the upper bound of Lemma 3.6 we get:

$|F_{d,\frac{k}{2}}| - 2(k-2h)(d-1)^h - \frac{d(d-1)^h-2}{d-2}$

$$= \frac{d(d-1)^{\frac{k}{2}} - 2}{d-2} - 2(k-2h)(d-1)^h - \frac{d(d-1)^h - 2}{d-2}$$

$$= \frac{(d-1)^h}{d-2}[d(d-1)^{\frac{k}{2}-h} - d - 2(k-2h)(d-2)].$$

It is enough to prove $d(d-1)^{\frac{k}{2}-h} - d - 2(k-2h)(d-2) \geq 0$. Letting $\frac{k}{2} - h = x$ transforms the above to $d(d-1)^x - d - 4(d-2)x \geq d((d-1)^x - 1 - 4x)$ so it is enough to show that $((d-1)^x - 1 - 4x) \geq 0$.

It is easy to see that this holds for all feasible values except for the pairs $(d, x) = (3{,}1)$, $(3{,}2)$, $(3{,}3)$, $(3{,}4)$, $(4{,}1)$ and $(5{,}1)$. Trying these pairs in $d(d-1)^x - d - 4(d-2)x$, only $(d = 3, x = 1)$ gives a negative value. We will next show that the pair $(d = 3, x = 1)$ does not give constructions with more nodes than the extremal tree of maximum degree 3 and diameter at most k, where k is even and $k \geq 4$.

We will consider cases on the number of hanging subtrees of height $\frac{k}{2} - 1$. With a little combinatorial analysis we end up with the following cases:

- C_5 with 5 $F''_{d,\frac{k}{2}-1}$ attached to it.
- C_5 with 4 $F''_{d,\frac{k}{2}-1}$ and one $F''_{d,\frac{k}{2}-2}$ attached to it.
- C_7 with 3 $F''_{d,\frac{k}{2}-1}$ and 4 $F''_{d,\frac{k}{2}-2}$ attached to it.
- C_k with 2 $F''_{d,\frac{k}{2}-1}$ and 5 $F''_{d,\frac{k}{2}-2}$ attached to it.
- In the case that we are dealing with one such subtree we get as extremal a C_{k+1} with one $F''_{d,\frac{k}{2}-1}$ attached to it and a sequence of $F''_{d,\frac{k}{2}-2}$, $F''_{d,\frac{k}{2}-2}$, $F''_{d,\frac{k}{2}-2}$, $F''_{d,\frac{k}{2}-3}$, $F''_{d,\frac{k}{2}-4}$, $\ldots$, $F''_{d,2}$, $F''_{d,1}$, $F''_{d,0}$ both on its left and right sides (order is $|F_{d,\frac{k}{2}-1}| + 4|F''_{d,\frac{k}{2}-2}|$).

Clearly, the first case gives the maximum order.

Subtracting the order of the first case from the order of the extremal tree we get

$|F_{3,\frac{k}{2}}| - 5|F''_{3,\frac{k}{2}-1}| = 3 \cdot 2^{\frac{k}{2}} - 2 - 5 \cdot 2^{\frac{k}{2}-1} = 2^{\frac{k}{2}-1} - 2 \geq 0.$ □

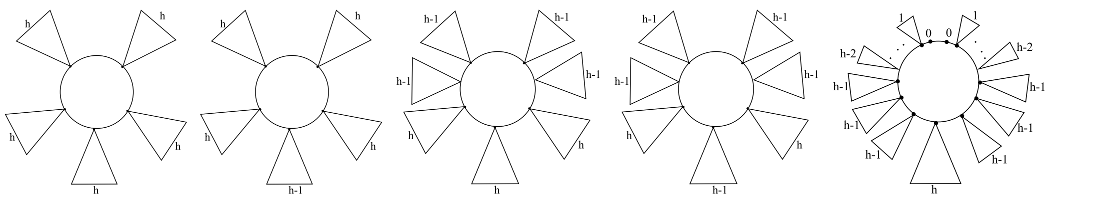

Figure 1: Cases considered in Lemma 3.7, a $\triangle$ with an h next to it stands for an $F''_{d,h}$ ($h = \frac{k}{2} - 1$).

The following lemma will be useful when considering the problem for k odd.

Lemma 3.8. *For k odd, a pseudotree of maximum degree $d \geq 4$, diameter at most k, having subtrees of maximum height $\frac{k-1}{2}$ on its cycle, and at least one subtree achieving that height, has at most $3|F''_{d,\frac{k-1}{2}}|$ nodes.*

Proof. We consider cases according to the number of hanging subtrees of height $\frac{k-1}{2}$. With a little combinatorial analysis we arrive at the following cases:

- C_3 with 3 $F''_{d,\frac{k-1}{2}}$ attached to it.
- C_5 with 2 $F''_{d,\frac{k-1}{2}}$ and 3 $F''_{d,\frac{k-3}{2}}$ attached to it or a C_{k+2} with two consecutive $F''_{d,\frac{k-1}{2}}$ attached to it and a sequence of $F''_{d,\frac{k-3}{2}}$, $F''_{d,\frac{k-5}{2}}$, …,$F''_{d,2}$, $F''_{d,1}$, $F''_{d,0}$ both on their left and right sides (order is $|F_{d,\frac{k-1}{2}}| + |F''_{d,\frac{k-1}{2}}|$).
- In the case that we are dealing with one such subtree we get as an upper bound a C_{k+2} with consecutive $F''_{d,\frac{k-3}{2}}$, $F''_{d,\frac{k-1}{2}}$, $F''_{d,\frac{k-3}{2}}$ attached to it and a sequence of $F''_{d,\frac{k-3}{2}}$, $F''_{d,\frac{k-5}{2}}$, …,$F''_{d,2}$, $F''_{d,1}$, $F''_{d,0}$ both on their left and right sides (order is $|F_{d,\frac{k-1}{2}}| + 2|F''_{d,\frac{k-3}{2}}|$).

Clearly, the first case gives the maximum order. □

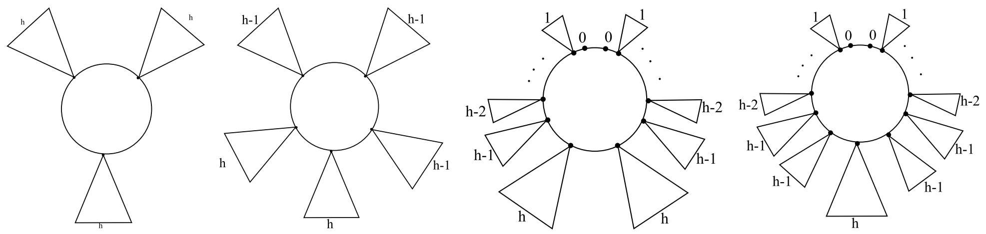

Figure 2: Cases considered in Lemma **??** a $\triangle$ with an h next to it stands for an $F''_{d,h}$ ($h = \frac{k-1}{2}$)

Using the upper bound of Lemma 3.6, the next lemma gives the extremal case for k odd, $k \geq 3$, and $d \geq 4$.

Lemma 3.9. *When k is odd, a pseudotree of maximum degree d and diameter at most k has at most $3|F''_{d,\frac{k-1}{2}}|$ nodes, where $d \geq 4$ and $k \geq 3$.*

Proof. Subtracting the extremal tree of height $\frac{k+1}{2}$ we get:

$$\begin{aligned}
3|F''_{d,\frac{k}{2}-1}| - \frac{2(d-1)^{\frac{k+1}{2}} - 2}{d-2} &= 3(d-1)^{\frac{k-1}{2}} - \frac{2(d-1)^{\frac{k+1}{2}} - 2}{d-2} \\
&= \frac{3(d-1)^{\frac{k-1}{2}}(d-2) + 2 - 2(d-1)^{\frac{k+1}{2}}}{d-2} \\
&= \frac{(d-1)^{\frac{k-1}{2}}(3d-10) + 2}{d-2} > 0.
\end{aligned}$$

Subtracting the upper bound of Lemma 3.6 we get:
$3|F''_{d,\frac{k}{2}-1}| - 2(k-2h)(d-1)^h + \frac{d(d-1)^h - 2}{d-2} = 3(d-1)^{\frac{k-1}{2}} - 2(k-2h)(d-1)^h + \frac{d(d-1)^h-2}{d-2}$
$> (d-1)^h[3(d-1)^{\frac{k-1}{2}-h} - 2(k-2h) - \frac{d}{d-2}]$.
Consider: $3(d-1)^{\frac{k-1}{2}-h} - 2(k-2h) - \frac{d}{d-2} \geq 3(d-1)^{\frac{k-1}{2}-h} - 2(k-2h) - 3$.

Substituting $x = \frac{k-1}{2} - h$ the above expression becomes $3(d-1)^x - 4x - 5$. It suffices to prove that $3(d-1)^x - 4x - 5 \geq 0$. It is easy to see that this holds for all valid values except for the pairs $(d = 3, x = 1)$ and $(d = 3, x = 2)$. $\square$

The following lemma proves useful when considering the problem for k odd, $d = 3$ and $k \geq 5$.

Lemma 3.10. *For k odd, a pseudotree of maximum degree 3, diameter at least k, having subtrees of maximum height $\frac{k-3}{2}$ on its cycle and at least one subtree having that height, has at most $7|F''_{3,\frac{k-3}{2}}|$ nodes.*

Proof. We consider cases according to the number of hanging subtrees of height $\frac{k-3}{2}$. With a little combinatorial analysis we identify all the possible cases as follows.

- C_7 with 7 $F''_{3,\frac{k-3}{2}}$ attached to it.
- C_7 with 6 $F''_{3,\frac{k-3}{2}}$ and one $F''_{3,\frac{k-5}{2}}$ attached to it.
- C_7 with 5 $F''_{3,\frac{k-3}{2}}$ and 2 $F''_{3,\frac{k-5}{2}}$ attached to it.
- C_9 with 4 $F''_{3,\frac{k-3}{2}}$ and 5 $F''_{3,\frac{k-5}{2}}$ attached to it.
- C_9 with 3 $F''_{3,\frac{k-3}{2}}$ and 6 $F''_{3,\frac{k-5}{2}}$ attached to it.
- In the case that we are dealing with 2 such subtrees we get as an upper bound the construction for the case below with one of $F''_{3,\frac{k-5}{2}}$ at a distance at most 3 from the central $F''_{3,\frac{k-3}{2}}$ replaced by one $F''_{3,\frac{k-3}{2}}$ (order is $|F_{3,\frac{k-3}{2}}| + 5|F''_{3,\frac{k-5}{2}}| + |F''_{3,\frac{k-3}{2}}|$).
- In the case that we are dealing with one such subtree we get as an upper bound a C_{k+2} with consecutive $F''_{3,\frac{k-5}{2}}$, $F''_{3,\frac{k-5}{2}}$, $F''_{3,\frac{k-5}{2}}$, $F''_{3,\frac{k-3}{2}}$, $F''_{3,\frac{k-5}{2}}$, $F''_{3,\frac{k-5}{2}}$, $F''_{3,\frac{k-5}{2}}$ attached to it and a sequence of $F''_{3,\frac{k-5}{2}}$, $F''_{3,\frac{k-7}{2}}$, $\ldots$, $F''_{3,2}$, $F''_{3,1}$, $F''_{3,0}$ both on its left and right sides (order is $|F_{3,\frac{k-3}{2}}| + 6|F''_{3,\frac{k-5}{2}}|$).

Clearly, the first case gives the maximum order. □

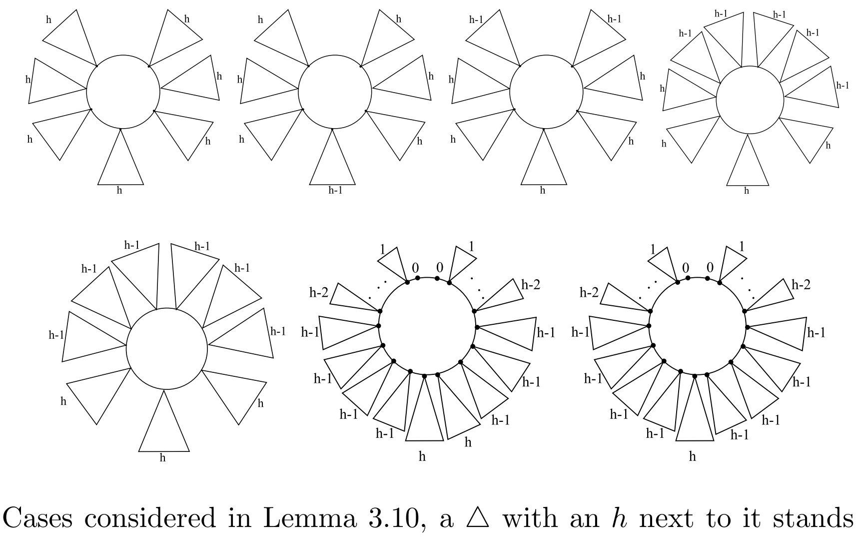

Figure 3: Cases considered in Lemma 3.10, a △ with an h next to it stands for an $F''_{d,h}$ ($h = \frac{k-3}{2}$.)

Using Lemmas 3.9 and 3.10, the following lemma gives the extremal case for k odd, $k \geq 5$, and $d = 3$.

Lemma 3.11. *For k odd, $k \geq 3$, a pseudotree of maximum degree 3 and diameter at most k has at most the same number of nodes as the corresponding extremal tree.*

Proof. The extremal construction with maximum height hanging subtree $F''_{3,\frac{k-1}{2}}$ (Lemma 3.8) gives the same order since:
$(2 \cdot 2^{\frac{k+1}{2}} - 2) - (|F_{\frac{k-1}{2}}| + |F''_{\frac{k-1}{2}}|) = 4 \cdot 2^{\frac{k-1}{2}} - 2 - 3 \cdot 2^{\frac{k-1}{2}} + 2 - 2^{\frac{k-1}{2}} = 0$
Similarly, the extremal construction with maximum height hanging subtree $F''_{3,\frac{k-3}{2}}$ gives:

$$\begin{aligned}(2 \cdot 2^{\frac{k+1}{2}} - 2) - 7|F''_{3,\frac{k-3}{2}}| &= (2 \cdot 2^{\frac{k+1}{2}} - 2) - 7 \cdot 2^{\frac{k-3}{2}} \\ &= 2^{\frac{k-3}{2}} - 2 \geq 0.\end{aligned}$$

Upper bound for the remaining cases is also less:

$$\begin{aligned}(2 \cdot 2^{\frac{k+1}{2}} - 2) - (2(k-2h)2^h + 3 \cdot 2^h - 2) &= (2 \cdot 2^{\frac{k+1}{2}} - 2) - 2(k-2h)2^h - 3 \cdot 2^h \\ &= 2^h(2^{\frac{k+3}{2}-h} - 2k + 4h - 3) - 2.\end{aligned}$$

Letting $x = \frac{k+1}{2} - h$ transforms the above expression to $2^h(2^{x+1} - 4x - 1) - 2$.
It suffices to show that $2^{x+1} - 4x - 1 > 0$, which clearly holds for all $x \geq 3$. □

We are now ready to present the main result of this paper.

Theorem 3.12. *A pseudotree of maximum degree d and diameter at most k, where $d \geq 2$ and $k \geq 1$, has order at most*

$$\begin{cases} 2k+1, & d = 2 \\ 3, & k = 1 \\ 5, & k = 2,\ d = 3 \\ 7, & k = 3,\ d = 3 \\ \frac{d(d-1)^{\frac{k}{2}} - 2}{d-2}, & (k \text{ is even},\ k \geq 2,\ d \geq 4) \\ & \text{or } (k \text{ is even},\ k \geq 4,\ d = 3) \\ 2 \cdot 2^{\frac{k+1}{2}} - 2, & k \text{ is odd},\ k \geq 5,\ d = 3 \\ 3(d-1)^{\frac{k-1}{2}}, & k \text{ is odd},\ k \geq 3,\ d \geq 4 \end{cases}$$

Proof. For $d = 2$ the only valid construction is a cycle and the extremal case is the C_{2k+1}.

For $k = 1$ the only valid construction the C_3.

For $k = 2$ and $d = 3$ it is easy to see that the extremal case is the C_5. (by considering cases on the girth of the construction).

For $k = 3$ and $d = 3$ it is easy to see that the extremal case is the C_7. (by considering cases on the girth of the construction).

The fifth case follows from Lemma 3.7. The sixth case follows from Lemma 3.11. The last case follows from Lemma 3.9. □

4. Conclusion and future work

The degree/diameter problem asks: Given a graph of maximum degree d and diameter at most k, what is the maximum number of vertices $n_{d,k}$ that can exist in the graph? In this paper we have considered the degree/diameter problem on trees, special types of trees such as Cayley trees, caterpillars, lobsters, banana trees and firecracker trees, as well as for tree-like structures such as pseudotrees, and we gave the extremal numbers and some constructions. We have shown that unlike the general case when the Moore bound is usually not achieved, in the case of banana trees, caterpillars and lobsters, we can always construct the corresponding Moore graph as well as for trees, Cayley trees and pseudotrees in the case that k is even.

Further research might concentrate on the degree/diameter problem for other special types of graphs or on finding better upper and/or lower bounds and extremal cases for the problem in its general case.

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