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(K, H)-Multifactorization of $K_{m,m}(\lambda)$

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Abstract

Let $G(\lambda)$ be a multigraph with uniform edge-multiplicity $\lambda > 0$. Factorization of $G(\lambda)$ into r copies of K-factors and s copies of H-factors, for some integers $r, s \geq 1$ is called a (K, H)-multifactorization of $G(\lambda)$. In this paper, the existence of (K, H)-multifactorization of $K_{m,m}(\lambda)$ has been proved when (i) $(K, H) = (K_{1,m-1}, P_m)$ and (ii) $(K, H) = (C_m, P_m)$. Further, $(C_m, K_{1,m-1})$-multifactorization of $K_{m,m}(\lambda)$ has been established for all $\lambda > 3$ except $(\lambda, m) = (2t + 1, 6)$, $(3, m)$, where $t > 1$, and the non existence has been shown when $\lambda = 1,\ 2$.

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1. Introduction

Let G be a simple, finite graph. A graph obtained from G by replacing every edge of G with λ edges is called a multigraph with uniform edge-multiplicity λ and we denote it by $G(\lambda)$. Let λG denotes λ edge-disjoint copies of G. Let K_m be a complete graph on 'm' vertices and $K_{m,m}$ be a complete bipartite graph with 'm' vertices in each partite set. Let P_k, C_k and S_k be respectively denote a path, a cycle and a star on k vertices.

Partition of G into edge-disjoint subgraphs $G_1, G_2, ..., G_r$ such that $E(G) = E(G_1) \cup E(G_2) \cup ... \cup E(G_r)$ is called *decomposition* of G and in this case, we write $G = G_1 \oplus$

$G_2 \oplus ... \oplus G_r = \oplus_{i=1}^{r} G_i$. In particular, if each $G_i \cong H$, $1 \le i \le r$, we say that G has a H-*decomposition* and we denote such existence by $H \mid G$.

A H-factor of G is a spanning subgraph of G such that each component of it is isomorphic to H. Decomposition of G into edge-disjoint H-factors is called a H-*factorization* of G and we denote such decomposition by $H \parallel G$.

Decomposition of $G(\lambda)$ into r copies of K and s copies of H for some integers $r, s \ge 1$ is called a (K,H)-*multidecomposition* of $G(\lambda)$, where K, H are subgraphs of G. The study of (K,H)-multidecomposition has been introduced by Atif Abueida and M.Daven [1, 2]. Moreover, Atif Abueida and Theresa O' Neil [3] have settled the existence of (K,H)-multidecomposition of $K_m(\lambda)$ when $(K,H) = (K_{1,n-1}, C_n)$ for $n = 3,4,5$ and $m \ge n$. Further, they conjectured that "For any integer $m \ge n \ge 3$, there is a (K,H)-multidecomposition of $K_m(\lambda)$, where $(K,H) = (K_{1,n-1}, C_n)$". The authors [6, 7] have settled the above conjecture for $m = n$ when $(K,H) = (K_{1,n-1}, C_n), (K_{1,n-1}, P_n), (P_n, C_n)$ and also for $m = n+1$ when $(K,H) = (K_{1,n-1}, P_n)$. Shyu [9, 10, 11] has obtained some necessary and sufficient conditions for the existence of (K,H)-multidecomposition of K_n, when $(K,H) = (P_l, S_k), (C_l, S_k), (C_l, P_k)$. Later, the authors [8] have settled the existence of (K,H)-multidecomposition of $K_{m,m}(\lambda)$ when $(K,H) = (K_{1,m-1}, P_m), (C_m, K_{1,m-1}), (C_m, P_m)$. Recently, Atif Abueida and Daven [4] have settled the existence of (K,H)-multidecomposition of several product graphs, when $(K,H) = (C_4, E_2)$, where E_2 be two vertex-disjoint edges.

The above facts motivate us to consider the similar study for a multifactorization of $G(\lambda)$. Decomposition of $G(\lambda)$ into r K-factors and s H-factors, for some integers $r, s \ge 1$ is called a (K,H)-*multifactorization* of $G(\lambda)$.

In this paper, the existence of (K,H)-multifactorization of $K_{m,m}(\lambda)$ has been proved when (i) $(K,H) = (K_{1,m-1}, P_m)$ and (ii) $(K,H) = (C_m, P_m)$. Further, $(C_m, K_{1,m-1})$-multifactorization of $K_{m,m}(\lambda)$ has been established for all $\lambda > 3$ except $(\lambda, m) = (2t+1, 6)$, $(3, m)$, where $t > 1$, and the non existence has been shown when $\lambda = 1,\ 2$.

To prove our results, we state the following:

Theorem 1.1. [5] *$K_{m,n}$ has a C_k-factorization if and only if (i) $m = n \equiv 0(\mod 2)$, (ii) $k \equiv 0(\mod 2)$, $k \ge 4$ and $2n \equiv 0(\mod k)$ with precisely one exception, namely $m = n = k = 6$.*

Theorem 1.2. [12] *Let F be any 1-factor of $K_{k,k}$. Then the following holds:*
(a) $K_{k,k} - F$ has a P_k-factorization, when k is even.
(b) $K_{k,k}(2) - F(2)$ has a P_k-factorization, when k is odd.

Lemma 1.3. *If $K_{m,m}(\lambda)$ has a (K,H)-multifactorization, then so does $K_{ms,ms}(\lambda)$.*

Proof. First partition each partite set of $K_{ms,ms}(\lambda)$ into m-subsets. Then we construct a new graph by identifying each m-subset into a single vertex and join two of them by an edge if the corresponding m-subsets form a $K_{m,m}(\lambda)$ in $K_{ms,ms}(\lambda)$. The resulting graph is isomorphic to $K_{s,s}$. We know that $K_{s,s}$ is 1-factorable. While we go back to the original graph, correspond to each 1-factor of $K_{s,s}$, we have a $K_{m,m}(\lambda)$-factor in $K_{ms,ms}(\lambda)$. Thus the 1-factorization of $K_{s,s}$ yields a $K_{m,m}(\lambda)$-factorization of $K_{ms,ms}(\lambda)$. If $K_{m,m}(\lambda)$ possess a (K, H)-multifactorization, then the $K_{m,m}(\lambda)$ factorization of $K_{ms,ms}(\lambda)$ gives a (K, H)-multifactorization of $K_{ms,ms}(\lambda)$. □

Definition 1.4. *A 1-factor of distance t in $K_{m,m}$ with partite sets $V_1 = \{u_1, u_2, u_3, ..., u_m\}$ and $V_2 = \{v_1, v_2, v_3, ..., v_m\}$ is defined as $\alpha_t = \{(u_i, v_{i+t-1}) : 1 \leq i \leq m\}$, $1 \leq t \leq m$, where the addition in the subscript is taken modulo m.*

Theorem 1.5. *The graph $K_{6,6}(2)$ has a C_6-factorization.*

Proof. We know that $K_{6,6}(2) = \oplus_{i=1}^{6} \alpha_i(2)$. Then $F_i = \alpha_i \oplus \alpha_{i+4}, 1 \leq i \leq 6$, is a C_6-factor and hence, $\{F_1, \ldots, F_6\}$ gives a required C_6-factorization of $K_{6,6}(2)$. □

Notation: Throughout the paper, let $V(K_{m,m}) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, u_3, ..., u_m\}$ and $V_2 = \{v_1, v_2, v_3, ..., v_m\}$. We denote a star $K_{1,m-1}$ of $K_{m,m}$ with center at u_i and end vertices $v_1, v_2, ..., v_{m-1}$ by $(u_i : v_1, v_2, ..., v_{m-1})$. Further a cycle and a path with m vertices $v_1, v_2, ..., v_m$ and edges $v_i v_{i+1}$, $i = 1, 2, \ldots, m$ are denoted by $v_1 v_2 ... v_m v_1$ and $v_1 v_2 ... v_m$ respectively, where addition in the subscripts are taken modulo m.

Remark 1.6. *Let $K_{m,m}(2)$ be the complete bipartite multigraph with partite sets $V_1 = \{u_1, u_2, u_3, ..., u_m\}$ and $V_2 = \{v_1, v_2, v_3, ..., v_m\}$. Now we define a star factor of $K_{m,m}(2)$ as follows. For $m > 2$, let $S_i^j = \{(u_i : v_i, v_{i+1}, ..., v_{i+j-2}, v_{i+j}, v_{i+j+1} ..., v_{i+m-1}), (v_{i+j-1} : u_{i+1}, u_{i+2}, ..., u_{i+m-1})\}$, where addition in the subscripts are taken with modulo m. Clearly, for the fixed i, j, $1 \leq i, j \leq m$, each S_i^j, is a $K_{1,m-1}$-factor of $K_{m,m}$ and $\cup_{i=1}^{m} S_i^j = K_{m,m}(2) \setminus \alpha_j(2)$, for any j. (i.e), $K_{m,m}(2) = \cup_{i=1}^{m} S_i^j \oplus \alpha_j(2)$. When $m = 2$, $K_{1,m-1} = P_m$ and hence $(K_{1,m-1}, P_m)$-factorization is nothing but P_2-factorization of $K_{m,m}(2)$, which is obvious by definition 1.4.*

2. $(K_{1,m-1}, P_m)$-multifactorization of $K_{m,m}(\lambda)$

Lemma 2.1. *For all even $m \geq 4$, $P_m \parallel R$ where $R = \alpha_1(\frac{m}{2}) \oplus \alpha_2(\frac{m-2}{2})$ and $\alpha_i(r)$ is a 1-factor of distance i of $K_{m,m}(\lambda)$ with edge-multiplicity r.*

Proof. We construct the required $\frac{m}{2}$ P_m-factors from R as follows:

$$\begin{aligned} \text{Let } H = & \{\alpha_1 \oplus \alpha_2 \setminus \{(u_1, v_2), (u_{\frac{m}{2}+1}, v_{\frac{m}{2}+2})\}\} \oplus \\ & \{\alpha_1 \oplus \alpha_2 \setminus \{(u_2, v_3), (u_{\frac{m}{2}+2}, v_{\frac{m}{2}+3})\}\} \oplus \\ & \{\alpha_1 \oplus \alpha_2 \setminus \{(u_3, v_4), (u_{\frac{m}{2}+3}, v_{\frac{m}{2}+4})\}\} \oplus \\ & \ldots \\ & \{\alpha_1 \oplus \alpha_2 \setminus \{(u_{\frac{m-2}{2}}, v_{\frac{m-2}{2}+1}), (u_{m-1}, v_m)\}\}. \end{aligned}$$

Clearly, each term in the above sum is a P_m-factor and hence, H consists of $(\frac{m-2}{2})$ edge-disjoint P_m-factors. Further the edges removed from α_2 *'s* in the above construction and the edges of α_1 (which are not used in H) together form another P_m-factor which is edge-disjoint from H. Hence, we get the required $\frac{m}{2}$ edge-disjoint P_m-factors of $K_{m,m}(\lambda)$ from R. □

Lemma 2.2. *For $m > 2$, there exists a $(K_{1,m-1}, P_m)$-multifactorization of $K_{m,m}(\lambda)$ if one of the following holds:*
(i) $\lambda \equiv 0 \ mod\ (m-1)$, when m is even or
(ii) $\lambda \equiv 0 \ mod\ (2(m-1))$, when m is odd.

Proof. **Case 1.** $\lambda \equiv 0 \ mod\ (m-1)$.

Let $\lambda = m-1$. Now we construct the required factors of $K_{m,m}(m-1)$ in 2 subcases, accordingly $m \equiv 2\ (mod\ 4)$ and $m \equiv 0\ (mod\ 4)$.

Subcase 1(i). $m \equiv 2\ (mod\ 4)$.

We write,

$$\begin{aligned}K_{m,m}(m-1) &= K_{m,m}(\tfrac{m-2}{2}) \oplus K_{m,m}(\tfrac{m-2}{2}) \oplus K_{m,m}\\ &= (\tfrac{m-2}{4})\ (\cup_{i=1}^{m} S_i^1 \oplus \alpha_1(2)) \oplus (\tfrac{m-2}{4})\ (\cup_{i=1}^{m} S_i^2 \oplus \alpha_2(2)) \oplus\ \alpha_1\\ &\quad \oplus \tfrac{m}{2} P_m\text{-factors, by Remark 1.6 and Theorem 1.2}(a),\\ &= (\tfrac{m-2}{4})\ (\cup_{i=1}^{m}(S_i^1 \oplus S_i^2)) \oplus (\tfrac{m-2}{4})\ (\alpha_1(2) \oplus \alpha_2(2)) \oplus \alpha_1 \oplus \tfrac{m}{2} P_m\text{-factors, where}\end{aligned}$$

S_i^1 and S_i^2 are $K_{1,m-1}$-factors. Clearly, the edges not covered by the above $K_{1,m-1}$-factors and the P_m-factors form the graph $\alpha_1(\frac{m}{2}) \oplus \alpha_2(\frac{m-2}{2})$. By Lemma 2.1, $\alpha_1(\frac{m}{2}) \oplus \alpha_2(\frac{m-2}{2})$ has a P_m-factorization and hence, $(K_{1,m-1}, P_m) \parallel K_{m,m}(\lambda)$ for all $\lambda \equiv 0 \ mod\ (m-1)$.

Subcase 1(ii). $m \equiv 0\ (mod\ 4)$.

First we construct the required factors of $K_{m,m}$, when $m = 4$. Let $V(K_{4,4}) = \{u_1, u_2, u_3, u_4\} \cup \{v_1, v_2, v_3, v_4\}$. For $1 \le i \le 4$, let $S_i = \{(u_i : v_{i+1}, v_{i+2}, v_{i+3}); (v_i : u_{i+1}, u_{i+2}, u_{i+3})\}$ be a $K_{1,3}$- factor of $K_{4,4}$. The 4 P_4-factors $\{v_1u_1v_2u_2, v_3u_3v_4u_4\}$, $\{v_1u_1v_4u_4, v_2u_2v_3u_3\}$, $\{u_1v_3u_4v_2, u_3v_1u_2v_4\}$ and $\{u_1v_1u_4v_4, u_2v_2u_3v_3\}$ together with $\cup_{i=1}^{4} S_i$ give the required factors of $K_{4,4}$.
Now consider $m \equiv 0\ (mod\ 4)$ and $m > 4$. We write,

$$\begin{aligned}K_{m,m}(m-1) &= K_{m,m}(\tfrac{m-4}{2}) \oplus K_{m,m}(\tfrac{m}{2}) \oplus K_{m,m}\\ &= (\tfrac{m-4}{4})\ (\cup_{i=1}^{m} S_i^1 \oplus \alpha_1(2)) \oplus (\tfrac{m}{4})\ (\cup_{i=1}^{m} S_i^2 \oplus \alpha_2(2)) \oplus\ \alpha_1\\ &\quad \oplus \tfrac{m}{2} P_m\text{-factors, by Remark 1.6 and Theorem 1.2}(a),\\ &= (\tfrac{m-4}{4})\ (\cup_{i=1}^{m} S_i^1) \oplus (\tfrac{m}{4})\ (\cup_{i=1}^{m} S_i^2) \oplus (\tfrac{m-4}{4})(\alpha_1(2)) \oplus (\tfrac{m}{4})(\alpha_2(2)) \oplus \alpha_1\\ &\quad \oplus\ \tfrac{m}{2} P_m\text{-factors, where } S_i^1 \text{ and } S_i^2 \text{ are } K_{1,m-1}\text{-factors. Clearly,}\end{aligned}$$

the edges not covered by the above $K_{1,m-1}$-factors and the P_m-factors form the graph $\alpha_1(\frac{m}{2}) \oplus \alpha_2(\frac{m-2}{2})$. By Lemma 2.1, $\alpha_1(\frac{m}{2}) \oplus \alpha_2(\frac{m-2}{2})$ has a P_m-factorization and Hence, $(K_{1,m-1}, P_m) \parallel K_{m,m}(\lambda)$ for all $\lambda \equiv 0 \ mod\ (m-1)$.

Case 2. $\lambda \equiv 0 \ mod\ (2(m-1))$.

Let $\lambda = 2(m-1)$. We write,

$$K_{m,m}(2(m-1)) = \underbrace{K_{m,m}(2) \oplus ... \oplus K_{m,m}(2)}_{(m-1)\ times}$$

$$= (\cup_{i=1}^{m} S_i^1 \oplus \alpha_1(2)) \oplus (\cup_{i=1}^{m} S_i^2 \oplus \alpha_2(2)) \oplus \cdots \oplus (\cup_{i=1}^{m} S_i^{m-1} \oplus \alpha_{m-1}(2)),$$
by Remark 1.6.
$$= \cup_{i=1}^{m}(S_i^1 \oplus S_i^2 \oplus \cdots \oplus S_i^{m-1}) \oplus (\alpha_1(2) \oplus \alpha_2(2) \cdots \oplus \alpha_{m-1}(2)),$$
$$= \cup_{i=1}^{m}(S_i^1 \oplus S_i^2 \oplus \cdots \oplus S_i^{m-1}) \oplus (\cup_{j=1}^{m-1} \alpha_j(2)),$$

where S_i^j $'s$, $1 \leq j \leq m-1$ are $K_{1,m-1}$-factors. As $\cup_{j=1}^{m-1}\alpha_j(2) = K_{m,m}(2) \setminus \alpha_m(2)$ is P_m-factorable by Theorem 1.2(b), we have the required factorization of $K_{m,m}(2(m-1))$. Hence, $(K_{1,m-1}, P_m) \parallel K_{m,m}(\lambda)$ for all $\lambda \equiv 0 \ mod\ (2(m-1))$, when m is odd. Hence, the Lemma. □

Theorem 2.3. *For $m > 2$, $K_{m,m}(\lambda)$ has a $(K_{1,m-1}, P_m)$-multifactorization if and only if*
(i) $\mid E(K_{m,m})(\lambda) \mid = 2r(m-1) + 2s(m-1),\ r, s \geq 1$,
(ii) $\lambda \equiv 0 \ mod\ (m-1)$, when m is even and
(iii) $\lambda \equiv 0 \ mod\ (2(m-1))$, when m is odd.

Proof. **Necessity:** Assume that $K_{m,m}(\lambda)$ has a $(K_{1,m-1}, P_m)$-multifactorization. Then there exist integers $r, s \geq 1$, such that $K_{m,m}(\lambda) = r\ K \oplus s\ P$, where K and P respectively denote $K_{1,m-1}$-factor and P_m-factor. Hence, $\mid E(K_{m,m})(\lambda) \mid = 2r(m-1) + 2s(m-1)$ for some $r, s \geq 1$. Thus (i) holds. By counting the number of edges in $K_{m,m}(\lambda)$ and by (i), we have

$$\mid E(K_{m,m})(\lambda) \mid = 2r(m-1) + 2s(m-1) = 2(m-1)(r+s) = \lambda m^2,$$

where $r, s \geq 1$. This shows that $\lambda \equiv 0 \ mod\ (m-1)$, when m is even, since $g.c.d.\ (\frac{m}{2}, m-1) = 1 = g.c.d.\ (m, m-1) = 1$ and $\lambda \equiv 0 \ mod\ (2(m-1))$, when m is odd, since $g.c.d.\ (m, m-1) = 1$. Thus (ii) and (iii) holds.
Sufficiency: Follows from Lemma 2.2. □

By Lemma 1.3 and Theorem 2.3, we have the following:

Corollary 2.4. *If $K_{m,m}(\lambda)$ has a $(K_{1,m-1}, P_m)$-multifactorization, then so does $K_{ms,ms}(\lambda)$.*

3. (C_m, P_m)-multifactorization of $K_{m,m}(\lambda)$

Theorem 3.1. *For $m > 2$, $K_{m,m}(\lambda)$ has a (C_m, P_m)-multifactorization if and only if*
(i) $m \equiv 0 \ (mod\ 2)$,
(ii) $\mid E(K_{m,m})(\lambda) \mid = 2rm + 2s(m-1),\ r, s \geq 1$ and
(iii) $\lambda > 1$.

Proof. **Necessity:** Since the graph under consideration is bipartite, cycles must be of even length and Hence, $m \equiv 0 \ (mod\ 2)$. This proves (i). By the hypothesis, since $K_{m,m}(\lambda)$ has a (C_m, P_m)-multifactorization, there exists integers $r, s \geq 1$, such that $K_{m,m}(\lambda) = r\ C \oplus s\ P$, where C and P respectively denote C_m-factor and P_m- factor. Hence, $\mid E(K_{m,m})(\lambda) \mid = 2rm + 2s(m-1)$ for $r, s \geq 1$. This proves (ii). From (ii), we have

$\lambda m^2 = 2rm + 2s(m-1)$. Since the left hand side (L.H.S) of the above equation is a multiple of 4, s should be even. Moreover, since L.H.S of the above equation is congruent to m, right hand side (R.H.S) will be congruent to m, only if $s \equiv 0 \ (mod\ m)$. In such case, (i.e), when $s \equiv 0 \ (mod\ m)$, we need $2m^2 - 2m$ edges to have at least minimum number of P_m-factors in $K_{m,m}(\lambda)$, which is possible only when $\lambda > 1$. Thus (iii) holds.

Sufficiency:

Case 1. λ even.

Subcase $1(i)$:$\lambda = 2$.

Now consider the subgraphs of $K_{m,m}(2)$ induced by $\alpha_1 \oplus \alpha_3$,$\alpha_2 \oplus \alpha_4$, $\alpha_3 \oplus \alpha_5$,$\ldots$, $\alpha_{m-3} \oplus \alpha_{m-1}$, $\alpha_{m-2} \oplus \alpha_m$, $\alpha_{m-1} \oplus \alpha_1$, $\alpha_m \oplus \alpha_2$. It is easy to see that each sum induces a C_m-factor of $K_{m,m}(2)$ and Hence, a C_m- factorization of $K_{m,m}(2)$ exists.

Now we construct the required number of P_m- factors from the C_m- factors constructed above, as follows:

$$\text{Let } P = \begin{cases} u_1v_mu_2v_{m-1}u_3v_{m-2}...u_{\frac{m}{2}-1}v_{\frac{m}{2}+2}u_{\frac{m}{2}}v_{\frac{m}{2}+1}, \\ u_mv_1u_{m-1}v_2u_{m-2}v_3...u_{\frac{m}{2}+2}v_{\frac{m}{2}-1}u_{\frac{m}{2}+1}v_{\frac{m}{2}}; \quad when \ \ m \equiv 0 \ (mod\ 4); \\ \\ u_1v_2u_mv_3u_{m-1}v_4...u_{\frac{m}{2}+3}v_{\frac{m}{2}-1}u_{\frac{m}{2}+2}v_{\frac{m}{2}+1}, \\ v_1u_2v_mu_3v_{m-1}u_4...v_{\frac{m}{2}+3}u_{\frac{m}{2}-1}v_{\frac{m}{2}+2}u_{\frac{m}{2}+1}; \quad when \ \ m \equiv 2 \ (mod\ 4). \end{cases}$$

Clearly, P is a P_m- factor of $K_{m,m}(2)$. One can observe that the edges of P_m in P are all from distinct distance 1-factors and also from distinct $(m-1)$ cycles (from $(m-1)$ C_m-factors) obtained above except the cycles from the C_m-factor induced by $\alpha_{m-1} \oplus \alpha_1$. Hence, $(C_m, P_m) \parallel K_{m,m}(2)$.

Subcase $1(ii)$: even $\lambda > 2$.

We write $K_{m,m}(\lambda) = K_{m,m}(\lambda-2) + K_{m,m}(2)$. By Subcase $1(i)$ and Theorems $1.1, 1.5$, $(C_m, P_m) \parallel K_{m,m}(2)$ and $C_m \parallel K_{m,m}(\lambda-2)$. Hence, $(C_m, P_m) \parallel K_{m,m}(\lambda)$

Case 2. λ odd.

Subcase $2(i)$: $\lambda = 3$.

First we construct the required factors of $K_{6,6}(3)$ as follows. Let $V(K_{6,6}) = \{u_1, u_2, u_3, u_4, u_5, u_6\} \cup \{v_1, v_2, v_3, v_4, v_5, v_6\}$. The 6 P_6-factors

$\{u_1v_2u_6v_3u_2v_4, u_4v_5u_3v_6u_5v_1\}$,$\{u_1v_3u_6v_4u_2v_5, u_4v_6u_3v_1u_5v_2\}$,
$\{u_2v_6u_4v_1u_3v_2, u_5v_3u_1v_4u_6v_5\}$,$\{u_2v_1u_4v_2u_3v_3, u_5v_4u_1v_5u_6v_6\}$,
$\{u_3v_4u_5v_5u_1v_6, u_6v_1u_2v_2u_4v_3\}$, $\{u_3v_5u_5v_6u_1v_1, u_6v_2u_2v_3u_4v_4\}$

and the 4 C_6-factors

$\{u_1v_6u_2v_4u_3v_2u_1, u_4v_3u_5v_1u_6v_5u_4\}$, $\{u_1v_4u_3v_6u_2v_2u_1, u_4v_5u_5v_3u_6v_1u_4\}$,
$\{u_1v_5u_2v_3u_3v_1u_1, u_4v_2u_5v_6u_6v_4u_4\}$, $\{u_1v_3u_3v_5u_2v_1u_1, u_4v_4u_5v_2u_6v_6u_4\}$,
together give the required factorization of $K_{6,6}(3)$.

When $m \neq 6$, we write $K_{m,m}(3) = K_{m,m}(2) \oplus K_{m,m}$. By Subcase $1(i)$ and Theorem 1.1, $(C_m, P_m) \parallel K_{m,m}(2)$ and $C_m \parallel K_{m,m}$ and Hence, the case.

Subcase $2(ii)$: odd $\lambda > 3$.

We write $K_{m,m}(\lambda) = K_{m,m}(\lambda-3) \oplus K_{m,m}(3)$. By Subcase 2($i$) and Theorems 1.1, 1.5, $(C_m, P_m) \parallel K_{m,m}(3)$ and $C_m \parallel K_{m,m}(\lambda-3)$. Hence, $(C_m, P_m) \parallel K_{m,m}(\lambda)$, for all $\lambda \geq 2$. □

By Lemma 1.3 and Theorem 3.1, we have the following:

Corollary 3.2. *If $K_{m,m}(\lambda)$ has a (C_m, P_m)-multifactorization, then so does $K_{ms,ms}(\lambda)$.*

4. $(C_m, K_{1,m-1})$-multifactorization of $K_{m,m}(\lambda)$

Theorem 4.1. *For $m > 2$, $K_{m,m}(\lambda)$ has a $(C_m, K_{1,m-1})$-multifactorization then*
(i) $m \equiv 0\ (mod\ 2)$,
(ii) $\mid E(K_{m,m})(\lambda) \mid = 2rm + 2s(m-1),\ r, s \geq 1$ and
(iii) $\lambda > 2$.

Proof. Since the graph under consideration is bipartite, cycles must be of even length and Hence, $m \equiv 0\ (mod\ 2)$. This proves (i). By the hypothesis, since $K_{m,m}(\lambda)$ has a $(C_m, K_{1,m-1})$-multifactorization, there exists integers $r, s \geq 1$, such that $K_{m,m}(\lambda) = r\ C \oplus s\ K$, where C and K respectively denote C_m-factor and $K_{1,m-1}$-factor. Hence, $\mid E(K_{m,m})(\lambda) \mid = 2rm + 2s(m-1)$ for $r, s \geq 1$. This proves (ii). From (ii), we have $\lambda m^2 = 2rm + 2s(m-1)$. Since L.H.S of the above equation is a multiple of 4, s should be even. Moreover, since L.H.S of the above equation is congruent to m, R.H.S will be congruent to m, only if $s \equiv 0\ (mod\ m)$. In such case, (i.e), when $s \equiv 0\ (mod\ m)$, we need $2m^2 - 2m$ edges, to have at least minimum number of $K_{1,m-1}$-factors in $K_{m,m}(\lambda)$, which is possible only when $\lambda > 1$.

When $\lambda = 2$, the only possibility is $(r, s) = (1, m)$. By the hypothesis, $K_{m,m}(2)$ can be factorized into m $K_{1,m-1}$-factors and a C_m- factor. After removing any arbitrary C_m-factor in $K_{m,m}(2)$, the degree of each vertex will be $2(m-1)$. Every vertex should be a center vertex for exactly one $K_{1,m-1}$ and pendant vertex for $(m-1)$ $K_{1,m-1}$, otherwise its degree will never be exhausted. Without loss of generality, let m $K_{1,m-1}$-factors be $\{(u_i : v_i, v_{i+1}, ..., v_{i+m-2}); (v_{i+m-1} : u_{i+1}, u_{i+2}, ..., u_{i+m-1}), 1 \leq i \leq m\}$, then the edges of multiplicity '2' between the center vertices of the $K_{1,m-1}$-factors cannot be used for the C_m-factor, a contradiction to our assumption. Hence, $\lambda > 2$. Thus (iii) holds. □

Lemma 4.2. *For $m > 2$, there exists a $(C_m, K_{1,m-1})$-multifactorization of $K_{m,m}(\lambda)$ if one of the following holds*
(i) $\lambda \equiv 0\ (mod\ 2) > 2$ or
(ii) $\lambda \equiv 1\ (mod\ 2) > 3$, and $m \neq 6$.

Proof. **Case 1.:** $\lambda \equiv 0\ (mod\ 2) > 2$.
Subcase 1(i)*:* $\lambda = 4$.

We can write
$$K_{m,m}(4) = K_{m,m}(2) \oplus K_{m,m}(2)$$
$$= (\cup_{i=1}^{m} S_i^1 \oplus \alpha_1(2)) \oplus (\cup_{i=1}^{m} S_i^3 \oplus \alpha_3(2)), \text{ by Remark 1.6,}$$

$$= \cup_{i=1}^{m}(S_i^1 \oplus S_i^3) \oplus (\alpha_1(2) \oplus \alpha_3(2)),$$

where S_i^1, S_i^3 are $K_{1,m-1}$-factors of $K_{m,m}$. The remaining edges of $(\alpha_1 \oplus \alpha_3)(2)$ provides 2 C_m-factors of $K_{m,m}$.

Subcase $1(ii)$**:** even $\lambda > 4$.

For $\lambda > 4$, we write $K_{m,m}(\lambda) = K_{m,m}(\lambda - 4) \oplus K_{m,m}(4)$. By Subcase $1(i)$ and Theorems 1.1, 1.5, we have $(K_{1,m-1}, C_m) \parallel K_{m,m}(\lambda)$.

Case 2. $\lambda \equiv 1 \ (mod\ 2) > 3$, and $m \neq 6$.

For $\lambda > 3$, we write $K_{m,m}(\lambda) = K_{m,m}(\lambda - 1) \oplus K_{m,m}$. By Case 1 and Theorem 1.1, we have $(K_{1,m-1}, C_m) \parallel K_{m,m}(\lambda)$. □

By Lemma 1.3 and Theorem 4.1, we have the following:

Corollary 4.3. *If $K_{m,m}(\lambda)$ has a $(C_m, K_{1,m-1})$-multifactorization, then so does $K_{ms,ms}(\lambda)$.*

The above results leads to raise the following:

Conjecture 4.4. *For all $m, n > 2$, there exists a (K, H)-multifactorization of $K_{m;n}(\lambda)$, the complete m-partite multi graph with n vertices in each part, where K, H are combination of path factor, star factor and cycle factor.*

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