AKCE Int. J. Graphs Comb., 10, No. 4 (2013), pp. 415-426

Monochromatic absorbency and independence in 3-quasi-transitive digraphs

Hortensia Galeana-Sánchez and Eugenia O'Reilly-Regueiro
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Área de la Investigacin Científica
Circuito Exterior, Ciudad Universitaria
Coyoacán 04510, México, D.F. México
e-mail: *hgaleana@matem.unam.mx, eugenia@matem.unam.mx*

Communicated by: T.W. Haynes
Received 8 December 2011; accepted 25 January 2013

Dedicated to the memory of Dr. B.D. Acharya

Abstract

We call the digraph D an m-coloured digraph if its arcs are coloured with m colours. In an m-coloured digraph D we say that a subdigraph H is: *monochromatic* whenever all of its arcs are coloured alike, and *almost monochromatic* if with at most 1 exception all of its arcs are coloured with the same colour.

If D is an m-coloured digraph a *kmp* or a *kernel by monochromatic paths* of D is a set K of vertices of D which is independent by monochromatic paths (for any two different vertices $u, v \in K$ there are no monochromatic paths between them) such that for every other vertex $x \in V(D) \setminus K$ there is a vertex $v \in K$ such that there is an xv monochromatic directed path in D.

A digraph D is *3-quasi-transitive* if whenever (x, y), (y, w), and $(w, z) \in A(D)$ with x, y, w and z pairwise different vertices, either (x, z) or (z, x) is in $A(D)$, and it is *asymmetric* if it has no symmetric arcs.

In 1982, Sands, Sauer, and Woodrow proved that every 2-coloured tournament has a kmp. They also posed the following problem: Let T be a 3-coloured tournament which does not contain $\hat{C}_3$ (the 3-coloured cyclic tournament of order 3). Then, must T contain a kmp?

In this paper we consider asymmetric 3-quasi-transitive digraphs, which not only generalise tournaments but also bipartite tournaments, and prove that if D is an m-coloured asymmetric 3-quasi-transitive digraph such that every C_4 (the directed cycle of length 4) is monochromatic and every C_3 (the directed cycle of length 3) is almost monochromatic, then D has a kernel by monochromatic paths.

We also note that the hypotheses on C_3 and C_4 are tight.

Keywords: monochromatic path, 3-quasi-transitive digraphs, monochromatic absorbency, monochromatic independence, kernel by monochromatic paths

2010 Mathematics Subject Classification: 05C20.

1. Introduction

For general concepts, we refer the reader to [1, 2]. Throughout this paper all paths and cycles will be directed paths and directed cycles. The topic of absorbency in graphs has been widely studied by several authors, and a complete study of this topic is presented in [16, 17]. A special class of absorbency is absorbency in digraphs, and it is defined as follows: Let D be a digraph. A set of vertices $S \subseteq V(D)$ is *absorbent* whenever for every $w \in (V(D) \setminus S)$ there exists a wS-arc in D. Absorbent independent sets in digraphs (kernels in digraphs) have found many applications in different topics of mathematics (see, for instance, [4, 5, 7, 8]) and they have been studied by several authors. Interesting surveys of kernels in digraphs can be found in [6, 9].

Let D be an m-*coloured* digraph. A set $N \subseteq V(D)$ is said to be a *kernel by monochromatic paths* (kmp) if it satisfies the following two conditions:

1. for every pair of different vertices $u, v \in N$ there is no monochromatic path between them, and
2. for every vertex $x \in (V(D) \setminus N)$ there is a vertex $y \in N$ such that there is a xy-monochromatic path.

Clearly the concepts of absorbency, independence, and kernel by monochromatic paths in edge-coloured digraphs are a generalisation of those of absorbency, independence, and kernel in digraphs. The study of the existence of kernels by monochromatic paths in edge-coloured digraphs starts with the theorem of Sands, Sauer, and Woodrow, proved in [18], which asserts that every 2-coloured digraph has a kernel by monochromatic paths. In several papers (see [10, 11, 12]), sufficient conditions for the existence of kernels by monochromatic paths in edge-coloured digraphs have been obtained mainly for tournaments and near tournaments, and require monochromaticity or almost monochromaticity of small subdigraphs (due to the difficulty of the problem). Other interesting results can be found in [13]. In [10] (resp. [14]) it was proved that if D is an m-coloured tournament (resp. bipartite tournament) such that every cycle of length 3 (resp. every cycle of length 4) is monochromatic, then D has a kernel by monochromatic paths.

Here we consider asymmetric, 3-quasi-transitive digraphs, which not only generalise tournaments, but also bipartite tournaments. In this paper we prove that if D is a coloured asymmetric 3-quasi-transitive digraph such that every C_4 is monochromatic and every C_3 is almost monochromatic, then D has a kernel by monochromatic paths. To do this we shift our attention to the *closure* of D, that is, $\mathcal{C}(D)$ which is a multidigraph defined thus: $V(\mathcal{C}(D)) = V(D)$, and the arc $(x, y) \in \mathcal{C}(D)$ if and only if in D there is a monochromatic path from x to y. We prove that every cycle in $\mathcal{C}(D)$ has a symmetric arc, which by Lemma 3.2 implies D has a kernel by monochromatic paths. Note that D has a kernel by monochromatic paths if and only if $\mathcal{C}(D)$ has a kernel.

We would like to point out that the hypotheses on the 3 and 4-cycles cannot be relaxed further. In the case of triangles, a 3-coloured triangle is itself a counterexample to the result, and an infinite family of counterexamples can be built as follows: Start with a 3-coloured triangle (x, y, z). Add a vertex w_0, and arcs $(w_0, x), (w_0, y)$, and (w_0, z) all the same colour. Next add a vertex w_1, and arcs $(w_1, w_0), (w_1, x), (w_1, y)$, and (w_1, z), all the same colour. This procedure can be repeated any number of times, in each step adding a vertex and arcs from that vertex to all previous vertices coloured in the same way. These graphs are 3-quasi-transitive, and do not have a kernel by monochromatic paths.

Regarding 4-cycles, as pointed out in [15], we can construct an infinite family of counterexamples. Start with D a 3-coloured bipartite tournament (hence 3-quasi-transitive and asymmetric) defined as follows:

$V(D) = \{u, v, w, x, y, z\}$ and

$A(D) = \{(u, x), (x, v), (u, y), (y, w), (w, z), (z, u), (x, w), (y, u), (z, v)\}$,

with $(x, w), (w, z)$, and (z, u) coloured 1, $(y, u), (u, x)$, and (x, v) coloured 2, and $(z, v), (v, y)$, and (y, w) coloured 3. The only 4-cycles in D are (u, x, w, z, u), (v, y, u, x, v), and (w, z, v, y, w) which are all almost monochromatic, and the digraph $\mathcal{C}(D)$ is a complete digraph with no kernel, so D has no kernel by monochromatic paths. For the infinite family of counterexamples, let D_n be the bipartite tournament (hence 3-quasi-transitive and asymmetric) obtained from D by adding vertices $z_1, \ldots, z_n$ and arcs of colour 3 from each of these vertices to u, v, and w respectively.

Finally with respect to the asymmetry hypothesis, we cannot claim that it is tight. There are instances of 3-quasi-transitive coloured digraphs such that their closure has a cycle γ with no symmetric arc. For example, define D as follows:

$V(D) = \{x, y, z, w\}$ and

$A(D) = \{(x, w), (w, x), (y, w), (w, y), (z, w), (w, z)\}$,

with (x, w) and (w, y) coloured 1, (y, w) and (w, z) coloured 2, and (z, w) and (w, y) coloured 3. The closure of D contains the cycle $\gamma = (x, y, z, x)$ which has no symmetric arc. These digraphs are sometimes referred to as "flowers", and any such flower with any number of "petals" with the same colouring pattern gives rise to a cycle in its closure with no symmetric arc.

In the method we use to prove that our digraph has a kernel by monochromatic paths, we use the fact that every cycle in the closure of D has a symmetric arc, which by Theorem 3.1 implies $\mathcal{C}(D)$ has a kernel, so by Lemma 3.2 D has a kernel by monochromatic paths. We must point out, however, that the failure of $\mathcal{C}(D)$ to have a symmetric arc in every cycle does not imply that $\mathcal{C}(D)$ does not have a kernel, it merely renders our method of proof useless for this case. We do not know whether our main result still holds if we drop the asymmetry hypothesis.

2. Preliminaries

An arc-coloured digraph D is said to be m-coloured if its arcs are coloured with m colours. The set of vertices of D will be denoted by $V(D)$, and the arcs of D will be

$A(D)$. An arc $(u,v) \in A(D)$ is called *asymmetrical* (resp. *symmetrical*) if $(v,u) \notin A(D)$ (resp. if $(v,u) \in A(D)$). The asymmetrical part of D (resp. symmetrical) which is denoted by $\mathrm{Asym}(D)$ (resp. $\mathrm{Sym}(D)$) is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D. If S is a non-empty subset of $V(D)$ then the subdigraph $D[S]$ of D induced by S is the digraph having vertex set S and whose arcs are the arcs of D joining vertices of S.

We call a subset $K \subseteq V(D)$ *independent* if there are no arcs between its vertices, and *independent by monochromatic paths* if for any two vertices in K, there are no monochromatic paths between them. The subset $K \subseteq V(D)$ is *absorbent* if for any vertex $x \notin K$ there is a vertex $y \in K$ such that the arc $(x,y) \in A(D)$, and *absorbent by monochromatic paths* if for any vertex x outside of K, there is a vertex y in K such that there is a monochromatic path from x to y. A subset $K \subset V(D)$ is a *kernel* of D if it is both independent and absorbent, and it is a *kernel by monochromatic paths* if it is both independent and absorbent by monochromatic paths.

If D is an m-coloured digraph then the *closure* of D, denoted by $\mathcal{C}(D)$ is the m-coloured multidigraph defined as follows: $V(\mathcal{C}(D)) = V(D)$, and $A(\mathcal{C}(D)) = A(D) \cup \{(u,v)_i |$ in D there exists a uv-monochromatic path with colour $i\}$, where $(u,v)_i$ denotes the arc (u,v) coloured with colour i. Notice that for any digraph D, $\mathcal{C}(\mathcal{C}(D)) \cong \mathcal{C}(D)$ and K is a kmp of D if and only if K is a kernel of $\mathcal{C}(D)$.

A *tournament* T is a digraph such that between any two vertices there is one and only one arc, and a digraph D is *3-quasi-transitive* if whenever $(x,y),(y,w),(w,z) \in A(D)$ with x,y,w and z pairwise different vertices then either (x,z) or (z,x) is in $A(D)$.

The following easy observation will be used throughout this paper, without further explanation: If D is an asymmetric digraph then every open directed walk of length 3 is a path, that is, all vertices and arcs are pairwise different.

3. Results

We will use the following theorem:

Theorem 3.1. [3] *Let D be a digraph. If every directed cycle of D has at least one symmetric arc, then D has a kernel.*

The following lemma is at the heart of the proof of our main result:

Lemma 3.2. *If D is an m-coloured digraph such that every cycle γ in $\mathcal{C}(D)$ has a symmetric arc, then D has a kernel by monochromatic paths.*

Proof. Let D be an m-coloured digraph such that every cycle γ in $\mathcal{C}(D)$ has a symmetric arc. This by Theorem 3.1 implies $\mathcal{C}(D)$ has a kernel, which in turn implies D has a kernel by monochromatic paths. □

Also, a well known result in digraphs:

Lemma 3.3. *Let D be a digraph and $x, y \in V(D)$. Then every xy-walk in D contains a xy-path.*

Lemma 3.4. *Let D be an asymmetric 3-quasi-transitive digraph, and $u, v \in V(D)$ such that there is a uv-path P and no vu-path. Then one of the following holds:*

1. *$(u, v) \in A(D)$ (when $l(P)$ is odd), or*
2. *There is a vertex $w \in P$ such that (u, w) and (w, v) are arcs in D (when $l(P)$ is even).*

Proof. Let D be an asymmetric 3-quasi-transitive digraph, and $u, v \in V(D)$ such that there is a path $P = (u = w_0, w_1, \ldots, w_n = v)$ and no vu-path. We have two cases:

1. $l(P)$ odd. We will prove, by induction on $l(P)$, that $(u, v) \in A(D)$. If $l(P) = 3$ then there are vertices w_1 and w_2 in $V(D)$ such that $P = (u, w_1, w_2, v)$ (and all these vertices are distinct). Since D is 3-quasi-transitive, either (u, v) or (v, u) is in $A(D)$. Since we assumed $(v, u) \notin A(D)$, we conclude $(u, v) \in A(D)$, which proves the basis of our induction.

Now suppose the result is true for $3 \leq m < n = 2k + 1$, and let $u, v \in V(D)$ such that there is a path $P = (u = w_0, w_1, \ldots w_n = v)$ with $l(P) = n = 2k + 1$ and there is no vu-path in D. Since D is 3-quasi-transitive, either (w_0, w_3) or (w_3, w_0) is in $A(D)$.

If $(w_0, w_3) \in A(D)$, then there is a path $P' = (w_0, w_3, \ldots, w_n)$ in D with $l(P') = 2k - 1$, with no $w_n w_0$-path, so by induction hypothesis, $(w_0 = u, v = w_n) \in A(D)$ and we are done.

Now suppose $(w_3, w_0) \in A(D)$, and consider the path (which is part of P) $Q = (w_2, \ldots, w_n)$, which is of length $2k - 1$. If there is a path from w_n to w_2, then there is a directed walk

$$(w_n, \ldots, w_2, w_3, w_0),$$

which by Lemma 3.3 contains a $w_n w_0$-path, contradicting our assumption. Therefore there is no path from w_n to w_2, and by our induction hypothesis, $(w_2, w_n) \in A(D)$, so there is a path (w_0, w_1, w_2, w_n) in D, and since there is no path from w_n to w_0 and D is 3-quasi-transitive, we conclude $(w_0, w_n) \in A(D)$.

2. $l(P)$ is even. As above, we will prove the result by induction on $n = l(P)$, so first let $n = 2$, and the proof is immediate. For the sake of clarity, now let $n = 4$, so $P = (w_0, w_1, w_2, w_3, w_4)$. Since P is a path, these vertices are all distinct, and D 3-quasi-transitive implies either $(w_0, w_3) \in A(D)$ or $(w_3, w_0) \in A(D)$, and either $(w_1, w_4) \in A(D)$ or $(w_4, w_1) \in A(D)$. If both (w_3, w_0) and (w_4, w_1) are in $A(D)$, then $(w_4, w_1, w_2, w_3, w_0)$ is a path from w_4 to w_0, a contradiction, so at least one of $(w_0, w_3) \in A(D)$ or $(w_1, w_4) \in A(D)$ holds. Then (w_0, w_3, w_4) and/or (w_0, w_1, w_4) are/is (a) path(s) in D, which proves the lemma for $n = 4$.

For the induction hypothesis, suppose the result is true for $6 \leq k \leq n-2$ and let $P = (u = w_0, w_1, \ldots, w_{n-2}, w_{n-1}, w_n = v)$ be a path in D such that there is no $w_n w_0$-path in D.

Since P is a path, $w_0, \ldots, w_n$ are all distinct, and since D is 3-quasi-transitive, for every w_i, w_{i+3} with $0 \leq i \leq n-3$ either $(w_i, w_{i+3}) \in A(D)$ or $(w_{i+3}, w_i) \in A(D)$. If there is $i \in \{0, \ldots, n-3\}$ such that $(w_i, w_{i+3}) \in A(D)$ then the path $P' = (w_0, \ldots, w_i, w_{i+3}, \ldots, w_n)$ has length $n-2$ and there is no path from w_n to w_0, so by our induction hypothesis there is $w \in V(D)$ such that (w_0, w) and (w, w_n) are in $A(D)$.

If, on the other hand, $(w_{i+3}, w_i) \in A(D)$ $\forall i = 0, \ldots, n-3$ then $(w_n, w_{n-3}, w_{n-2}, w_{n-1}, w_{n-4}, w_{n-3}, w_{n-2}, \ldots, w_1, w_2, w_3, w_0)$ is a directed walk from w_n to w_0, which by Lemma 3.3 contains a $w_n w_0$-path, a contradiction. □

Lemma 3.5. *Let D be a coloured asymmetric 3-quasi-transitive digraph such that every C_4 is monochromatic, and $u, v \in V(D)$ such that there is a monochromatic uv-path P and no monochromatic vu-path. Then either:*

1. $(u, v) \in A(D)$ *(when $l(P)$ is odd), or*
2. *There exists $w \in V(D)$ such that (u, w) and (w, v) are arcs in D (when $l(P)$ is even).*

Proof. Let D be a coloured asymmetric 3-quasi-transitive digraph with every C_4 monochromatic, and suppose $u, v \in V(D)$ are such that there is a monochromatic uv-path P of length n and there is no monochromatic vu-path. Let $P = (u = w_0, w_1, \ldots, w_n = v)$ be a monochromatic uv-path in D of minimum length, and suppose there is no monochromatic vu-path in D.

Note if $n = 2$ the result follows, and if $n = 3$ then D 3-quasi-transitive implies either $(w_0, w_3) \in A(D)$ or $(w_3, w_0) \in A(D)$, the latter is a monochromatic vu-path, a contradiction, so $(w_0, w_n) \in A(D)$, proving the lemma for $n = 3$, and so we assume $n \geq 4$.

First we will prove by induction that either the lemma holds, or for every w_i, w_j with $0 \leq i < j \leq n$ and $j - i \geq 2$, $(w_i, w_j) \notin A(D)$, that is, if there is an arc between w_i and w_j then it is (w_j, w_i), that is, it goes "backwards".

We will do the basis of our induction for both w_0 and w_1. Consider the set of vertices $\{w_i\}$ of P such that there is an arc between w_0 and w_i, with $i \geq 2$, and note w_3 is one such vertex since D is 3-quasi-transitive, so this set is not empty. If for all such w_i the arc (w_i, w_0) is in $A(D)$, then we are done, so suppose there is at least one vertex w_i such that $(w_0, w_i) \in A(D)$, and let j be the maximum subindex such that $(w_0, w_j) \in A(D)$. If $j = n$ then $(w_0, w_n) \in A(D)$ and the lemma is proved, similarly, if $j = n-1$ then (w_0, w_{n-1}) and (w_{n-1}, w_n) are in $A(D)$, also proving the lemma, so suppose $j \leq n-2$.

Since D is 3-quasi-transitive $(w_0, w_j) \in A(D)$ implies there is an arc between w_0 and w_{j+2}, and j being the maximum subindex such that the arc goes "forwards" implies

$(w_{j+2}, w_0) \in A(D)$. Then $(w_0, w_j, w_{j+1}, w_{j+2})$ is a C_4 in D, and is therefore monochromatic, which makes $(w_0, w_j, \ldots, w_n)$ a monochromatic uv-path of length shorter than $l(P)$, a contradiction. We conclude that for every $w_i \in P$ such that there is an arc between w_0 and w_i and $i \geq 2$, $(w_i, w_0) \in A(D)$.

We now prove the result for w_1 in the same way. If for every $w_i \in P$ (with $i > 2$) such that there is an arc between w_1 and w_i $(w_i, w_1) \in A(D)$, then we are done, so consider w_j to be the vertex furthest from w_1 such that $(w_1, w_j) \in A(D)$ and suppose $j > 2$. As above, if $j = n$ then the lemma follows. If $j = n-1$ then since D is 3-quasi-transitive, there is an arc between w_0 and w_n. If $(w_0, w_n) \in A(D)$ we have a monochromatic path between w_0 and w_n of length shorter than $l(P)$, a contradiction, and if $(w_n, w_0) \in A(D)$ then there is a monochromatic vu-path in D, another contradiction. Therefore $2 < j \leq n-2$. Since D is 3-quasi-transitive, there is an arc between w_1 and w_{j+2}, and since w_j is the vertex in P furthest from w_1 such that $(w_1, w_j) \in A(D)$, we conclude $(w_{j+2}, w_1) \in A(D)$ so $(w_1, w_j, w_{j+1}, w_{j+2})$ is a C_4 in D and therefore monochromatic, so $(w_0, w_1, w_j, \ldots, w_n)$ is a monochromatic uv-path in D of length shorter than $l(P)$, a contradiction.

Now for our induction hypothesis suppose that for every $i < k$, if there is an arc between w_i and w_j and $j > i+1$, then $(w_j, w_i) \in A(D)$, and consider w_k. If $k = n$ or $n-1$ then we have nothing to prove, so suppose $k < n-1$. Suppose also there is $j > k+1$ such that $(w_k, w_j) \in A(D)$, and let m be the maximum of these subindices.

If $m = n$ then $(w_k, w_n) \in A(D)$, also (w_{k-2}, w_{k-1}) and (w_{k-1}, w_k) are in $A(D)$, and since D is 3-quasi-transitive, there is an arc between w_{k-2} and w_n, which by our induction hypothesis must be (w_n, w_{k-2}).

Then $(w_{k-2}, w_{k-1}, w_k, w_n)$ is a C_4 in D which must therefore be monochromatic, which implies $(w_0, \ldots, w_k, w_n)$ is a monochromatic uv-path of length shorter than P, a contradiction.

If $m = n-1$ then (w_{k-1}, w_k), (w_k, w_{n-1}), and (w_{n-1}, w_n) are arcs in D which is 3-quasi-transitive, this implies there is an arc between w_{k-1} and w_n, which by induction hypothesis must be (w_n, w_{k-1}). Then $(w_{k-1}, w_k, w_{n-1}, w_n)$ is a C_4 in D, which is monochromatic, so $(w_0, \ldots, w_k, w_{n-1}, w_n)$ is a monochromatic uv-path of length shorter than that of P, a contradiction.

Finally, if $m < n-1$ consider the arcs (w_k, w_m), (w_m, w_{m+1}), and (w_{m+1}, w_{m+2}). Since D is 3-quasi-transitive, there is an arc between w_k and w_{m+2}. Since m is the maximum subindex such that $(w_k, w_m) \in A(D)$, we conclude $(w_{m+2}, w_k) \in A(D)$, so $(w_k, w_m, w_{m+1}, w_{m+2})$ is a C_4 in D, so it is monochromatic. This implies $(w_0, \ldots, w_k, w_m, \ldots, w_n)$ is a monochromatic uv-path in D of shorter length than P, a contradiction, with which the proof or our claim is complete.

Since D is 3-quasi-transitive, for every $i = 0, \ldots, n-3$ either (w_i, w_{i+3}) or (w_{i+3}, w_i) is in $A(D)$, and by the claim we have just proved, $(w_{i+3}, w_i) \in A(D)$. This implies, for each $i = 0, \ldots, n-3$, there is a C_4 in D, namely $(w_i, w_{i+1}, w_{i+2}, w_{i+3})$, which is monochromatic.

This yields a monochromatic vu-walk in D, namely

$$(w_n, w_{n-3}, w_{n-2}, w_{n-1}, w_{n-4}, \dots, w_1, w_2, w_3, w_0),$$

which by Lemma 3.3 contains a monochromatic uv-path, a contradiction.

Therefore the conclusions of the lemma hold. □

Lemma 3.6. *Let D be a coloured asymmetric 3-quasi-transitive digraph such that every C_4 is monochromatic, and every C_3 is almost monochromatic. Suppose there is an asymmetric cycle γ in $\mathcal{C}(D)$. Then $l(\gamma) \geq 4$.*

Proof. If γ is an asymmetric cycle in $\mathcal{C}(D)$ then it has length at least 3, so suppose $l(\gamma) = 3$ and $\gamma = (x, y, z)$ is asymmetric. Then by Lemma 3.5 there are xy-, yz-, and zx-paths with length 1 or 2, so we have four cases:

1. They all have length 1, and (x, y, z) is a directed triangle in D. In this case, since all directed triangles in D are almost monochromatic, there are two arcs of the same colour, say, (x, y) and (y, z). This means there is a monochromatic xz-path, which induces the arc (x, z) in $\mathcal{C}(D)$, which is a contradiction as we assumed the cycle γ to be asymmetric.

2. One of the paths, say, xy has length 2 in D, and the others have length 1, that is, there is a vertex x_0 in D such that (x, x_0, y, z) is a cycle in D. This is a C_4, so it must be monochromatic, so there is a monochromatic path from any vertex to any other vertex, that is, γ is symmetric, a contradiction.

3. Two of the paths, say xy and yz are of length 2, the other of length 1 in D. Then there are vertices x_0 and y_0 in D such that (x, x_0, y, y_0, z) is a cycle in D. Since D is 3-quasi-transitive and (y, y_0, z, x) is a path in D, either (x, y) or (y, x) is in $A(D)$. If $(y, x) \in A(D)$ then this is a symmetric arc in γ, a contradiction. If, on the other hand, $(x, y) \in A(D)$ then (y, y_0, z, x) is a C_4 in D, and must therefore be monochromatic, which implies the arc (y, x) is in γ contradicting the assumption of γ being asymmetric.

4. Finally, the three paths have length 2, so there are vertices x_0, y_0, and z_0 in D such that (x, x_0, y, y_0, z, z_0) is a cycle in D. Since D is 3-quasi-transitive, either (x, y_0) or (y_0, x) is in $A(D)$. Suppose first that $(x, y_0) \in A(D)$, so (x, y_0, z, z_0) is a C_4 in D, and hence monochromatic. Similarly, either (z, x_0) or (x_0, z) is in $A(D)$. If $(z, x_0) \in A(D)$ then (z, x_0, y, y_0) is a C_4 in D, and therefore also monochromatic. Given the overlap of these two cycles, the path (x_0, y, y_0, z, z_0, x) is monochromatic, so the arc $(y, x) \in \gamma$, a contradiction. Suppose now that $(x_0, z) \in A(D)$. Then (x_0, z, z_0, x) is a C_4 in D, and hence monochromatic. Again, the overlap of these two cycles implies the path (y_0, z, z_0, x, x_0) is monochromatic. Also, either (y, z_0) or (z_0, y) is in $A(D)$. Following the same reasoning as above, if $(y, z_0) \in A(D)$ then there is a monochromatic path in D from z to y, so in γ the arc (y, z) is symmetric,

a contradiction. If, however, $(z_0, y) \in A(D)$ then in D there is a monochromatic path from y to x, so the arc (x, y) in γ is symmetric, again, a contradiction.

Now suppose $(y_0, x) \in A(D)$. The proof is analogous, due to symmetry.

□

Lemma 3.7. *Let D be a coloured asymmetric 3-quasi-transitive digraph such that every C_4 is monochromatic and every C_3 is almost monochromatic, and $\mathcal{C}(D)$ the closure of D. Suppose there is a asymmetric cycle γ in $\mathcal{C}(D)$ and consider γ' to be the corresponding closed directed walk in D, that is, the vertices u, v and arcs (u, v) of γ when $(u, v) \in A(D)$ plus the vertices w and arcs (u, w) and (w, v) in D when $(u, v) \notin A(D)$.*

If the vertices of the closed directed walk γ' are $x_0, x_1, \ldots, x_n$, then $(x_0, x_{2k+1}) \in A(D)$ for every k such that $3 \leq 2k + 1 < n$ and for any $x_0 \in \gamma$.

Proof. Since D is 3-quasi-transitive there is an arc between x_0 and x_3. Suppose $(x_3, x_0) \in A(D)$. Then (x_0, x_1, x_2, x_3) is a C_4 in D (all vertices are distinct) and is therefore monochromatic. If $x_1 \in \gamma$ then there is a monochromatic path from x_1 to x_0, contradicting the assumption that γ is asymmetric. If $x_1 \notin \gamma$, then $x_2 \in \gamma$ and the same reasoning applies. Therefore $(x_0, x_3) \in A(D)$.

If $n = 4$ then we are done, so suppose $n > 4$, consider x_5 so $x_5 \neq x_0$ and note $x_5 \neq x_4$. Also, D asymmetric implies $x_5 \neq x_3$. If $x_5 = x_1$ then $(x_0, x_5) \in A(D)$. If $x_5 = x_2$ then this vertex is not in γ, which forces x_1, x_3, and x_4 to be all in γ. Since D is 3-quasi-transitive and (x_0, x_3), (x_3, x_4), and (x_4, x_2) are all arcs in D, there must be an arc between x_0 and x_2. If $(x_2, x_0) \in A(D)$ then (x_0, x_3, x_4, x_2) is a C_4 in D, and so it must be monochromatic, which makes the arc $(x_3, x_4) \in \gamma$ symmetric, a contradiction.

Finally if x_5 is none of the previous vertices since $(x_0, x_3) \in A(D)$ and D is 3-quasi-transitive then either $(x_0, x_5) \in A(D)$ or $(x_5, x_0) \in A(D)$. If $(x_5, x_0) \in A(D)$ then (x_0, x_3, x_4, x_5) is a C_4 in D, hence monochromatic, which implies an arc in γ is symmetric, a contradiction. Therefore $(x_0, x_5) \in A(D)$.

Following the same reasoning, by induction, suppose the lemma is not true and let j be the first subindex such that $(x_0, x_{2j+1}) \notin A(D)$ (with $2j+1 < n$). Since $(x_0, x_{2j-1}) \in A(D)$ and D is 3-quasi-transitive, we conclude $(x_{2j+1}, x_0) \in A(D)$. We observe $x_0 \neq x_{2j-1}, x_{2j}$, and x_{2j+1}. Also, $x_{2j} \neq x_{2j-1}$ and x_{2j+1}, and since D is asymmetric $x_{2j-1} \neq x_{2j+1}$. That is, all four vertices are distinct. Then $(x_0, x_{2j-1}, x_{2j}, x_{2j+1})$ is a C_4 in D, hence monochromatic. If $x_{2j-1} \in \gamma$ then so is at least one of x_{2j} and x_{2j+1}, this implies there is a symmetric arc in γ, a contradiction. If $x_{2j-1} \notin \gamma$ then $x_{2j} \in \gamma$. If $x_{2j+1} \in \gamma$ then again we get a symmetric arc in γ, a contradiction. Now suppose neither x_{2j-1} nor x_{2j+1} are in γ (and $x_{2j} \in \gamma$).

We go back to x_{2j-3} and note that $(x_0, x_{2j-3}) \in A(D)$ (our first two steps of induction allow us to do this). We also note $x_{2j-2} \in \gamma$. Since D is 3-quasi-transitive, and

$(x_{2j+1}, x_0), (x_0, x_{2j-3})$, and (x_{2j-3}, x_{2j-2}) are all in $A(D)$, there must be an arc between x_{2j+1} and x_{2j-2} (and these are distinct vertices as one is in γ and the other one is not). If $(x_{2j+1}, x_{2j-2}) \in A(D)$ then $(x_{2j-2}, x_{2j-1}, x_{2j}, x_{2j+1})$ is a C_4 in D, and hence monochromatic. This makes the arc (x_{2j-2}, x_{2j}) in γ symmetric, a contradiction.

If, on the other hand, $(x_{2j-2}, x_{2j+1}) \in A(D)$, then $(x_0, x_{2j-3}, x_{2j-2}, x_{2j+1})$ is a C_4 in D, so it is monochromatic, and as it intersects $(x_0, x_{2j-1}, x_{2j}, x_{2j+1})$ in the arc (x_{2j+1}, x_0), the arcs in these two C_4s all have the same colour. This yields a monochromatic path from x_{2j} to x_{2j-2} which makes the arc $(x_{2j-2}, x_{2j}) \in \gamma$ symmetric, a contradiction with which our proof is now complete. □

Now we prove our main result.

Theorem 3.8. *Let D be a coloured asymmetric 3-quasi-transitive digraph such that every C_4 is monochromatic and every C_3 is almost monochromatic. Then D has a kernel by monochromatic paths.*

Proof. Let D be a coloured asymmetric 3-quasi-transitive digraph such that every C_4 is monochromatic and every C_3 is almost monochromatic, and consider $\mathcal{C}(D)$, the closure of D. If every cycle in $\mathcal{C}(D)$ has a symmetric arc, then by Theorem3.1 $\mathcal{C}(D)$ has a kernel, which by Lemma 3.2 implies D has a kernel by monochromatic paths, and we are done. So suppose in $\mathcal{C}(D)$ there is an asymmetric cycle, and let γ be such a cycle of minimum length, which, by Lemma 3.6 has length at least 4.

We consider $\gamma' = (x_0, \ldots, x_n)$ the corresponding closed directed walk in D, that is, the vertices and arcs of γ which are in D plus the vertices w and arcs (u, w) and (w, v) in D when $u, v \in V(\gamma)$, $(u, v) \in A(\gamma)$, and $w \notin V(\gamma)$, $(u, v) \notin A(D)$. We can assume w.l.o.g. that $x_0 \in V(\gamma)$, and by Lemma 3.7 $(x_0, x_{2j+1}) \in A(D)$ for every j such that $1 \leq 2j+1 < n$. We now consider two cases, according to the parity of $n \geq 4$.

First suppose n is odd, and consider the vertices x_0, x_n, x_{n-1}, and x_{n-2}. Note x_0 is different from any of the other vertices, otherwise the length of γ would be shorter. Also, $x_n \neq x_{n-1} \neq x_{n-2}$ since they are adjacent, and $x_n \neq x_{n-2}$ because D is asymmetric. Therefore $(x_0, x_{n-2}, x_{n-1}, x_n)$ is a C_4 and so it is monochromatic. Since $x_0 \in \gamma$ then at least one of x_n and x_{n-1} is in γ. If $x_n \in \gamma$ then the arc $(x_n, x_0) \in \gamma$ is symmetric, and if $x_n \notin \gamma$ then the arc $(x_{n-1}, x_n) \in \gamma$ is symmetric, in both cases we have a contradiction.

Now suppose n is even. As above, all the vertices x_0, x_n, x_{n-1}, and x_{n-2} are distinct. Since D is 3-quasi-transitive, there is an arc between x_0 and x_{n-2}. If $(x_0, x_{n-2}) \in A(D)$ then as in the previous paragraph there is a symmetric arc in γ, a contradiction. Therefore $(x_{n-2}, x_0) \in A(D)$. Since $(x_0, x_{n-1}) \in A(D)$, if $x_n \notin \gamma$ then $x_{n-1} \in \gamma$ and γ has a symmetric arc, a contradiction which implies $x_n \in \gamma$. Also, the vertices (x_0, x_{n-1}, x_n) form a C_3, which is almost monochromatic. If the arcs (x_0, x_{n-1}) and (x_{n-1}, x_n) have the same colour, then there is a monochromatic path in D from x_0 to x_n, which are consecutive vertices in γ, so in γ there is a symmetric arc, a contradiction.

Now suppose (x_n, x_0) and (x_0, x_{n-1}) have the same colour. Then there is a monochromatic path from x_n to x_{n-1}. This forces $x_{n-1} \notin \gamma$, otherwise (x_{n-1}, x_n) would be a symmetric arc in γ, a contradiction. Now $x_{n-1} \notin \gamma$, forces $x_{n-2} \in \gamma$. Since $x_{n-2} \neq x_0, x_n$, and x_{n-1}, and D is 3-quasi-transitive, there is an arc between x_{n-2} and x_n (because of the path $(x_{n-2}, x_0, x_{n-1}, x_n)$). If $(x_n, x_{n-2}) \in A(D)$ then it is a symmetric arc in γ, and if $(x_{n-2}, x_n) \in A(D)$ then in D these two vertices are at distance 1, so the existence of x_{n-1} in the cycle γ' is a contradiction. We conclude the arcs (x_{n-1}, x_n) and (x_n, x_0) have the same colour (and $x_{n-2} \in \gamma$).

We have proved that if n is even and we consider a vertex in γ' which is also in γ, then the preceding vertex is also in γ and the arc in γ' of which this vertex is an end point has the same colour than the preceding arc. Going backwards by induction, we conclude every vertex of γ' is in γ and every arc in $\gamma'(= \gamma)$ has the same colour, and therefore a symmetric arc, a contradiction.

We have proved every directed cycle in $\mathcal{C}(D)$ has a symmetric arc. This, by Theorem3.1 implies $\mathcal{C}(D)$ has a kernel, which in turn by Lemma 3.2 implies D has a kernel by monochromatic paths. □

References

[1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer, London, 2001.

[2] C. Berge, *Graphs*, North-Holland, Amsterdam, 1985.

[3] C. Berge and P. Duchet, Recent problems and results about kernels in directed graphs, *Discrete Math.*, **86** (1990), 27–31.

[4] J. M. Le Bars, Counterexample of the 0-1 law for fragments of existential second-order logic; an overview, *Bull. Symbolic Logic*, **9** (2000), 67–82.

[5] J. M. Le Bars, *The 0-1 law fails for frame satisfiability of propositional model logic, in:* Proceedings of the 17th Symposium on Logic in Computer Science, (2002), 225–234.

[6] E. Boros and V. Gurvich, Perfect graphs, kernels and cores of cooperative games, *Discrete Math.*, **306** (2006), 2336–2354.

[7] P. Duchet, Kernels in directed graphs: a poison game, *Discrete Math.*, **115** (1993), 273–276.

[8] A. S. Fraenkel, Combinatorial game theory foundations applied to digraph kernels, *Electron. J. Combin.*, **4** (1997), 17.

[9] A. S. Fraenkel, Combinatorial games: selected bibliography with a succinct gourmet introduction, *Electron. J. Combin.*, **14** (2007), #DS2.

[10] H. Galeana-Sánchez, Kernels in edge-coloured digraphs, *Discrete Math.*, **184** (1998), 87–99.

[11] H. Galeana-Sánchez and R. Rojas-Monroy, On monochromatic paths and monochromatic 4-cycles in edge coloured bipartite tournaments, *Discrete Math.*, **285** (2004), 313–318.

[12] H. Galeana-Sánchez and R. Rojas-Monroy, Monochromatic paths and at most 2-coloured arc sets in edge-coloured tournaments, *Graphs Combin.*, **21** (2005), 307–317.

[13] H. Galeana-Sánchez and R. Rojas-Monroy, Kernels and some operations in edge-coloured digraphs, *Discrete Math.*, **308** (2008), 6036–6046.

[14] H. Galeana-Sánchez and R. Rojas-Monroy, Independent domination by monochromatic paths in arc coloured bipartite tournaments, *AKCE Int. J. Graphs Combin.*, **6**(2) (2009), 267–285.

[15] H. Galeana-Sánchez and R. Rojas-Monroy, On monochromatic paths and monochromatic 4-cycles in edge coloured bipartite tournaments, *Discrete Math.*, **285** (2004), 313–318.

[16] T. W. Haynes, T. Hedetniemi and P. J. Slater, *Domination in Graphs, Advanced Topics*, Marcel Dekker Inc., (1998).

[17] T. W. Haynes, T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., (1998).

[18] B. Sands, N. Sauer and R. Woodrow, On monochromatic paths in edge-coloured digraphs, *J. Combin. Theory Ser. B*, **33** (1982), 271–275.