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10-Shredders in 10-Connected Graphs

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Dedicated to the memory of Dr. B.D. Acharya

Abstract

For a graph G, a subset S of $V(G)$ is called a shredder if $G - S$ consists of three or more components. We show that if G is a 10-connected graph of order at least 4227, then the number of shredders of cardinality 10 of G is less than or equal to $(2|V(G)| - 11)/3$.

Keywords: graph, connectivity, shredder, upper bound.

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1. Introduction

In this paper, we consider only finite, undirected, simple graphs with no loops and no multiple edges. Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, we let $N_G(x)$ denote the set of vertices adjacent to x in G. For $S \subseteq V(G)$, $N_G(S)$ denotes the union of $N_G(x)$ as x ranges over S. For $S \subseteq V(G)$, we let $G[S]$ denote the subgraph induced by S in G, and let $G - S$ denote the subgraph obtained from G by deleting all vertices in S together with the edges incident with them; thus $G - S = G[V(G) - S]$.

As is introduced by Cheriyan and Thurimella in [1], a subset S of $V(G)$ is called a *shredder* if $G - S$ consists of three or more components. A shredder of cardinality k is referred to as a k-shredder. In [2; Theorem 1], it is proved that if $k \geq 5$ and G is a k-connected graph, then the number of k-shredders of G is less than $2|V(G)|/3$, and it is shown that for each fixed $k \geq 5$, the coefficient $2/3$ in the upper bound is best possible.

For $k = 5$, it is shown in [2; Theorem 3] that if G is a 5-connected graph of order at least 135, then the number of 5-shredders of G is less than or equal to $(2|V(G)| - 10)/3$; for $k = 6$, it is shown in [8] that if G is a 6-connected graph of order at least 325, then the number of 6-shredders of G is less than or equal to $(2|V(G)|-9)/3$; for $k=7$, it is shown in [6] that if G is a 7-connected graph of order at least 42, then the number of 7-shredders of G is less than or equal to $(2|V(G)|-8)/3$; for $k=8$, it is shown in [7] that if G is a 8-connected graph of order at least 177, then the number of 8-shredders of G is less than or equal to $(2|V(G)|-10)/3$; for $k=9$, it is shown in [4] that if G is a 9-connected graph of order at least 67, then the number of 9-shredders of G is less than or equal to $(2|V(G)|-9)/3$. It is also shown that each of these five boundsis attained by infinitely many graphs.

For $k \geq 11$, it is shown in [2; Theorem 1] that if G is a k-connected graph of order at least $10k$, then the number of k-shredders of G is less than or equal to $(2|V(G)|-6)/3$, and the upper bound $(2|V(G)|-6)/3$ is believed to be best possible. If this bound is in fact best possible for $k \geq 11$, then the case where $k=10$ will be the only case for which the best possible bound has not been obtained (for results concerning the case where $1 \leq k \leq 4$, the reader is referred to [4] and [2; Theorem 2]). In this paper, we take up this remaining case, and prove the following theorem.

Theorem 1.1. *Let G be a 10-connected graph of order at least 4227. Then the number of 10-shredders of G is less than or equal to*

$$(2|V(G)|-11)/3.$$

We here construct an infinite family of graphs G which attain the bound $(2|V(G)|-11)/3$ in the Theorem. Let $m \geq 10$. We first define auxiliary graphs H_m and H'_m having orders m and $3m-6$, respectively. Define H_m by letting

$$\begin{aligned} V(H_m) &= \{v_i | 1 \leq i \leq m\}, \\ E(H_m) &= \{v_i v_{i+4} | 1 \leq i \leq m-4\} \\ &\cup \{v_1v_2,\ v_1v_3,\ v_1v_4,\ v_2v_3,\ v_2v_5,\ v_3v_4\} \\ &\cup \{v_{m-4}v_{m-1},\ v_{m-3}v_{m-2},\ v_{m-3}v_m,\ v_{m-2}v_{m-1},\ v_{m-2}v_m,\ v_{m-1}v_m\}. \end{aligned}$$

We define H'_m by adding $m-6$ vertices to the so-called lexicographic product of H_m and the null graph of order 2. More precisely, we let

$$V(H'_m) = \{x_{i,j} | 1 \leq i \leq m,\ 1 \leq j \leq 2\} \cup \{\alpha_i | 4 \leq i \leq m-4\} \cup \{a\},$$

$$\begin{aligned} E(H'_m) = &\{x_{i,j}x_{i+4,k} | 1 \leq i \leq m-4,\ 1 \leq j,k \leq 2\} \\ \cup &\{x_{i-1,j}\alpha_i,\ x_{i,j}\alpha_i,\ x_{i+1,j}\alpha_i,\ x_{i+2,j}\alpha_i | 4 \leq i \leq m-4,\ 1 \leq j \leq 2\} \\ \cup &\{a\alpha_i | 4 \leq i \leq m-4\} \\ \cup &\{ax_{i,j} | 1 \leq i \leq m \textit{ and } i \neq 3,5,m-4,m-2,\ 1 \leq j \leq 2\} \\ \cup &\{x_{1,j}x_{2,k},\ x_{1,j}x_{3,k},\ x_{1,j}x_{4,k},\ x_{2,j}x_{3,k},\ x_{2,j}x_{5,k},\ x_{3,j}x_{4,k} | 1 \leq j,k \leq 2\} \\ \cup &\{x_{m-4,j}x_{m-1,k},\ x_{m-3,j}x_{m-2,k},\ x_{m-3,j}x_{m,k},\ x_{m-2,j}x_{m-1,k}, \\ &x_{m-2,j}x_{m,k},\ x_{m-1,j}x_{m,k} | 1 \leq j,k \leq 2\}. \end{aligned}$$

Now define a graph G_m of oder $3m-5$ by

$$\begin{aligned} V(G_m) &= V(H'_m) \cup \{b\} \\ E(G_m) &= E(H'_m) \cup \{bv \mid v \in V(H'_m)\}. \end{aligned}$$

It is shown in [3] that H'_m is 9-connected. Hence G_m is 10-connected. Also G_m has $2m-7$ 10-shredders.

$\{x_{i-4,1}, x_{i-4,2}, x_{i+4,1}, x_{i+4,2}, \alpha_{i-2}, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, a, b\}$ $(6 \le i \le m-5)$,

$\{x_{i-1,1}, x_{i-1,2}, x_{i,1}, x_{i,2}, x_{i+1,1}, x_{i+1,2}, x_{i+2,1}, x_{i+2,2}, a, b\}$ $(4 \le i \le m-4)$,

$\{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{9,1}, x_{9,2}, \alpha_4, \alpha_5, \alpha_6, b\}$,

$\{x_{m-8,1}, x_{m-8,2}, x_{m-1,1}, x_{m-1,2}, x_{m,1}, x_{m,2}, \alpha_{m-6}, \alpha_{m-5}, \alpha_{m-4}, b\}$,

$\{x_{1,1}, x_{1,2}, x_{3,1}, x_{3,2}, x_{8,1}, x_{8,2}, \alpha_4, \alpha_5, a, b\}$,

$\{x_{m-7,1}, x_{m-7,2}, x_{m-4,1}, x_{m-4,2}, x_{m,1}, x_{m,2}, \alpha_{m-5}, \alpha_{m-4}, a, b\}$,

$\{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{4,1}, x_{4,2}, x_{7,1}, x_{7,2}\alpha_4, b\}$,

$\{x_{m-6,1}, x_{m-6,2}, x_{m-3,1}, x_{m-3,2}, x_{m-1,1}, x_{m-1,2}, x_{m,1}, x_{m,2}, \alpha_{m-4}, b\}$,

$\{x_{1,1}, x_{1,2}, x_{3,1}, x_{3,2}, x_{5,1}, x_{5,2}, x_{6,1}, x_{6,2}, a, b\}$,

$\{x_{m-5,1}, x_{m-5,2}, x_{m-4,1}, x_{m-4,2}, x_{m-1,1}, x_{m-1,2}, x_{m,1}, x_{m,2}, a, b\}$,

$\{x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{4,1}, x_{4,2}, x_{5,1}, x_{5,2}, a, b\}$,

$\{x_{m-4,1}, x_{m-4,2}, x_{m-3,1}, x_{m-3,2}, x_{m-2,1}, x_{m-2,2}, x_{m-1,1}, x_{m-1,2}, a, b\}$.

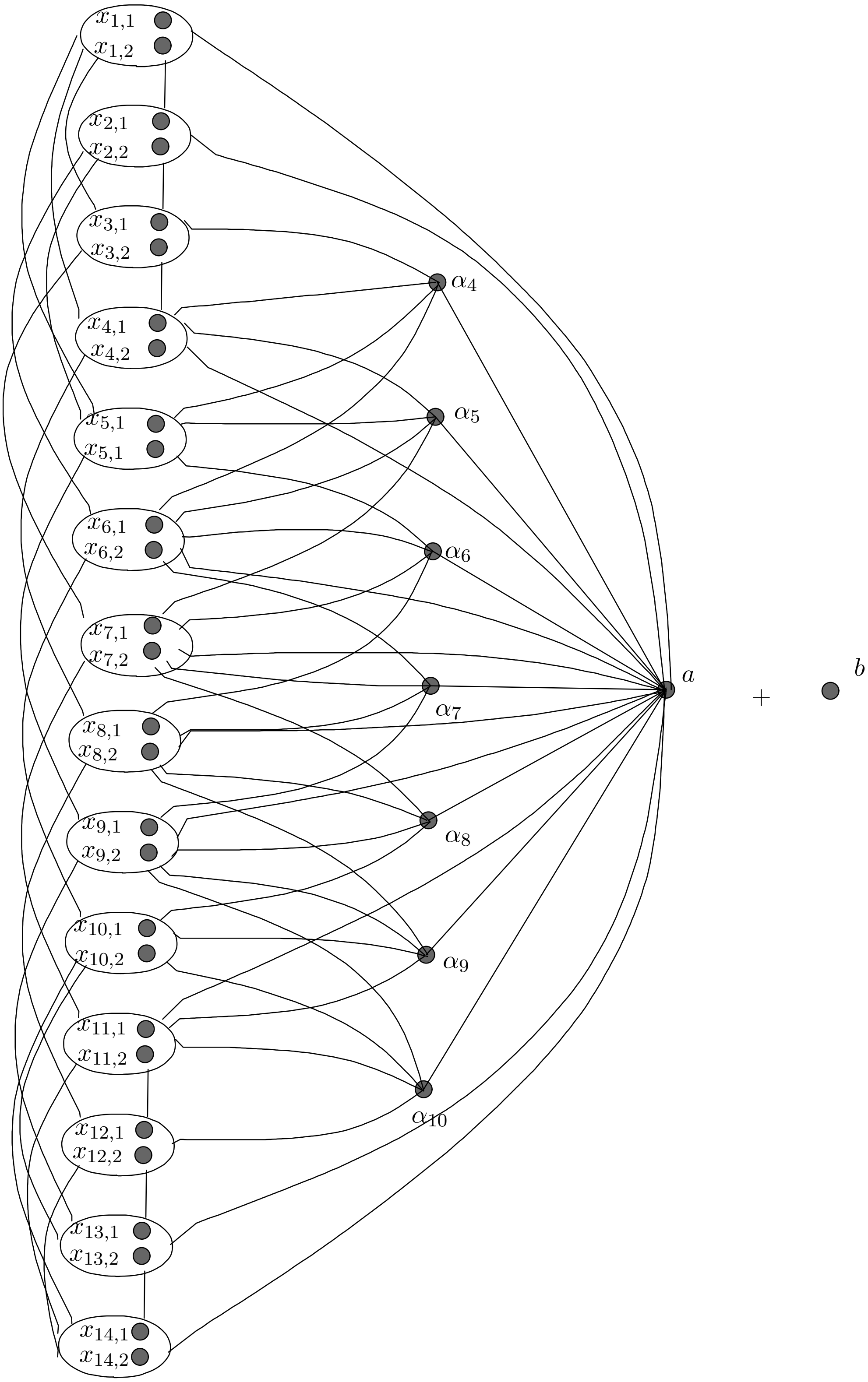
Thus the number of 10-shredders of G_m is $2m-7 = (2(3m-5)-11)/3 = (2|V(G_m)|-11)/3$.

2. Preliminary result

Throughout the rest of this paper, let G be a 10-connected graph, and let $\mathscr{S}$ denote the set of 10-shredders of G. For each $S \in \mathscr{S}$, we define $\mathscr{K}(S)$, $\mathscr{L}(S)$ and $L(S)$ as follows. Let $S \in \mathscr{S}$. We let $\mathscr{K}(S)$ denote the set of componetns of $G-S$. Write $\mathscr{K}(S) = \{H_1, \cdots, H_s\}$ $(s = |\mathscr{K}(S)|)$. We may assume $|V(H_1)| \ge |V(H_2)| \ge \cdots \ge |V(H_s)|$ (any such labeling will do). Under this notation, we let $\mathscr{L}(S) = \mathscr{K}(S) - \{H_1\}$ and $L(S) = \cup_{2 \le i \le s} V(H_i)$; thus $L(S) = \cup_{C \in \mathscr{L}(S)} V(C)$. Now let $\mathscr{L} = \cup_{S \in \mathscr{S}} \mathscr{L}(S)$. A member F of $\mathscr{L}$ is said to be *saturated* if there exists a subset $\mathscr{C}$ of $\mathscr{L} - \{F\}$ such that $V(F) = \cup_{C \in \mathscr{C}} V(C)$.

Let $S, T \in \mathscr{S}$ with $S \ne T$. We say that S *meshes* with T if S intersects with at least two member of $\mathscr{K}(T)$. It is easy to see that if S meshes with T, then T intersects with all members of $\mathscr{K}(S)$, and hence T meshes with S and S intersects with all members of $\mathscr{K}(T)$ (see [1; Lemma 4.3 (1)]).

The following two lemmas are proved in [5; Lemmas 2.1 and 3.1] (see also [2; Lemmas 3.2 and 3.4]).



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α_8
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α_{10}
a
$+$
b

Figure 1: $m = 14$

Lemma 2.1. *Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that S does not mesh with T. Then one of the following holds:*

(i) *$L(S) \cap L(T) = \emptyset$, $(L(S) \cup L(T)) \cap (S \cup T) = \emptyset$, and no edge of G joins a vertex in $L(S)$ and a vertex in $L(T)$;*

(ii) *there exists $C \in \mathscr{L}(S)$ such that $V(C) \supseteq L(T)$ (so $L(S) \supseteq L(T)$); or*

(iii) *there exists $D \in \mathscr{L}(T)$ such that $V(D) \supseteq L(S)$ (so $L(T) \supseteq L(S)$).*

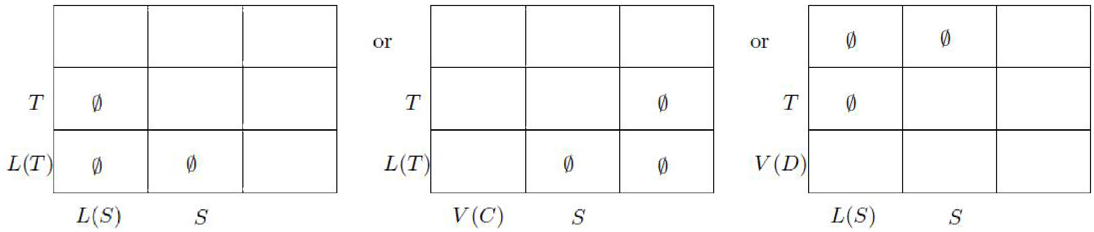

Figure 2: Lemma 2.1

Lemma 2.2. *Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that S meshes with T. Then $S \supseteq L(T)$ or $T \supseteq L(S)$ (so $L(S) \cap L(T) = \emptyset$).*

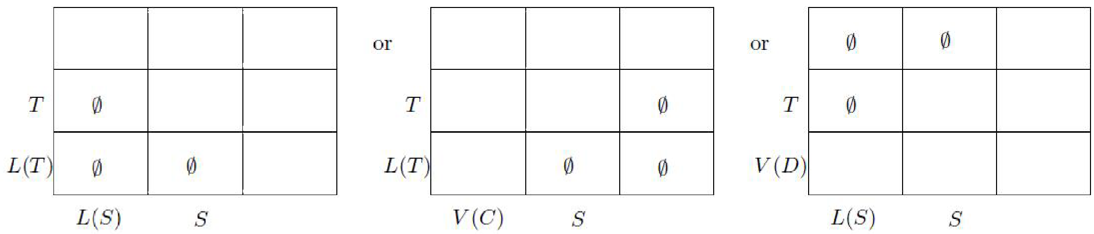

Figure 3: Lemma 2.2

The following lemma follows from Lemmas 2.1 and 2.2.

Lemma 2.3. *Let $S, T \in \mathscr{S}$. Then we have $L(S) \supseteq L(T)$, $L(T) \supseteq L(S)$ or $L(S) \cap L(T) = \emptyset$.*

As corollaries of Lemmas 2.1 and 2.2, we obtain the following four lemmas.

Lemma 2.4. *Let $S, T \in \mathscr{S}$, and suppose that $L(S) \cap L(T) = \emptyset$. Then S meshes with T if and only if $S \cap L(T) \neq \emptyset$*

Proof. The "only if" part follows from the definition of meshing; the "if" part follows from Lemma 2.1. □

Lemma 2.5. *Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that $L(S) \supseteq L(T)$. Then there exists $C \in \mathscr{L}(S)$ such that $V(C) \supseteq L(T)$.*

Proof. Since $L(S) \supseteq L(T)$, S does not mesh with T by Lemma 2.2, and neither (i) nor (iii) of Lemma 2.1 holds. Hence (ii) of Lemma 2.1 holds. □

Lemma 2.6. *Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that $L(S) \supseteq L(T)$. Then $|L(S)| \geq |L(T)| + 2$. Further if there exists $R \in \mathscr{S}$ such that $L(R) \cap L(T) = \emptyset$ and $L(S) \supseteq L(R)$, then $|L(S)| \geq |L(T)| + 3$.*

Proof. By Lemma 2.5, there exists $C \in \mathscr{L}(S)$ such that $V(C) \supseteq L(T)$. Since $|\mathscr{L}(T)| \geq 2$, $G[L(T)]$ is disconnected. Since C is connected, this implies $|V(C)| \geq |L(T)| + 1$. Since $|\mathscr{L}(S)| \geq 2$, we also have $|L(S)| \geq |V(C)| + 1$. Hence $|L(S)| \geq |L(T)| + 2$. Now assume that there exists $R \in \mathscr{S}$ such that $L(R) \cap L(T) = \emptyset$ and $L(S) \supseteq L(R)$. By Lemma 2.5, there exists $C' \in \mathscr{L}(S)$ such that $V(C') \supseteq L(R)$. If $C' = C$, then $|V(C)| \geq |L(T)| + |L(R)| \geq |L(T)| + 2$, and hence $|L(S)| \geq |V(C)| + 1 \geq |L(T)| + 3$; if $C' \neq C$, then $V(C) \cap L(R) = \emptyset$, and hence $|L(S)| \geq |V(C)| + |L(R)| \geq |L(T)| + 1 + |L(R)| \geq |L(T)| + 3$. □

Lemma 2.7. *Let $S \in \mathscr{S}$ and $T_1, \cdots, T_r \in \mathscr{S} - \{S\}$, and suppose that $L(S) = \cup_{1 \leq i \leq r} L(T_i)$. Then $r \geq 4$.*

Proof. Note that $|\mathscr{L}(S)| \geq 2$. Write $\mathscr{L}(S) = \{C_1, \cdots, C_p\}$ $(p = |\mathscr{L}(S)| \geq 2)$. By Lemma 2.5, we can write $\{1, \cdots, r\} = I_1 \cup \cdots \cup I_p$ so that $V(C_j) = \cup_{i \in I_j} L(T_i)$ for each $1 \leq j \leq p$. For each $1 \leq i \leq r$, $G[L(T_i)]$ is disconnected. Hence for each $1 \leq j \leq p$, $|I_j| \geq 2$ because C_j is connected. Consequently $r = \sum_{1 \leq j \leq p} |I_j| \geq 2p \geq 4$. □

The following three lemmas are proved in [3, Lemmas 2.8 and 2.9] and [7, Lemma 2.6] (in [7], the corresponding lemma is proved for 8-connected graphs, but the proof works for 10-connected graphs with no change).

Lemma 2.8. *Let $F \in \mathscr{L}$, and set $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq V(F)\}$. Then the following hold.*

(I) $|\mathscr{T}| \leq (2|V(F)| - 2)/3$.

(II) *If $|\mathscr{T}| = (2|V(F)| - 2)/3$, then one of the following holds:*

- (i) *F is trivial (i.e., $|V(F)| = 1$); or*
- (ii) *F is saturated, and there exist $T_1, T_2 \in \mathscr{T}$ such that $V(F) = L(T_1) \cup L(T_2)$, T_1 meshes with T_2, and $|\{T \in \mathscr{T} | L(T) \subseteq L(T_i)\}| = (2|L(T_i)| - 1)/3$ for each $i = 1, 2$.*

(III) *If $|\mathscr{T}| = (2|V(F)| - 3)/3$, then one of the following holds:*

(i) *F is saturated, and there exist $T_1, T_2 \in \mathscr{T}$ such that $V(F) = L(T_1) \cup L(T_2)$, T_1 meshes with T_2, and $|\{T \in \mathscr{T} \,|L(T) \subseteq L(T_1)\}| = (2|L(T_1)| - 1)/3$ and $|\{T \in \mathscr{T} \,|L(T) \subseteq L(T_2)\}| = (2|L(T_2)| - 2)/3$.*

(ii) *F is saturated, and there exist $T_1, T_2, T_3 \in \mathscr{T}$ such that $V(F) = L(T_1) \cup L(T_2) \cup L(T_3)$, T_3 meshes with T_1 and T_2, and $|\{T \in \mathscr{T} \,|L(T) \subseteq L(T_i)\}| = (2|L(T_i)| - 1)/3$ for each $i = 1, 2, 3$; or*

(iii) *F is not saturated, and there exists $T_0 \in \mathscr{T}$ such that $|L(T_0)| = |V(F)| - 1$, and $|\{T \in \mathscr{T} \,|L(T) \subseteq L(T_0)\}| = (2|L(T_0)| - 1)/3$.*

Lemma 2.9. *Let $S \in \mathscr{S}$, and write $\mathscr{L}(S) = \{F_1, \cdots, F_p\}$ ($p = |\mathscr{L}(S)|$). Set $\mathscr{R} = \{T \in \mathscr{S} | L(T) \subseteq L(S)\}$, and set $\mathscr{T}_i = \{T \in \mathscr{T} | L(T) \subseteq V(F_i)\}$ for each $1 \le i \le p$. Then the following hold.*

(I) *$|\mathscr{R}| \le (2|L(S)| - 2p + 3)/3 \le (2|L(S)| - 1)/3$.*

(II) *If $|\mathscr{R}| = (2|L(S)| - 1)/3$, then $p = 2$ and $|\mathscr{T}_i| = (2|V(F_i)| - 2)/3$ for each i.*

(III) *If $|\mathscr{R}| = (2|L(S)| - 2)/3$, then $p = 2$, and either $|\mathscr{T}_1| = (2|V(F_1)| - 2)/3$ and $|\mathscr{T}_2| = (2|V(F_2)| - 3)/3$, or $|\mathscr{T}_1| = (2|V(F_1)| - 3)/3$ and $|\mathscr{T}_2| = (2|V(F_2)| - 2)/3$.*

Lemma 2.10. *Let $F \in \mathscr{L}$, and suppose that $|V(F)| \neq 1$. Set $\mathscr{T} = \{T \in \mathscr{S} \,|L(T) \subseteq V(F)\}$, and set $\mathscr{T}_M = \{T \in \mathscr{T} \,|L(T) \subseteq V(M)\}$ for each $M \in (\bigcup_{T \in \mathscr{T}} \mathscr{L}(T)) \cup \{F\}$ (thus $\mathscr{T} = \mathscr{T}_F$).*

(I) *Suppose that $|\mathscr{T}| = (2|V(F)| - 2)/3$. Then the following hold.*

   (i) *$|\mathscr{T}_M| = (2|V(M)| - 2)/3$ for each $M \in (\bigcup_{T \in \mathscr{T}} \mathscr{L}(T)) \cup \{F\}$.*

   (ii) *$|\mathscr{L}(T)| = 2$ and $|\{S \in \mathscr{T} \,|L(S) \subseteq L(T)\}| = (2|L(T)| - 1)/3$ for each $T \in \mathscr{T}$.*

(II) *Suppose that $|\mathscr{T}| = (2|V(F)| - 3)/3$. Then the following hold.*

   (i) *$|\mathscr{T}_M| \ge (2|V(M)| - 3)/3$ for each $M \in (\bigcup_{T \in \mathscr{T}} \mathscr{L}(T)) \cup \{F\}$.*

   (ii) *$|\mathscr{L}(T)| = 2$ and $|\{S \in \mathscr{T} \,|L(S) \subseteq L(T)\}| \ge (2|L(T)| - 2)/3$ for each $T \in \mathscr{T}$.*

We need the following result which is related Lemma 2.10.

Lemma 2.11. *Let $F \in \mathscr{L}$ with $|V(F)| \neq 1$, set $\mathscr{T} = \{T \in \mathscr{S} \,|L(T) \subseteq V(F)\}$, and suppose that $|\mathscr{T}| \ge (2|V(F)| - 3)/3$. Set $\mathscr{T}_M = \{T \in \mathscr{T} \,|L(T) \subseteq V(M)\}$ for each $M \in (\cup_{T \in \mathscr{T}} \mathscr{L}(T)) \cup \{F\}$.*

(I) *Let $R, R' \in \mathscr{T}$, and suppose that $L(R) \cap L(R') = \emptyset$. Then we have $|\{T \in \mathscr{T} \,|L(T) \subseteq L(R)\}| = (2|L(R)| - 1)/3$ or $|\{T \in \mathscr{T} \,|L(T) \subseteq L(R')\}| = (2|L(R')| - 1)/3$.*

(II) *Let* $R \in \mathscr{T}$ *and* $M \in \cup_{T\in\mathscr{T}} \mathscr{L}(T)$*, and suppose that* $L(R) \cap V(M) = \emptyset$*. Then we have* $|\{T \in \mathscr{T}\,|L(T) \subseteq L(R)\}| = (2|L(R)| - 1)/3$ *or* $|\,\mathscr{T}_M\,| = (2|V(M)| - 2)/3$.

Proof. Let R, R' be as in (I). Let M be a member of $(\cup_{T\in\mathscr{T}} \mathscr{L}(T)) \cup \{F\}$ such that $L(R), L(R') \subseteq V(M)$. We choose M so that $|V(M)|$ is as small as possible. By Lemma 2.10, $|\,\mathscr{T}_M\,| \geq (2|V(M)| - 3)/3$, and hence it follows from Lemma 2.8 that there exist members T_i $(1 \leq i \leq s)$ of $\mathscr{T}_M$ with $s = 2$ or 3 such that $V(M) = \cup_{1\leq i\leq s} L(T_i)$ and $|\{T \in \mathscr{T}_M\,|L(T) \subseteq L(T_i)\}| \geq (2|L(T_i)| - 2)/3$ for each i. In view of Lemma 2.3, we have $L(R) \subseteq L(T_i)$ for some i (note that since $s \leq 3$, Lemma 2.8 shows that $L(R)$ cannot be the union of two or more members of $\{L(T_i)|1 \leq i \leq 3\}$). We may assume $L(R) \subseteq L(T_1)$. First assume that we also have $L(R') \subseteq L(T_1)$. By Lemma 2.9, $|\,\mathscr{L}(T_1)| = 2$. Write $\mathscr{L}(T_1) = \{M_1, M_2\}$. By Lemma 2.5, each of $L(R)$ and $L(R')$ is contained in $V(M_1)$ or $V(M_2)$. By the minimality of $|V(M)|$ and by symmetry, we may assume $L(R) \subseteq V(M_1)$ and $L(R') \subseteq V(M_2)$. By Lemma 2.9, we have $|\,\mathscr{T}_{M_1}\,| = (2|V(M_1)| - 2)/3$ or $|\,\mathscr{T}_{M_2}\,| = (2|V(M_2)| - 2)/3$. We may assume $|\,\mathscr{T}_{M_1}\,| = (2|V(M_1)| - 2)/3$. Then by Lemma 2.10(I)(ii), $|\{T \in \mathscr{T}_{M_1}\,|L(T) \subseteq L(R)\}| = (2|L(R)| - 1)/3$, as desired. Next assume $L(R') \not\subseteq L(T_1)$. By Lemmas 2.3 and 2.8, we may assume $L(R') \subseteq L(T_2)$. By Lemma 2.8 and by symmetry, we may assume $|\{T \in \mathscr{T}_M\,|L(T) \subseteq L(T_1)\}| = (2|L(T_1)| - 1)/3$. If $R = T_1$, then we clearly have $|\{T \in \mathscr{T}_M\,|L(T) \subseteq L(R)\}| = (2|L(R)| - 1)/3$. Thus we may assume $R \neq T_1$. Since $L(R) \subseteq L(T_1)$, it follows from Lemma 2.5 that there exists $M_1 \in \mathscr{L}(T_1)$ such that $L(R) \subseteq V(M_1)$. Consequently we obtain $|\,\mathscr{T}_{M_1}\,| = (2|V(M_1)| - 2)/3$ by Lemma 2.9(II), and hence $|\{T \in \mathscr{T}_M\,|L(T) \subseteq L(R)\}| = (2|L(R)| - 1)/3$ by Lemma 2.10(I)(ii).

Now let R, M be as in (II). Let $M \in \mathscr{L}(Q)$ $(Q \in \mathscr{T})$. By Lemma 2.10, $|\{T \in \mathscr{T}\,|L(T) \subseteq L(Q)\}| \geq (2|L(Q)| - 2)/3$. First assume $L(R) \subseteq L(Q)$. By Lemma 2.9, $|\,\mathscr{L}(Q)| = 2$. Write $\mathscr{L}(Q) = \{M, M'\}$. Then $R \in \mathscr{T}_{M'}$. By Lemma 2.9, $|\,\mathscr{T}_M\,| = (2|V(M)| - 2)/3$ or $|\,\mathscr{T}_{M'}\,| = (2|V(M')| - 2)/3$ and, in the case where $|\,\mathscr{T}_{M'}\,| = (2|V(M')| - 2)/3$, we have $|\{T \in \mathscr{T}_{M'}\,|L(T) \subseteq L(R)\}| = (2|L(R)|) - 1)/3$ by Lemma 2.10(I)(ii). Next assume $L(R) \not\subseteq L(Q)$. Then $L(R) \cap L(Q) = \emptyset$ by Lemma 2.3. By (I), $|\{T \in \mathscr{T}\,|L(T) \subseteq L(R)\}| = (2|L(R)| - 1)/3$ or $|\{T \in \mathscr{T}\,|L(T) \subseteq L(Q)\}| = (2|L(Q)| - 1)/3$ and, in the case where $|\{T \in \mathscr{T}\,|L(T) \subseteq L(Q)\}| = (2|L(Q)| - 1)/3$, we have $|\,\mathscr{T}_M\,| = (2|V(M)| - 2)/3$ by Lemma 2.9(II). This completes the proof of the lemma. ☐

The following two lemmas are proved in [3; Lemmas 2.11 and 2.12].

Lemma 2.12. *Let* $S, T \in \mathscr{S}$*, and suppose that* S *meshes with* T *and* $L(S) \not\subseteq T$*. Then* $L(T) \subseteq S$ *and* $|L(T)| \leq 4$.

Lemma 2.13. *Suppose that* $|V(G)| \geq 21$*. Let* $S, T \in \mathscr{S}$*, and suppose that* S *meshes with* T*,* $L(S) \subseteq T$ *and* $L(T) \subseteq S$*. Then* $|L(S)| + |L(T)| \leq 10$.

The following lemma follows from Lemmas 2.12 and 2.13.

Lemma 2.14. *Suppose that* $|V(G)| \geq 21$*. Let* $S, T \in \mathscr{S}$*, and suppose that* S *meshes with* T *and* $|L(S)| \geq 6$*. Then* $L(T) \subseteq S$ *and* $|L(T)| \leq 4$.

As an immediate corollary of Lemma 2.14, we obtain the following lemma.

Lemma 2.15. *Suppose that* $|V(G)| \geq 21$*. Let* $S, T \in \mathscr{S}$ *with* $S \neq T$*, and suppose that either* $|L(S)| \geq 5$ *and* $|L(T)| \geq 6$*, or* $|L(S)| \geq 6$ *and* $|L(T)| \geq 5$*. Then* S *does not mesh with* T*.*

The following two lemmas are proved in [2; Lemma 3.6] and [3; Lemma 2.10].

Lemma 2.16. *Let* $F \in \mathscr{L}$*, and suppose that* F *is saturated. Then* $|V(F)| \geq 4$*.*

Lemma 2.17. *Let* $X \subseteq V(G)$*. Set* $\mathscr{T} = \{T \in \mathscr{S} \,|\, L(T) \subseteq X\}$*, and suppose that no component in* $\cup_{T \in \mathscr{T}} \mathscr{L}(T)$ *is saturated. Then* $|\mathscr{T}| \leq |X|/2$*.*

Lemma 2.18. *Suppose that* $|V(G)| \geq 21$*. Let* $S \in \mathscr{S}$*, and suppose that* $|L(S)| \geq 6$*. Then for each* $X \subseteq S$*, the number of those members* P *of* $\mathscr{S}$ *which mesh with* S *and satisfy* $L(P) \subseteq X$ *is at most* $|X|/2$*; in particular, the number of those members of* $\mathscr{S}$ *which mesh with* S *is at most* 5*.*

Proof. By Lemmas 2.4 and 2.14, the family of members of $\mathscr{S}$ meshing with S is the same as the family of members P of $\mathscr{S}$ satisfying $L(P) \subseteq S$. Let $X \subseteq S$, and set $\mathscr{Q} = \{P \in \mathscr{S} \,|\, L(P) \subseteq X\}$. For each $P \in \mathscr{Q}$ and each $M \in \mathscr{L}(P)$, we have $|L(P)| \leq 4$ by Lemma 2.14, and hence $|V(M)| \leq |L(P)| - 1 \leq 3$ because $|\mathscr{L}(P)| \geq 2$. By Lemma 2.16, this implies that no component in $\cup_{P \in \mathscr{Q}} \mathscr{L}(P)$ is saturated. Consequently $|\mathscr{Q}| \leq |X|/2$ by Lemma 2.17. ☐

Lemma 2.19. *Let* $S, T \in \mathscr{S}$ *and* $F \in \mathscr{L}(S)$*, and suppose that* $L(T) \cap L(S) = \emptyset$ *and* $T \cap L(S) \neq \emptyset$*. Then we have* $|T \cap V(F)| \geq \min\{|V(F)|, |L(T)|\}$ and $|T \cap L(S)| \geq \min\{|L(S)|, |L(T)| + 1\}$.

Proof. By Lemma 2.4, T meshes with S. If $V(F) \subseteq T$, then $|T \cap V(F)| = |V(F)| \geq \min\{|V(F)|, |L(T)|\}$. Thus we may assume $V(F) \not\subseteq T$. By Lemma 2.2, $L(T) \subseteq S$. Note that we have $V(F) - T \neq \emptyset$ and $V(F) - T = V(F) - (T \cup L(T))$. Set $K = (T \cap V(F)) \cup (S - L(T))$. Then K separates $V(F) - T$ from the rest. Hence $|K| \geq 10$. Therefore $|T \cap V(F)| = |K| - |S - L(T)| \geq 10 - (|S| - |L(T)|) = |L(T)| \geq \min\{\, |V(F)|, |L(T)|\}$, as desired. Similarly, if $L(S) \subseteq T$, then $|T \cap L(S)| = |L(S)| \geq \min\{|L(S)|, |L(T)| + 1\}$. Thus we may assume $L(S) \not\subseteq T$. Then there exists $F' \in \mathscr{L}(S)$ such that $V(F') \not\subseteq T$. Arguing as above, we get $|T \cap V(F')| \geq |L(T)|$. Since T mesh with S, we also have $|T \cap L(S)| \geq |T \cap V(F')| + 1$. Consequently $|T \cap L(S)| \geq |L(T)| + 1 \geq \min\{|L(S)|, |L(T)| + 1\}$. ☐

Lemma 2.20. *Suppose that* $|V(G)| \geq 41$*. Let* $S, T \in \mathscr{S}$*, and suppose that* T *meshes with* S *and* $L(T) \subseteq S$*. Then* $|T \cap L(S)| \leq 10 - |L(T)|$*.*

Proof. Write $\mathscr{K}(T) - \mathscr{L}(T) = \{C_T\}$ and $\mathscr{K}(S) - \mathscr{L}(S) = \{C_S\}$. Since $L(T) \subseteq S$, $L(T) \cap V(C_S) = \emptyset$. Suppose that $V(C_T) \cap V(C_S) = \emptyset$. Then $T \supseteq V(C_S)$, and hence $|V(C_S)| = |T \cap V(C_S)|$. By the definition of $\mathscr{L}(S)$, this implies that $|V(C)| \leq |T \cap V(C_S)|$

for all $C \in \mathscr{K}(S)$. Since T meshes with S, $|\mathscr{L}(S)| \leq |T - (T \cap V(C_S))|$. Hence $|L(S)| \leq (|T| - |T \cap V(C_S)|)|T \cap V(C_S)|$. Consequently $|V(G)| = |L(S)| + |S| + |V(C_S)| \leq (|T| - |T \cap V(C_S)|)|T \cap V(C_s)| + |S| + |T \cap V(C_S)|$. Since $|S| = |T| = 10$ and $|T \cap V(C_S)|$ is an integer, this implies $|V(G)| \leq 40$, which contradicts the assumption that $|V(G)| \geq 41$. Thus $V(C_T) \cap V(C_S) \neq \emptyset$. Set $K = (S - L(T)) \cup (T \cap V(C_S))$. Then K separates $V(C_T) \cap V(C_S)$ from the rest. Hence $|K| \geq 10$. This implies $|T \cap V(C_S)| = |K| - |S - L(T)| \geq 10 - (|S| - |L(T)|) = |L(T)|$. Therefore $|T \cap L(S)| \leq |T| - |T \cap V(C_S)| \leq 10 - |L(T)|$. □

We often use the following result in connection with Lemmas 2.19 and 2.20.

Lemma 2.21. *Let $S, T \in \mathscr{S}$ and $F \in \mathscr{L}(S)$, and suppose that $L(T) \subseteq V(F)$. Then $T \subseteq (V(F) - L(T)) \cup S$.*

Proof. Note that $T = N_G(L(T)) - L(T)$ and $S = N_G(V(F)) - V(F)$. Since $L(T) \subseteq V(F)$, we have $N_G(L(T)) \subseteq N_G(V(F))$. Hence $T \subseteq N_G(V(F)) - L(T) \subseteq (V(F) \cup S) - L(T) = (V(F) - L(T)) \cup S$. □

Finally we prove two lemmas concerning ”small” cases.

Lemma 2.22. *Let $M \in \mathscr{L}$, and suppose that $|V(M)| = 3$ and $\{S \in \mathscr{S} \,|L(S) \subseteq V(M)\} \neq \emptyset$. Then $|\{S \in \mathscr{S} \,|L(S) \subseteq V(M)\}| = 1$ and , if we write $\{S \in \mathscr{S} \,|L(S) \subseteq V(M)\} = \{T\}$, we have $|L(T)| = 2$ and $|T \cap (V(M) - L(T))| = 1$.*

Proof. Since $|\{S \in \mathscr{S} \,|L(S) \subseteq V(M)\}| \leq 4/3$ by Lemma 2.8(I), we get $|\{S \in \mathscr{S} \,|L(S) \subseteq V(M)\}| = 1$. Write $\{S \in \mathscr{S} \,|L(S) \subseteq V(M)\} = \{T\}$. Since $|\mathscr{L}(T)| \geq 2$, $G[L(T)]$ is disconnected. Since M is connected and $|V(M)| = 3$, this forces $|L(T)| = 2$, $|V(M) - L(T)| = 1$ and $V(M) - L(T) \subseteq N_G(L(T))$. Since $T = N_G(L(T)) - L(T)$, we get $V(M) - L(T) \subseteq T$, and hence $|T \cap (V(M) - L(T))| = |V(M) - L(T)| = 1$. □

Lemma 2.23. *Let $T \in \mathscr{S}$, and suppose that $|L(T)| = 4$ and $|\{S \in \mathscr{S} \,|L(S) \subseteq L(T)\}| = 2$. Then there exists $T' \in \mathscr{S}$ such that $L(T') \subseteq L(T)$, $|L(T')| = 2$ and $|T' \cap (L(T) - L(T'))| = 1$*

Proof. By Lemma 2.9(III), there exists $M \in \mathscr{L}(T)$ such that $|\{S \in \mathscr{S} \,|L(S) \subseteq V(M)\}| = (2|V(M)| - 3)/3$. Then $|V(M)| \equiv 0 \pmod 3$. Since $|L(T)| = 4$, we get $|V(M)| = 3$. By Lemma 2.22, there exists $T' \in \mathscr{S}$ such that $L(T') \subseteq V(M)$, $|L(T')| = 2$ and $|T' \cap (V(M) - L(T'))| = 1$. Since M is a component of $G[L(T)]$, we have $N_G(L(T')) \cap L(T) = N_G(L(T')) \cap V(M)$. Since $T' = N_G(L(T')) - L(T')$, we get $|T' \cap (L(T) - L(T'))| = |T' \cap (V(M) - L(T'))| = 1$. □

3. Shredders meshing with five shredders

We continue with the notation of the preceding section. In this section we investigate members of $\mathscr{S}$ meshing with five other members of $\mathscr{S}$. Throughout this section, we assume that $|V(G)| \geq 41$, let $S \in \mathscr{S}$ and $F \in \mathscr{L}(S)$, set $\mathscr{T} = \{T \in \mathscr{S} \,|L(T) \subseteq V(F)\}$ and suppose that $|\mathscr{T}| \geq (2|V(F)|-3)/3$, and set $\mathscr{T}_M = \{T \in \mathscr{T} \,|L(T) \subseteq V(M)\}$ for each $M \in (\cup_{T\in\mathscr{T}} \mathscr{L}(T)) \cup \{F\}$, and let P_1, P_2, P_3, P_4, P_5 be five distinct members of $\mathscr{S}$ with $|L(P_1)| \leq \cdots \leq |L(P_5)|$ which mesh with S. Our main result in this section is Lemma 3.5.

Lemma 3.1. *Suppose that $|V(F)| \geq 5$. Then $S = \bigcup_{1\leq i\leq 5} L(P_i)$, and one of the following holds:*

(i) $|L(P_i)| = 2$ *for each* $1 \leq i \leq 5$*;*

(ii) $|L(P_5)| = 4$*,* $|L(P_i)| = 2$ *for each* $1 \leq i \leq 4$*, and one of* P_1, P_2, P_3 *and* P_4*, say* P_1*, satisfies* $L(P_1) \subseteq L(P_5)$*; or*

(iii) $|L(P_4)| = |L(P_5)| = 4$*,* $|L(P_i)| = 2$ *for each* $1 \leq i \leq 3$*, and some two of* P_1, P_2 *and* P_3*, say* P_1 *and* P_2*, satisfy* $L(P_1) \subseteq L(P_5)$ *and* $L(P_2) \subseteq L(P_4)$*.*

Proof. Since $|L(S)| \geq 1 + |V(F)| \geq 6$, it follows from Lemma 2.14 that $L(P_i) \subseteq S$ and $|L(P_i)| \leq 4$ for each $1 \leq i \leq 5$. If $|L(P_5)| = 2$, then $|L(P_i)| = 2$ for every $1 \leq i \leq 5$, and hence it follows from Lemma 2.3 that $\cup_{1\leq i\leq 5} L(P_i) = S$ and (i) holds. Thus we may assume $|L(P_5)| \geq 3$. Then by Lemma 2.3, there exist i_1, i_2 with $i_1 < i_2$ such that $L(P_{i_1}) \subseteq L(P_{i_2})$. Since $|L(P_{i_2})| \leq 4$, it follows from Lemma 2.6 that $\{i | i \neq i_2, L(P_i) \subseteq L(P_{i_2})\} = \{i_1\}$, $|L(P_{i_1})| = 2$ and $|L(P_{i_2})| = 4$. We may assume $i_1 = 1$ and $i_2 = 5$. Then by Lemma 2.3, $L(P_i) \subseteq S - L(P_5)$ for each $2 \leq i \leq 4$. If $|L(P_4)| = 2$, then it follows from Lemma 2.3 that $\cup_{1\leq i\leq 5} L(P_i) = S$ and (ii) holds. Thus we may assume $|L(P_4)| \geq 3$. Then arguing as above, we see that some two of P_2, P_3 and P_4, say P_2 and P_4 satisfy $L(P_2) \subseteq L(P_4)$, $|L(P_2)| = 2$ and $|L(P_4)| = 4$. Now $L(P_3) \subseteq S - L(P_5) - L(P_4)$ by Lemma 2.3, and hence $|L(P_3)| = 2$, $\cup_{1\leq i\leq 5} L(P_i) = S$, and (iii) holds. □

Lemma 3.2. *Suppose that $|V(F)| \geq 5$. Then F is saturated.*

Proof. Suppose that F is not saturated. Then by Lemma 2.8, there exists $T_0 \in \mathscr{T}$ with $|L(T_0)| = |V(F)| - 1$ such that $L(T_0) \subseteq V(F)$. Since $G[L(T_0)]$ is disconnected and F is connected, $V(F) - L(T_0) \subseteq N_G(L(T_0))$. Note that $T_0 = N_G(L(T_0)) - L(T_0)$. Hence $V(F) - L(T_0) \subseteq T_0$. Therefore it follows from Lemma 2.21 that $|T_0| = |T_0 \cap (V(F) - L(T_0))| + |T_0 \cap S| = 1 + |T_0 \cap S|$. By Lemma 3.1, $|L(P_i)| = 2$ or 4 for each $1 \leq i \leq 5$. Since $|L(T_0)| = |V(F)| - 1 \geq 4$, it follows from Lemma 2.19 that we have $T_0 \supseteq L(P_i)$ or $T_0 \cap L(P_i) = \emptyset$ for each i. Consequently we see from the structure of $P_1, \cdots, P_5$ described in Lemma 3.1 that $|T_0 \cap S|$ is even. This implies that $|T_0| = 1 + |T_0 \cap S|$ is odd, which contradicts the fact that $|T_0| = 10$. □

Lemma 3.3. *Suppose that $|V(F)| \geq 7$. Then one of following holds:*

(i) *there exist $T_1, T_2 \in \mathscr{T}$ with $|L(T_1)| = 2$ or 4, such that $V(F) = L(T_1) \cup L(T_2)$; or*

(ii) *there exist $T_1, T_2, T_3 \in \mathscr{T}$ $(T_1 \neq T_2)$ with $|L(T_1)| = |L(T_2)| = 2$ such that $V(F) = L(T_1) \cup L(T_2) \cup L(T_3)$.*

Proof. By Lemma 3.2, F is saturated. Hence by Lemma 2.8, either there exist $T_1, T_2 \in \mathscr{T}$ such that $V(F) = L(T_1) \cup L(T_2)$, or there exist $T_1, T_2, T_3 \in \mathscr{T}$ such that $V(F) = L(T_1) \cup L(T_2) \cup L(T_3)$. First assume that there exist $T_1, T_2 \in \mathscr{T}$ such that $V(F) = L(T_1) \cup L(T_2)$. We may assume $|L(T_1)| \leq |L(T_2)|$. By Lemma 2.8, $|\{T \in \mathscr{T} \,|\, L(T) \subseteq L(T_j)\}| = (2|L(T_j)| - 2)/3$ or $(2|L(T_j)| - 1)/3$ for each $j = 1, 2$, which implies that $|L(T_j)| \equiv 1, 2 \pmod 3$ for each j. Suppose that $|L(T_1)| \neq 2, 4$. Then $|L(T_1)| \geq 5$. By Lemma 2.8, T_1 meshes with T_2. In view of Lemma 2.15, this forces $|L(T_1)| = |L(T_2)| = 5$. Hence it follows from Lemma 2.4 and Lemma 2.19 that $T_1 \supseteq L(T_2)$. Since $T_1 \subseteq (V(F) - L(T_1)) \cup S = L(T_2) \cup S$ by Lemma 2.21, we obtain $|T_1| = |T_1 \cap (L(T_2) \cup S)| = |L(T_2)| + |T_1 \cap S| = 5 + |T_1 \cap S|$. By Lemma 2.19, we have $T_1 \supseteq L(P_i)$ or $T_1 \cap L(P_i) = \emptyset$ for each $1 \leq i \leq 5$. Hence it follows from Lemma 3.1 that $|T_1 \cap S|$ is even. Consequently $|T_1| = 5 + |T_1 \cap S|$ is odd, a contradiction. Thus $|L(T_1)| = 2$ or 4, and (i) holds.

Throughout the rest of the proof of the lemma, we assume that there exist $T_1, T_2, T_3 \in \mathscr{T}$ such that $V(F) = L(T_1) \cup L(T_2) \cup L(T_3)$. We may assume $|L(T_1)| \leq |L(T_2)| \leq |L(T_3)|$. By Lemma 2.8, $|\{T \in \mathscr{T} \,|\, L(T) \subseteq L(T_j)\}| = (2|L(T_j)| - 1)/3$ for each $1 \leq j \leq 3$, which implies that $|L(T_j)| \equiv 2 \pmod 3$ for each j. Suppose that $|L(T_1)| \neq 2$. Then $|L(T_1)| \geq 5$. By Lemma 2.8, T_3 meshes with T_1 or T_2. Since $5 \leq |L(T_1)| \leq |L(T_2)| \leq |L(T_3)|$, this together with Lemma 2.15 forces $|L(T_1)| = |L(T_2)| = |L(T_3)| = 5$. Suppose that T_3 meshes with precisely one of T_1 and T_2. We may assume that T_3 meshes with T_1 but does not mesh with T_2. Then $T_3 \supseteq L(T_1)$ by Lemmas 2.4 and 2.19, and $T_3 \cap L(T_2) = \emptyset$ by Lemma 2.4. Since $T_3 \subseteq (V(F) - L(T_3)) \cup S = L(T_1) \cup L(T_2) \cup S$ by Lemma 2.21, we obtain $|T_3| = |T_3 \cap (L(T_1) \cup L(T_2) \cup S)| = |T_3 \cap (L(T_1) \cup S)| = |L(T_1)| + |T_3 \cap S|$. By Lemma 2.19, we have $T_3 \supseteq L(P_i)$ or $T_3 \cap L(P_i) = \emptyset$ for each $1 \leq i \leq 5$. Hence it follows from Lemma 3.1 that $|T_3 \cap S|$ is even. Consequently $|T_3| = 5 + |T_3 \cap S|$ is odd, a contradiction. Thus T_3 meshes with T_1 and T_2. By Lemmas 2.4 and 2.19, this implies $T_3 \supseteq L(T_1)$ and $T_3 \supseteq L(T_2)$, and hence $T_3 = L(T_1) \cup L(T_2)$. Since $T_3 = N_G(L(T_3)) - L(T_3)$, we obtain $N_G(L(T_3)) \subseteq L(T_1) \cup L(T_2) \cup L(T_3) = V(F)$. Note that the roles of T_1, T_2 and T_3 are symmetric because $|L(T_1)| = |L(T_2)| = |L(T_3)|$. Thus we also get $N_G(L(T_1)) \subseteq V(F)$ and $N_G(L(T_2)) \subseteq V(F)$. Consequently $N_G(V(F)) \subseteq V(F)$, which contradicts the fact that G is connected.

Thus $|L(T_1)| = 2$. Suppose that $|L(T_2)| \geq 5$. Then $|L(T_3)| \geq 5$. We show that this leads to a contradiction (in deriving a contradiction, we do not make use of the assumption that $|L(T_2)| \leq |L(T_3)|$; thus the roles of T_2 and T_3 are symmetric). We first show that T_2 does not mesh with T_3. Suppose that T_2 meshes with T_3. Then $|L(T_2)| = |L(T_3)| = 5$ by Lemma 2.15, and $T_2 \supseteq L(T_3)$ by Lemma 2.19. Since $T_2 \subseteq L(T_1) \cup L(T_3) \cup S$ by

Lemma 2.21, we get $|T_2| = 5 + |T_2 \cap (L(T_1) \cup S)|$. Recall that $|L(T_1)| = 2$. By Lemma 2.19, we have $T_2 \supseteq L(P)$ or $T_2 \cap L(P) = \emptyset$ for each $P \in \{T_1, P_1, \cdots, P_5\}$. Hence it follows from Lemma 3.1 that $|T_2 \cap (L(T_1) \cup S)|$ is even. Consequently $|T_2| = 5 + |T_2 \cap (L(T_1) \cup S)|$ is odd, a contradiction.

Thus T_2 does not mesh with T_3. By Lemma 2.4, this implies $T_2 \cap L(T_3) = \emptyset$. Hence $T_2 \subseteq L(T_1) \cup S$ by Lemma 2.21. Consequently $|T_2 \cap S| \geq |S| - 2$. By Lemma 2.19, we have $T_2 \cap S \supseteq L(P_i)$ or $(T_2 \cap S) \cap L(P_i) = \emptyset$ for each i. Set $I = \{i | 1 \leq i \leq 5, L(P_i) \subseteq T_2 \cap S\}$. Then by Lemma 3.1, $|I| \geq 4$. Set $J = \{i | 1 \leq i \leq 5, L(P_i) \subseteq T_3 \cap S\}$. By symmetry, we also have $|J| \geq 4$. By Lemma 2.9(II), $|\mathscr{L}(T_2)| = 2$. Write $\mathscr{L}(T_2) = \{F_1, F_2\}$ with $|V(F_1)| \leq |V(F_2)|$. For each $j = 1, 2$, $|\mathscr{T}_{F_j}| = (2|V(F_j)| - 2)/3$ by Lemma 2.9(II), and hence $|V(F_j)| \equiv 1 \pmod 3$. Since $|L(T_2)| \geq 5$, this implies $|V(F_2)| \geq 4$. By Lemma 2.8(II), there exist $T_{2,1}, T_{2,2} \in \mathscr{T}_{F_2}$ with $|L(T_{2,1})| \leq |L(T_{2,2})|$ and $|L(T_{2,1})| \equiv |L(T_{2,2})| \equiv 2$ (mod 3) such that $V(F_2) = L(T_{2,1}) \cup L(T_{2,2})$. We now show that $|L(T_{2,1})| = 2$, and then show that $|V(F_2)| \geq 5$, and then derive a final contradiction. Suppose that $|L(T_{2,1})| \neq 2$. Then $|L(T_{2,1})| \geq 5$. Since $T_{2,1}$ meshes with $T_{2,2}$ by Lemma 2.8, we get $|L(T_{2,1})| = |L(T_{2,2})| = 5$ by Lemma 2.15. Since $T_{2,1} \subseteq (V(F_2) - L(T_{2,1})) \cup T_2 = L(T_{2,2}) \cup T_2$ by Lemma 2.21, $T_{2,1} = L(T_{2,2}) \cup (T_{2,1} \cap T_2)$ by Lemma 2.19. Since we already know $T_2 \subseteq L(T_1) \cup S$ and we clearly have $L(T_{2,2}) \cap (L(T_1) \cup S) = \emptyset$, we obtain $|T_{2,1}| = |L(T_{2,2})| + |T_{2,1} \cap (L(T_1) \cup S)| = 5 + |T_{2,1} \cap (L(T_1) \cup S)|$. Since we have $T_{2,1} \supseteq L(P)$ or $T_{2,1} \cap L(P) = \emptyset$ for each $P \in \{T_1, P_1, \cdots, P_5\}$ by Lemma 2.19, we see from Lemma 3.1 that $|T_{2,1}|$ is odd, a contradiction.

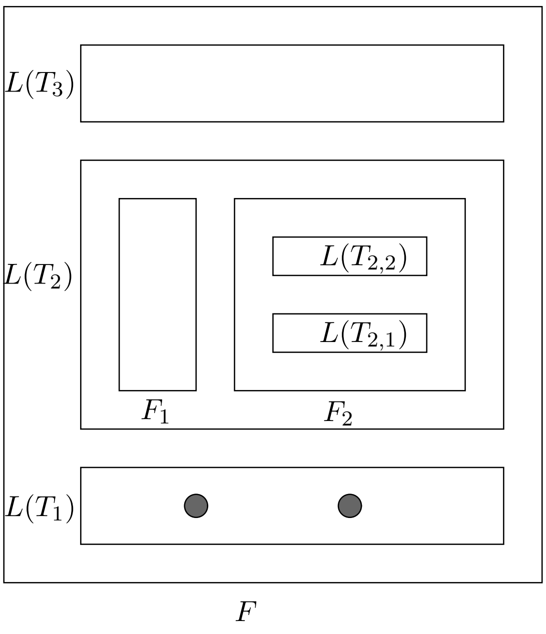

Figure 4: Case where $|L(T_2)| \geq 5$

Thus $|L(T_{2,1})| = 2$. By Lemma 2.21, $T_{2,2} \subseteq L(T_{2,1}) \cup T_2$, and hence $|T_{2,2} \cap T_2| \geq |T_2| - 2$, which implies $|T_{2,2} \cap T_2 \cap S| \geq |T_2 \cap S| - 2$. Hence $|\{i \in I | P_i \cap (T_{2,2} \cap T_2 \cap S) \neq \emptyset\}| \geq |I| - 1$. By Lemma 2.4, this implies $|\{i \in I | P_i \text{ meshes with } T_{2,2}\}| \geq |I| - 1$. Suppose that $|V(F_2)| = 4$. Then $|L(T_{2,2})| = 2$. Hence, arguing as above, we get $|\{i \in I | P_i \text{ meshes with } T_{2,1}\}| \geq |I| - 1$. Consequently $|\{i \in I | P_i \text{ meshes with } T_{2,1} \text{ and } T_{2,2}\}| \geq |I| - 2$. Since $|I| \geq 4$ and $|J| \geq 4$, this implies that there exists $i \in J$ such that P_i meshes with $T_{2,1}$ and $T_{2,2}$. Then by Lemma 2.4, P_i meshes with $T_{2,1}, T_{2,2}, T_2, T_3$ and S. By Lemma 2.4 and Lemma 2.19, $P_i \supseteq L(T_{2,1})$, $P_i \supseteq L(T_{2,2})$ and $|P_i \cap L(T_3)| \geq 3$. Since P_i meshes with T_2, it follows that $|P_i \cap L(T_2)| \geq |P_i \cap V(F_1)| + |P_i \cap V(F_2)| \geq 1 + (|L(T_{2,1})| + |L(T_{2,2})|) = 5$, and hence $|P_i \cap V(F)| \geq |P_i \cap L(T_2)| + |P_i \cap L(T_3)| \geq 8$. Since P_i meshes with S, we now obtain $|P_i \cap L(S)| \geq 1 + |P_i \cap V(F)| \geq 9$. Since $L(P_i) \subseteq S$, this contradicts Lemma 2.20.

Thus $|V(F_2)| \geq 5$. Hence $|L(T_{2,2})| \geq 3$. Recall that $|\{i \in I | P_i \text{ meshes with } T_{2,2}\}| \geq |I| - 1 \geq 3$. For each $i \in I$ such that P_i meshes with $T_{2,2}$, P_i meshes with $T_{2,2}$ and T_2 by Lemma 2.4, and hence it follows from Lemmas 2.4 and 2.19 that $|P_i \cap L(T_2)| \geq |P_i \cap V(F_1)| + |P_i \cap V(F_2)| \geq 1 + |P_i \cap L(T_{2,2})| \geq 4$. This implies that $|I'| \geq 3$, where $I' = \{i \in I || P_i \cap L(T_2)| \geq 4\}$. Set $J' = \{i \in J || P_i \cap L(T_3)| \geq 4\}$. By the symmetry of T_2 and T_3, we get $|J'| \geq 3$. Hence $I' \cap J' \neq \emptyset$. Take $i \in I' \cap J'$. Then since P_i meshes with S, we obtain $|P_i \cap L(S)| \geq 1 + |P_i \cap V(F)| \geq 1 + |P_i \cap L(T_2)| + |P_i \cap L(T_3)| \geq 9$, which contradicts Lemma 2.20. Thus $|L(T_2)| = 2$, which means that (ii) holds.

This completes the proof of Lemma 3.3 □

Lemma 3.4. *Suppose that $|V(F)| \geq 7$, and let $T_0 = T_2$ or T_3 according as* (i) or (ii) of Lemma 3.3 holds. Then there exist five members R_1, R_2, R_3, R_4, R_5 of $\mathscr{T} \cup \{P_1, \cdots, P_5\}$ which mesh with T_0 and satisfy $T_0 = \cup_{1 \leq i \leq 5} L(R_i)$, and satisfy one of the following three conditions:

(i) $|L(R_i)| = 2$ for each $1 \leq i \leq 5$;

(ii) $|L(R_5)| = 4$, $|L(R_i)| = 2$ for each $1 \leq i \leq 4$, and one of R_1, R_2, R_3 and R_4, say R_1, satisfies $L(R_1) \subseteq L(R_5)$; or

(iii) $|L(R_4)| = |L(R_5)| = 4$, $|L(R_i)| = 2$ for each $1 \leq i \leq 3$, and some two of R_1, R_2 and R_3, say R_1 and R_2, satisfy $L(R_1) \subseteq L(R_5)$ and $L(R_2) \subseteq L(R_4)$.

Proof. If (i) of Lemma 3.3 holds with $|L(T_1)| = 4$, then $|\{T \in \mathscr{T} | L(T) \subseteq L(T_1)\}| = (2|L(T_1)| - 2)/3$ by Lemma 2.8, and hence it follows from Lemma 2.23 that there exists $T_1' \in \mathscr{T}$ such that $L(T_1') \subseteq L(T_1)$ and $|L(T_1')| = 2$. Now if Lemma 3.3(i) holds with $|L(T_1)| = 4$, set $\mathscr{R} = \{T_1, T_1'\}$; if Lemma 3.3(i) holds with $|L(T_1)| = 2$, set $\mathscr{R} = \{T_1\}$; if Lemma 3.3(ii) holds, set $\mathscr{R} = \{T_1, T_2\}$. Then $\mathscr{R} \subseteq \mathscr{T}$. By Lemma 2.21, $T_0 = (T_0 \cap (V(F) - L(T_0))) \cup (T_0 \cap S)$. Since $T_0 \neq S$, this implies that $T_0 \cap (V(F) - L(T_0)) \neq \emptyset$.

Claim 3.1.

(i) $|\{P \in \mathscr{R} | L(P) \subseteq L(T_0)\}| = |T_0 \cap (V(F) - L(T_0))|/2$.

(ii) $|\{P \in \{P_1, \cdots, P_5\}|L(P) \subseteq L(T_0)\}| = |T_0 \cap S|/2$.

Proof. First we prove (i). Since $|L(T_0)| \geq |V(F)| - 4 \geq 3$, it follows from Lemma 2.19 that we have $T_0 \supseteq L(P)$ or $T_0 \cap L(P) = \emptyset$ for each $P \in \mathscr{R}$. Also recall that $T_0 \cap (V(F) - L(T_0)) \neq \emptyset$. Assume for the moment that Lemma 3.3(i) holds. Then $T_0 \cap (V(F) - L(T_0)) = L(T_1)$. Now if $|L(T_1)| = 4$, then $|\{P \in \mathscr{R}|L(P) \subseteq L(T_0)\}| = |\{T_1, T_1'\}| = 2 = |T_0 \cap (V(F) - L(T_0))|/2$; if $|L(T_1)| = 2$, then $|\{P \in \mathscr{R}|L(P) \subseteq L(T_0)\}| = |\{T_1\}| = 1 = |T_0 \cap (V(F) - L(T_0))|/2$. Thus we may assume that Lemma 3.3(ii) holds. Then either $T_0 \cap (V(F) - L(T_0)) = L(T_j)$ for $j = 1$ or 2, or $T_0 \cap (V(F) - L(T_0)) = L(T_1) \cup L(T_2)$. If $T_0 \cap (V(F) - L(T_0)) = L(T_j)$ for $j = 1$ or 2, then $|\{P \in \mathscr{R}|L(P) \subseteq L(T_0)\}| = |\{T_j\}| = 1 = |T_0 \cap (V(F) - L(T_0))|/2$; if $T_0 \cap (V(F) - L(T_0)) = L(T_1) \cup L(T_2)$, then $|\{P \in \mathscr{R}|L(P) \subseteq L(T_0)\}| = |\{T_1, T_2\}| = 2 = |T_0 \cap (V(F) - L(T_0))|/2$. Next we prove (ii). Since $|L(T_0)| \geq 3$, it follows from Lemma 2.19 that we have $T_0 \supseteq L(P_i)$ or $T_0 \cap L(P_i) = \emptyset$ for each $1 \leq i \leq 5$. Set $I = \{i|1 \leq i \leq 5, L(P_i) \subseteq T_0\}$. If Lemma 3.1(i) holds, let $L'(P_i) = L(P_i)$ for each $1 \leq i \leq 5$; if Lemma 3.1(ii) holds, let $L'(P_i) = L(P_i)$ for each $1 \leq i \leq 4$ and let $L'(P_5) = L(P_5) - L(P_1)$; if Lemma 3.1(iii) holds, let $L'(P_i) = L(P_i)$ for each $1 \leq i \leq 3$ and let $L'(P_4) = L(P_4) - L(P_2)$ and $L'(P_5) = L(P_5) - L(P_1)$. Then we have $T_0 \supseteq L'(P_i)$ or $T_0 \cap L'(P_i) = \emptyset$ for each $1 \leq i \leq 5$, and $I = \{i|1 \leq i \leq 5, L'(P_i) \subseteq T_0\}$. Since $|L'(P_i)| = 2$ for every i and $L'(P_i) \cap L'(P_j) = \emptyset$ for any i, j with $i \neq j$, it follows that $|T_0 \cap S| = |\cup_{i \in I} L'(P_i)| = 2|I| = 2|\{P \in \{P_1, \cdots, P_5\}|L(P) \subseteq L(T_0)\}|$. □

By Claim 3.1, $|\{P \in \mathscr{R} \cup \{P_1, \cdots, P_5\}|L(P) \subseteq T_0\}| = |T_0|/2 = 5$. Write $\{P \in \mathscr{R} \cup \{P_1, \cdots, P_5\}|L(P) \subseteq T_0\} = \{R_1, \cdots, R_5\}$ with $|L(R_1)| \leq \cdots \leq |L(R_5)|$. By Lemma 2.4, $R_1, \cdots, R_5$ mesh with T_0. Further, if $|\{P \in \mathscr{R}|L(P) \subseteq T_0\}| = 1$, then the unique member R of $\{P \in \mathscr{R}|L(P) \subseteq T_0\}$ satisfies $|L(R)| = 2$; if $|\{P \in \mathscr{R}|L(P) \subseteq T_0\}| = 2$ and if we write $\{P \in \mathscr{R}|L(P) \subseteq T_0\} = \{R, R'\}$ with $|L(R)| \leq |L(R')|$, then R and R' satisfy either $|L(R)| = |L(R')| = 2$ and $L(R) \cap L(R') = \emptyset$ or $|L(R)| = 2$, $|L(R')| = 4$ and $L(R) \subseteq L(R')$. Since $P_1, \cdots, P_5$ are as in Lemma 3.1, this implies that $T_0 = \cup_{1 \leq i \leq 5} L(R_i)$ and $R_1, \cdots, R_5$ satisfy one of the three conditions stated in the lemma. □

Lemma 3.5. *Let $T \in \mathscr{T} \cup \{S\}$ and $M \in \mathscr{L}(T)$ and, in the case where $T = S$, assume that $M = F$. Suppose that $|V(M)| \geq 6$. Then the following hold.*

(I) *There exist five members R_1, R_2, R_3, R_4, R_5 of $\mathscr{T} \cup \{P_1, \cdots, P_5\}$ which mesh with T and satisfy $T = \cup_{1 \leq i \leq 5} L(R_i)$, and satisfy one of the following three conditions:*

- (i) $|L(R_i)| = 2$ *for each* $1 \leq i \leq 5$*;*
- (ii) $|L(R_5)| = 4$, $|L(R_i)| = 2$ *for each* $1 \leq i \leq 4$*, and one of* R_1, R_2, R_3 *and* R_4*, say* R_1*, satisfies* $L(R_1) \subseteq L(R_5)$*; or*
- (iii) $|L(R_4)| = |L(R_5)| = 4$, $|L(R_i)| = 2$ *for each* $1 \leq i \leq 3$*, and some two of* R_1, R_2 *and* R_3*, say* R_1 *and* R_2*, satisfy* $L(R_1) \subseteq L(R_5)$ *and* $L(R_2) \subseteq L(R_4)$*.*

(II) *The component M is saturated. Further if $|V(M)| \geq 7$, then one of the following holds:*

(i) *there exist $T_0, T_1 \in \mathscr{T}_M$ with $|L(T_1)| = 2$ or 4 such that $V(M) = L(T_0) \cup L(T_1)$; or*

(ii) *there exist $T_0, T_1, T_1' \in \mathscr{T}_M$ $(T_1 \neq T_1')$ with $|L(T_1)| = |L(T_1')| = 2$ such that $V(M) = L(T_0) \cup L(T_1) \cup L(T_1')$.*

Proof. We proceed by backward induction on $|L(T)|$. If $T = S$, then the desired conclusion follows from Lemmas 3.1, 3.2 and 3.3. Thus we may assume $L(T) \subsetneq L(S)$. Choose $S' \in \mathscr{T} \cup \{S\}$ with $L(T) \subsetneq L(S') \subseteq L(S)$ so that $|L(S')|$ is as small as possible. By Lemma 2.5, there exists $F' \in \mathscr{L}(S')$ such that $L(T) \subseteq V(F')$ (in the case where $S' = S$, we have $F' = F$). It follows that $|V(F')| \geq |V(M)| + 1 \geq 7$. By Lemma 2.10, we have $|\mathscr{T}_{F'}| \geq (2|V(F')| - 3)/3$ and $|\mathscr{T}_M| \geq (2|V(M)| - 3)/3$. By the induction hypothesis, there exist five members $Q_1, \cdots, Q_5$ of $\mathscr{T} \cup \{P_1, \cdots, P_5\}$ which mesh with S'. By the induction assumption, we also see that either there exist $S_0, S_1 \in \mathscr{T}_{F'}$ with $|L(S_1)| = 2$ or 4 such that $V(F') = L(S_0) \cup L(S_1)$, or there exist $S_0, S_1, S_1' \in \mathscr{T}_{F'}$ with $|L(S_1)| = |L(S_1')| = 2$ such that $V(F') = L(S_0) \cup L(S_1) \cup L(S_1')$. By Lemma 2.3 and Lemma 2.7, $L(T) \subseteq L(S_0)$, which implies $S_0 = T$ by the minimality of $|L(S')|$. Consequently applying Lemma 3.4 to S' and F', we see that there exist five members $R_1, \cdots, R_5$ of $\mathscr{T}_{F'} \cup \{Q_1, \cdots, Q_5\}$ with $|L(R_1)| \leq \cdots \leq |L(R_5)|$ which mesh with T and satisfy $T = \cup_{1 \leq i \leq 5} L(R_i)$, and satisfy one of the three conditions in (I). Since $\mathscr{T}_{F'} \subseteq \mathscr{T}$ and $\{Q_1, \cdots, Q_5\} \subseteq \mathscr{T} \cup \{P_1, \cdots, P_5\}$, we also get $\{R_1, \cdots, R_5\} \subseteq \mathscr{T} \cup \{P_1, \cdots, P_5\}$. Thus (I) holds. Now, applying Lemmas 3.2 and 3.3 to T and M, we see that (II) holds. □

We conclude the section with the following technical lemma.

Lemma 3.6. *Let $T, T', T'' \in \mathscr{T} \cup \{S\}$ with $L(T) \subsetneq L(T'') \subsetneq L(T')$, and suppose that there exists $M \in \mathscr{L}(T)$ such that $|V(M)| \geq 7$. Let R be a member of $\mathscr{S}$ meshing with T and T'. Then $|L(R)| = 2$.*

Proof. By Lemma 2.14, $L(R) \subseteq T'$, and hence $L(R) \cap L(T') = \emptyset$, which implies $L(R) \cap L(T'') = \emptyset$ and $L(R) \cap L(T) = \emptyset$. By Lemma 2.4, $R \cap L(T) \neq \emptyset$, which implies $R \cap L(T'') \neq \emptyset$. Hence R meshes with T'' by Lemma 2.4. Now in view of Lemma 2.18, it follows from Lemma 3.5(I) that $|L(R)| = 2$ or 4. Suppose that $|L(R)| = 4$. Then $|R \cap L(T)| \geq 5$ by Lemma 2.19. By Lemma 2.5, there exists $M'' \in \mathscr{L}(T'')$ such that $L(T) \subseteq V(M'')$. Since R meshes with T''. We get $|R \cap L(T'')| \geq 1 + |R \cap V(M'')| \geq 1 + |R \cap L(T)| \geq 6$. Similarly $|R \cap L(T')| \geq 1 + |R \cap L(T'')| \geq 7$. Since $L(R) \subseteq T'$, this contradicts Lemma 2.20. □

4. Proof of the theorem

We follow the notation of Section 2, and prove the Theorem. Thus let $|V(G)| \geq 4227$ and, by way of contradiction, suppose that

$$|\mathscr{S}| \geq (2|V(G)| - 10)/3. \tag{4.1}$$

We define an order relation $\leq$ in $\mathscr{S}$ by

$$S \leq T \Longleftrightarrow L(S) \subseteq L(T) \ (S, T \in \mathscr{S}).$$

Let $S_1, \cdots, S_m$ be the maximal members of $\mathscr{S}$ with respect to this order relation. We may assume $|L(S_1)| \geq \cdots \geq |L(S_m)|$. Let $p_i = |\mathscr{L}(S_i)|$ for each i, and let $W = V(G) - (L(S_1) \cup \cdots \cup L(S_m))$. Arguing as in [3, Claims 3.2 through 3.4], we obtain the following three claims. We include sketches of their proofs for the convenience of the reader.

Claim 4.1.

(i) $m + 2|W| \leq 10$.

(ii) $2p_1 + (m-1) + 2|W| \leq 13$.

Proof. By (4.1) and Lemma 2.9 (I), $(2|V(G)| - 10)/3 \leq \sum_{1 \leq i \leq m} (2|L(S_i)| - 2p_i + 3)/3$, and hence $2(p_1 + \cdots + p_m) - 3m + 2|W| \leq 10$. Since $p_i \geq 2$ for all i, both (i) and (ii) follow from this. □

Claim 4.2. $|L(S_1)| \geq 153$.

Proof. If $|L(S_1)| \leq 152$, then by Claim 4.1 (i), $|V(G)| \leq 152m + |W| \leq 152m + 304|W| \leq 1520$, which contradicts the assumption that $|V(G)| \geq 4227$. □

Claim 4.3. *$m \geq 2$ and $|L(S_2)| \geq 153$.*

Proof. Suppose that $m = 1$ or $|L(S_2)| \leq 152$. Then by Claim 4.1 (ii), $|V(G) - L(S_1)| = |L(S_2)| + \cdots + |L(S_m)| + |W| \leq 152(m-1) + |W| \leq 152((m-1) + 2|W|) \leq 1976 - 304p_1$, and hence $|V(G) - (S_1 \cup L(S_1))| \leq 1966 - 304p_1$. Since $G[V(G) - (S_1 \cup L(S_1))]$ is the largest component of $G - S_1$, this implies $|L(S_1)| \leq p_1(1966 - 304p_1)$. Consequently $|V(G)| \leq p_1(1966 - 304p_1) + 1976 - 304p_1 \leq 4226$ because p_1 is an integer, which contradicts the assumption that $|V(G)| \geq 4227$. □

In what follows, we do not make use of the fact that $|L(S_1)| \geq |L(S_2)|$; thus the roles of S_1 and S_2 are symmetric.

Claim 4.4. *Let $T \in \mathscr{S}$.*

(I) (i) *If T does not mesh with S_1, then $L(T) \cap S_1 = \emptyset$.*
(ii) *If T does not mesh with S_2, then $L(T) \cap S_2 = \emptyset$.*

(II) (i) *If T meshes with S_1, then $L(T) \subseteq S_1$.*
(ii) *If T meshes with S_2, then $L(T) \subseteq S_2$.*

Proof. In view of the maximality of $L(S_1)$ and $L(S_2)$, (i) follows from Lemma 2.1. By Claims 4.2 and 4.3, (ii) follows from Lemma 2.14 □

By Lemma 2.15, Claims 4.2 and 4.3 imply that S_1 does not mesh with S_2. Since $L(S_1) \cap L(S_2) = \emptyset$ by the maximality of $L(S_1)$ and $L(S_2)$, $L(S_1) \cap S_2 = L(S_2) \cap S_1 = \emptyset$ by Lemma 2.1. Write $\mathscr{K}(S_1) - \mathscr{L}(S_1) = \{C_1\}$ and $\mathscr{K}(S_2) - \mathscr{L}(S_2) = \{C_2\}$; thus $C_1 = G - S_1 - L(S_1)$ and $C_2 = G - S_2 - L(S_2)$. We classify members of $\mathscr{S}$ into $\mathscr{T}_1$ and $\mathscr{T}_2$, and further into $\mathscr{T}_{1,1}$, $\mathscr{T}_{1,2}$, $\mathscr{T}_{1,3}$, $\mathscr{T}_{2,1}$, $\mathscr{T}_{2,2}$ and $\mathscr{T}_{2,3}$ as follows:

$$\begin{aligned}
\mathscr{T}_1 &= \{T \in \mathscr{S} | L(T) \cap (S_1 \cup S_2) = \emptyset\},\\
\mathscr{T}_2 &= \{T \in \mathscr{S} | L(T) \subseteq S_1 \cup S_2\},\\
\mathscr{T}_{1,1} &= \{T \in \mathscr{T}_1 | L(T) \subseteq L(S_1)\},\\
\mathscr{T}_{1,2} &= \{T \in \mathscr{T}_1 | L(T) \subseteq L(S_2)\},\\
\mathscr{T}_{1,3} &= \{T \in \mathscr{T}_1 | L(T) \subseteq V(C_1) \cap V(C_2)\},\\
\mathscr{T}_{2,1} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_1 - S_2\},\\
\mathscr{T}_{2,2} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_2 - S_1\},\\
\mathscr{T}_{2,3} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_1 \cap S_2\}.
\end{aligned}$$

By Claim 4.4, $\mathscr{T}_1$ is the set of those members of $\mathscr{S}$ which mesh with neither S_1 nor S_2, and $\mathscr{T}_2$ is the set of those members of $\mathscr{S}$ which mesh with S_1 or S_2. Thus $\mathscr{S} = \mathscr{T}_1 \cup \mathscr{T}_2$ (disjoint union). Further by Lemma 2.1, $\mathscr{T}_1 = \mathscr{T}_{1,1} \cup \mathscr{T}_{1,2} \cup \mathscr{T}_{1,3}$ (disjoint union) and, by Claim 4.4, $\mathscr{T}_2 = \mathscr{T}_{2,1} \cup \mathscr{T}_{2,2} \cup \mathscr{T}_{2,3}$ (disjoint union).

C_2	$\mathscr{T}_{1,1}$	$\mathscr{T}_{2,1}$	$\mathscr{T}_{1,3}$
S_2	$\emptyset$	$\mathscr{T}_{2,3}$	$\mathscr{T}_{2,2}$
$L(S_2)$	$\emptyset$	$\emptyset$	$\mathscr{T}_{1,2}$
	$L(S_1)$	S_1	C_1

Figure 5: Classification of members of $\mathscr{S}$

The following two claims follow from Lemma 2.9 (I) (see also [3, Claim 3.6]).

Claim 4.5. $|\mathscr{T}_{1,i}| \leq (2|L(S_i)| - 1)/3$ $(i = 1, 2)$.

Claim 4.6. $|\mathscr{T}_{1,3}| \leq 2|V(C_1) \cap V(C_2)|/3$.

Proof. Let $I = \{1 \leq i \leq m | S_i \in \mathscr{T}_{1,3}\}$. Then by Lemma 2.9 (I), $|\mathscr{T}_{1,3}| = \sum_{i \in I} |\{T \in \mathscr{T}_{1,3} | L(T) \subseteq L(S_i)\}| \leq \sum_{i \in I}(2|L(S_i)| - 1)/3 \leq (2|V(C_1) \cap V(C_2)| - |I|)/3 \leq 2|V(C_1) \cap V(C_2)|/3$. □

The following claim immediately follows from Lemma 2.18 (see also [3, Claim 3.8]).

Claim 4.7.

(i) $|\mathscr{T}_{2,1}| \leq |S_1 - S_2|/2$.

(ii) $|\mathscr{T}_{2,2}| \leq |S_2 - S_1|/2$.

(iii) $|\mathscr{T}_{2,3}| \leq |S_1 \cap S_2|/2$.

Claim 4.8. $|S_1 \cap S_2|$ *is even.*

Proof. Suppose that $|S_1 \cap S_2|$ is odd. Then it follows from Claim 4.7 that $|\mathscr{T}_2| \leq (|S_1 \cup S_2| - 3)/2$, and it follows from Claims 4.5 and 4.6 that $|\mathscr{T}_1| \leq (2(|V(G)| - |S_1 \cup S_2|) - 2)/3$. Hence $|\mathscr{S}| \leq (2|V(G)| - (|S_1 \cup S_2| + 13)/2)/3$. Since $|S_1 \cup S_2| \geq 11$, this contradicts (4.1). □

Write $|S_1 \cap S_2| = 2x$. Then $|S_1 \cup S_2| = 20 - 2x$. Hence it follows from Claim 4.7 that

$$|\mathscr{T}_2| \leq 10 - x \tag{4.2}$$

and it follows from Claims 4.5 and 4.6 that

$$|\mathscr{T}_1| \leq (2|V(G)| - 42 + 4x)/3. \tag{4.3}$$

By (4.2) and (4.3),

$$|\mathscr{S}| \leq (2|V(G)| - 12 + x)/3. \tag{4.4}$$

In view of (4.1) and (4.3), (4.4) implies that equality holds in (4.2) (note that $x \leq 4$). Thus it follows from Claim 4.7 that

$$|\mathscr{T}_{2,1}| = 5 - x, |\mathscr{T}_{2,2}| = 5 - x, |\mathscr{T}_{2,3}| = x. \tag{4.5}$$

By (4.1) and (4.2), (4.4) also implies $|\mathscr{T}_1| \geq (2|V(G)| - 44 + 4x)/3$ which, in view of Claims 4.5 and 4.6, implies that we have $|\mathscr{T}_{1,1}| \geq (2|L(S_1)| - 2)/3$ or $|\mathscr{T}_{1,2}| \geq (2|L(S_2)| - 2)/3$. By the symmetry of S_1 and S_2, we may assume

$$|\mathscr{T}_{1,1}| \geq (2|L(S_1)| - 2)/3. \tag{4.6}$$

Note that (4.5) implies that there exist five distinct members P_1, P_2, P_3, P_4, P_5 of $\mathscr{S}$ such that

$$P_1, P_2, P_3, P_4, P_5 \text{ mesh with } S_1. \tag{4.7}$$

By (4.6) and Lemma 2.9, $|\mathscr{L}(S_1)| = 2$. Let F be a member of $\mathscr{L}(S_1)$ with $|V(F)| \geq |L(S_1)|/2$. Then

$$|V(F)| \geq 77. \tag{4.8}$$

Set $\mathscr{T} = \{T \in \mathscr{T}_{1,1} \,|\, L(T) \subseteq V(F)\}$. Then

$$|\,\mathscr{T}\,| \geq (2|V(F)| - 3)/3. \tag{4.9}$$

by (4.6) and Lemma 2.9. Set $\mathscr{T}_M = \{T \in \mathscr{T} \,|\, L(T) \subseteq V(M)\}$ for each $M \in (\bigcup_{T\in\mathscr{T}} \mathscr{L}(T)) \cup \{F\}$. Then by (4.9) and Lemma 2.10,

$$|\,\mathscr{T}_M\,| \geq (2|V(M)| - 3)/3 \tag{4.10}$$

for each $M \in (\bigcup_{T\in\mathscr{T}} \mathscr{L}(T)) \cup \{F\}$, and

$$|\,\mathscr{L}(T)| = 2 \text{ and } |\{S \in \mathscr{T} \,|\, L(S) \subseteq L(T)\}| \geq (2|L(T)| - 2)/3 \tag{4.11}$$

for each $T \in \mathscr{T}$.

Set $S^{(0)} = S_1$ and $F^{(0)} = F$. We define a sequence $S^{(1)}, \cdots, S^{(k)}$ of members of $\mathscr{T}$ with $S^{(1)} > \cdots > S^{(k)}$ and a sequence $F^{(1)}, \cdots, F^{(k)}$ of members of $\cup_{T\in\mathscr{T}} \mathscr{L}(T)$ inductively according to the following procedure. Let $i \geq 1$, and assume that $S^{(i-1)}$ and $F^{(i-1)}$ have been defined. Assume for the moment that $|V(F^{(i-1)})| \geq 7$. We define $S^{(i)}$ to be the unique member of $\mathscr{T}_{F^{(i-1)}}$ such that $|L(S^{(i)})| = |V(F^{(i-1)})| - 2$ or $|V(F^{(i-1)})| - 4$, whose existence follows from (4.7), (4.9) and Lemma 3.5(II). By (4.11), $|\,\mathscr{L}(S^{(i)})| = 2$. Let $F^{(i)}$ be a member of $\mathscr{L}(S^{(i)})$ such that $|V(F^{(i)})| \geq |L(S^{(i)})|/2$. If $|V(F^{(i-1)})| \leq 6$, we terminate the procedure, and set $k = i - 1$.

Claim 4.9. $k \geq 4$.

Proof. By definition, $|V(F^{(i)})| \geq (|V(F^{(i-1)})| - 4)/2$ for each i, and we have $|V(F^{(0)})| \geq 77$ by (4.8). Hence $|V(F^{(k)})| \geq ((|V(F^{(0)})| + 4)/2^k) - 4 \geq (81/2^k) - 4$. Since $|V(F^{(k)})| \leq 6$, this implies $k \geq 4$. □

For notational simplicity, set $T_0 = S^{(k)}$, $T = S^{(k-1)}$, $T' = S^{(k-2)}$, $M = F^{(k-1)}$ and $M' = F^{(k-2)}$. Thus $|V(M)| \geq 7$, $|V(M)| = |L(T_0)| + 2$ or $|L(T_0)| + 4$, and $|V(M')| = |L(T)| + 2$ or $|L(T)| + 4$. By (4.11), $|\,\mathscr{L}(T_0)| = 2$. Write $\mathscr{L}(T_0) = \{M_1, M_2\}$ with $M_2 = F^{(k)}$. Thus $|V(M_1)| \leq |V(M_2)| \leq 6$ and $L(T_0) = V(M_1) \cup V(M_2)$ (we henceforth do not use notations $F^{(i)}$ and $S^{(i)}$ except that we make use of the fact that $k \geq 4$ in the proof of Claim 4.11). If $|V(M_2)| = 1$, then $|V(M)| \leq |L(T_0)| + 4 \leq 2|V(M_2)| + 4 = 6$, which contradicts the fact that $|V(M)| \geq 7$. Thus $|V(M_2)| \neq 1$. Since $|\,\mathscr{T}_M\,| = (2|V(M)| - 2)/3$ or $(2|V(M)| - 3)/3$ by (4.10), we also have $|V(M_2)| \neq 2$. Hence $|V(M_2)| \geq 3$.

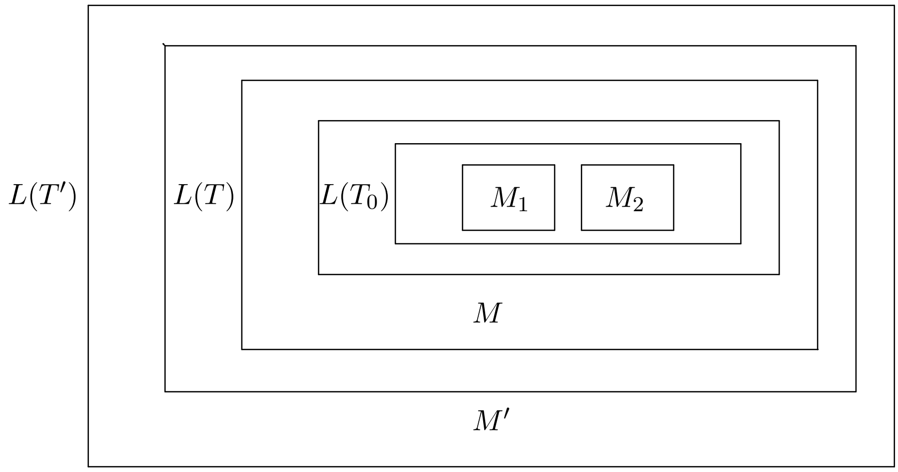

Figure 6: T_0, T, T' and M_1, M_2, M, M'

Claim 4.10. *One of the following holds:*

(i) $|V(M_1)| = 1$ *or* 4, $|V(M_2)| = 4$, *and there exist* $T_1, T_1' \in \mathscr{T}_M$ *with* $T_1 \neq T_1'$ *and* $|L(T_1)| = |L(T_1')| = 2$ *such that* $V(M) = L(T_0) \cup L(T_1) \cup L(T_1')$;

(ii) $|V(M_1)| = 1$ *or* 4, $|V(M_2)| = 4$, *and there exists* $T_1 \in \mathscr{T}_M$ *with* $|L(T_1)| = 4$ *such that* $V(M) = L(T_0) \cup L(T_1)$;

(iii) $|V(M_1)| = 1$ *or* 4, $|V(M_2)| = 6$, *and there exists* $T_1 \in \mathscr{T}_M$ *with* $|L(T_1)| = 2$ *such that* $V(M) = L(T_0) \cup L(T_1)$; *or*

(iv) $|V(M_2)| = 4$, *and there exists* $T_1 \in \mathscr{T}_M$ *with* $|L(T_1)| = 2$ *such that* $V(M) = L(T_0) \cup L(T_1)$.

Proof. By Lemma 3.5(II), either there exist $T_1, T_1' \in \mathscr{T}_M$ with $T_1 \neq T_1'$ and $|L(T_1)| = |L(T_1')| = 2$ such that $V(M) = L(T_0) \cup L(T_1) \cup L(T_1')$, or there exists $T_1 \in \mathscr{T}_M$ with $|L(T_1)| = 2$ or 4 such that $V(M) = L(T_0) \cup L(T_1)$.

First we consider the case where there exist $T_1, T_1' \in \mathscr{T}_M$ with $|L(T_1)| = |L(T_1')| = 2$ such that $V(M) = L(T_0) \cup L(T_1) \cup L(T_1')$. Since $|\mathscr{T}_M| = (2|V(M)| - 2)/3$ or $(2|V(M)| - 3)/3$ by (4.10), it follows from Lemma 2.8 that $|\{S \in \mathscr{T}_M \,|\, L(S) \subseteq L(T_0)\}| = (2|L(T_0)| - 1)/3$, and hence $|\mathscr{T}_{M_2}| = (2|V(M_2)| - 2)/3$ and $|\mathscr{T}_{M_1}| = (2|V(M_1)| - 2)/3$ by Lemma 2.9. This implies $|V(M_2)| \equiv 1 \pmod 3$ and $|V(M_1)| \equiv 1 \pmod 3$. Since $3 \leq |V(M_2)| \leq 6$ and $|V(M_1)| \leq |V(M_2)|$, we see that (i) holds.

Next we consider the case where there exists $T_1 \in \mathscr{T}_M$ with $|L(T_1)| = 2$ or 4 such that $V(M) = L(T_0) \cup L(T_1)$. If $|L(T_1)| = 4$, then since $|\mathscr{T}_M| = (2|V(M)| - 2)/3$ or $(2|V(M)| - 3)/3$, it follows from Lemma 2.8 that $|\{S \in \mathscr{T}_M \,|\, L(S) \subseteq L(T_0)\}| = (2|L(T_0)| - 1)/3$, and hence we can argue as in the preceding paragraph, to see that (ii) holds. Thus we may assume $|L(T_1)| = 2$. Suppose that $|V(M_2)| = 3$. Since $|\{S \in \mathscr{T}_M \,|L(S) \subseteq L(T_0)\}| = (2|L(T_0)| - 1)/3$ or $(2|L(T_0)| - 2)/3$ by (4.11), it follows from Lemma 2.9 that $|\mathscr{T}_{M_1}| = (2|V(M_1)| - 2)/3$. Since $1 \leq |V(M_1)| \leq 3$, this forces $|V(M_1)| = 1$, and hence $|V(M)| = |V(M_1)| + |V(M_2)| + |L(T_1)| = 6$, which contradicts the fact that $|V(M)| \geq 7$. Thus $4 \leq |V(M_2)| \leq 6$. Since $|\mathscr{T}_{M_2}| = (2|V(M_2)| - 2)/3$ or $(2|V(M_2)| - 3)/3$, we get $|V(M_2)| = 4$ or 6. If $|V(M_2)| = 6$, then since $|\{S \in \mathscr{T}_M \,|L(S) \subseteq L(T_0)\}| = (2|L(T_0)|-1)/3$ or $(2|L(T_0)|-2)/3$, it follows from Lemma 2.9 that $|\mathscr{T}_{M_1}| = (2|V(M_1)|-2)/3$, and hence (iii) holds; if $|V(M_2)| = 4$, then (iv) holds. □

Since $|V(M')| > |V(M)| \geq 7$, if follows from (4.7), (4.8), (4.9) and Lemma 3.5(I) that there exist five members Q_1, Q_2, Q_3, Q_4, Q_5 of $\mathscr{T} \cup \{P_1, P_2, P_3, P_4, P_5\}$ meshing with T', and there exist five members R_1, R_2, R_3, R_4, R_5 of $\mathscr{T} \cup \{P_1, P_2, P_3, P_4, P_5\}$ meshing with T, We first show that $|L(Q_i)| = 2$ and $|L(R_j)| = 2$ for all $1 \leq i, j \leq 5$ (Claims 4.13 and 4.14).

Claim 4.11. *Suppose that there exists $R \in \{Q_1, \cdots, Q_5, R_1, \cdots R_5\}$ such that $|L(R)| = 4$. Then $R \in \mathscr{T}$,* (iv) of Claim 4.10 holds, and $|V(M_1)| = 1$ or 4.

Proof. Note that $T < T'$. Since $k \geq 4$ by Claim 4.9, there exists $T'' \in \mathscr{T}$ such that $T' < T'' < S_1$. By Lemma 3.6, this implies $R \notin \{P_1, \cdots P_5\}$. Since $R \in \mathscr{T} \cup \{P_1, \cdots, P_5\}$, we get $R \in \mathscr{T}$. Since we have $L(R) \subseteq T$ or $L(R) \subseteq T'$, we see that $L(R) \cap V(M) = \emptyset$. By (4.11), $|\{S \in \mathscr{T} \,|L(S) \subseteq L(R)\}| = (2|L(R)| - 2)/3$. Consequently it follows from (4.9) and Lemma 2.11(II) that $|\mathscr{T}_M| = (2|V(M)| - 2)/3$ and $|\mathscr{T}_{M_1}| = (2|V(M_1)| - 2)/3$. This implies $|V(M)| \equiv 1 \pmod 3$, and hence none of (i), (ii) and (iii) of Claim 4.10 can hold. Hence (iv) of Claim 4.10 holds. Since we also get $|V(M_1)| \equiv 1 \pmod 3$, we have $|V(M_1)| = 1$ or 4. □

Claim 4.12. *Suppose that there exists $R \in \{R_1, \cdots R_5\}$ such that $|L(R)| = 4$, and let T_1, M_1, M_2 be as in* (iv) of Claim 4.10. Then $|V(M_1)| = 1$ and $T_1 \cap L(R) = \emptyset$.

Proof. Since R meshes with T, $|R \cap V(M)| \geq 4$ by Lemma 2.4 and Lemma 2.19. Since $|L(T_1)| = 2$, it follows that $R \cap L(T_0) \neq \emptyset$, and hence we get $R \supseteq V(M_1)$ and $R \supseteq V(M_2)$ by Lemma 2.19. Consequently $R \supseteq L(T_0)$. Suppose that $|V(M_1)| = 4$. Then $|R \cap L(T)| > |R \cap V(M)| \geq |R \cap L(T_0)| = |L(T_0)| = 8$. Since $L(R) \subseteq T$, this contradicts Lemma 2.20. Thus $|V(M_1)| = 1$. If R meshes with T_1, then $R \supseteq L(T_1)$ by Lemmas 2.4 and 2.19, and hence $|R \cap L(T)| > |R \cap V(M)| = |L(T_0)| + |L(T_1)| = 7$, which again contradicts Lemma 2.20. Thus R does not mesh with T_1. By Lemma 2.4, this implies $T_1 \cap L(R) = \emptyset$. □

Claim 4.13. *We have $|L(R_i)| = 2$ for every $1 \leq i \leq 5$.*

Proof. Suppose that one of $R_1, \cdots, R_5$, say R_5, satisfies $|L(R_5)| = 4$. By Claims 4.11 and 4.12, (iv) of Claim 4.10 holds, $|V(M_1)| = 1$ and $T_1 \cap L(R_5) = \emptyset$. By Lemma 2.21, $T_1 \subseteq (V(M) - L(T_1)) \cup T = L(T_0) \cup T$. Hence $T_1 \subseteq L(T_0) \cup (T - L(R_5))$. In view of Lemma 3.5(I), we may assume $L(R_1) \subseteq L(R_5)$. Now suppose that there exists

$R \in \{R_2, R_3, R_4\}$ such that $|L(R)| = 4$. Then $T_1 \cap L(R) = \emptyset$ by Claim 4.12, and hence $T \subseteq L(T_0) \cup (T - L(R_5) - L(R))$. By Lemma 3.5(I), $|T - L(R_5) - L(R)| = 2$. Consequently $|T| \leq |L(T_0)| + |T - L(R_5) - L(R)| = 7$, which contradicts the fact that $|T| = 10$. Thus $|L(R_i)| = 2$ for each $2 \leq i \leq 4$. Note that $|\mathscr{T}_{M_2}| = (2|V(M_2)| - 2)/3$. By Lemma 2.8, there exist $T_{2,1}, T_{2,1} \in \mathscr{T}_{M_2}$ such that $V(M_2) = L(T_{2,1}) \cup L(T_{2,2})$. Since $|V(M_2)| = 4$, $|L(T_{2,1})| = |L(T_{2,2})| = 2$. By Lemma 2.8, T_1 meshes with T_0, and hence $T_1 \cap V(M_1) \neq \emptyset$. Since $|V(M_1)| = 1$, this implies $|T_1 \cap V(M_1)| = 1$. Hence $|T_1 \cap L(T_0)| = 1 + |T_1 \cap V(M_2)|$. Since $T_1 \subseteq L(T_0) \cup (T - L(R_5))$, we now obtain $|T_1| = |T_1 \cap L(T_0)| + |T_1 \cap (T - L(R_5))| = 1 + |T_1 \cap (V(M_2) \cup (T - L(R_5)))|$. Set $\mathscr{R} = \{T_{2,1}, T_{2,2}, R_2, R_3, R_4\}$. Then $V(M_2) \cup (T - L(R_5)) = \cup_{R \in \mathscr{R}} L(R)$, and $|L(R)| = 2$ for all $R \in \mathscr{R}$. By Lemma 2.19, we have $T_1 \supseteq L(R)$ or $T_1 \cap L(R) = \emptyset$ for each $R \in \mathscr{R}$. Therefore $|T_1 \cap (V(M_2) \cup (T - L(R_5)))|$ is even, which implies that $|T_1| = 1 + |T_1 \cap (V(M_2) \cup (T - L(R_5)))|$ is odd. This contradicts the fact that $|T_1| = 10$. □

Claim 4.14. *We have* $|L(Q_i)| = 2$ *for every* $1 \leq i \leq 5$.

Proof. Suppose that there exists $Q \in \{Q_1, \cdots, Q_5\}$ such that $|L(Q)| = 4$. By Claim 4.11, $Q \in \mathscr{T}$. In view of (4.9), this together with Lemma 2.11(II) implies that $|\mathscr{T}_{M'}| = (2|V(M')| - 2)/3$. Hence it follows from Lemma 3.5(II) and Lemma 2.8 that there exists $T_2 \in \mathscr{T}_{M'}$ with $|L(T_2)| = 2$ such that $V(M') = L(T) \cup L(T_2)$. Since Q meshes with T', $|Q \cap V(M')| \geq 4$ by Lemmas 2.4 and 2.19. Since $|L(T_2)| = 2$, it follows that $Q \cap L(T) \neq \emptyset$, and hence Q meshes with T by Lemma 2.4. By Lemma 2.18, this implies $Q \in \{R_1, \cdots, R_5\}$, which contradicts Claim 4.13. □

We now take up cases (iii), (iv), (i), (ii) of Claim 4.10 in this order, and derive a contradiction.

Case 1. (iii) of Claim 4.10 holds.

We have $|\mathscr{T}_{M_2}| = (2|V(M_2)| - 3)/3$, $|\mathscr{T}_{M_1}| = (2|V(M_1)| - 2)/3$, and hence $|\{S \in \mathscr{T}_{M'} \,|L(S) \subseteq L(T)\}| = (2|L(T)| - 2)/3$ by Lemma 2.9 (II) and Lemma 2.10 (I). By Lemma 3.5(II), M_2 is saturated. We divide the proof into two subcases according to Lemma 2.8(III).

Subcase $1 - 1$. There exist $T_{2,1}, T_{2,2} \in \mathscr{T}_{M_2}$ such that $V(M_2) = L(T_{2,1}) \cup L(T_{2,2})$, $|\{S \in \mathscr{T}_{M_2} \,|L(S) \subseteq L(T_{2,1})\}| = (2|L(T_{2,1})| - 1)/3$, and $|\{S \in \mathscr{T}_{M_2} \,|L(S) \subseteq L(T_{2,2})\}| = (2|L(T_{2,2})| - 2)/3$.

Since $|V(M_2)| = 6$, we have $|L(T_{2,1})| = 2$ and $|L(T_{2,2})| = 4$ (note that $|L(T_{2,2})| \geq 2$). By Lemma 2.23, there exists $T_{2,2,1} \in \mathscr{T}_{M_2}$ such that $L(T_{2,2,1}) \subseteq L(T_{2,2})$ and $|T_{2,2,1} \cap (L(T_{2,1}) - L(T_{2,2,1}))| = 1$.

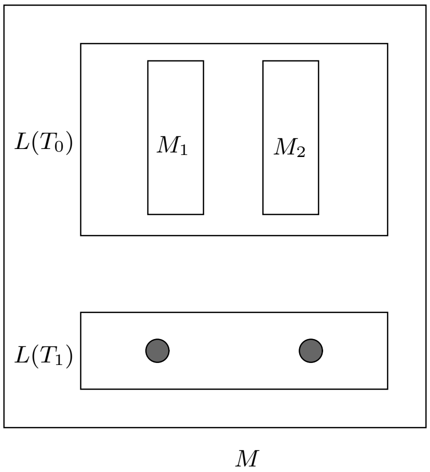

Figure 7: Case 1

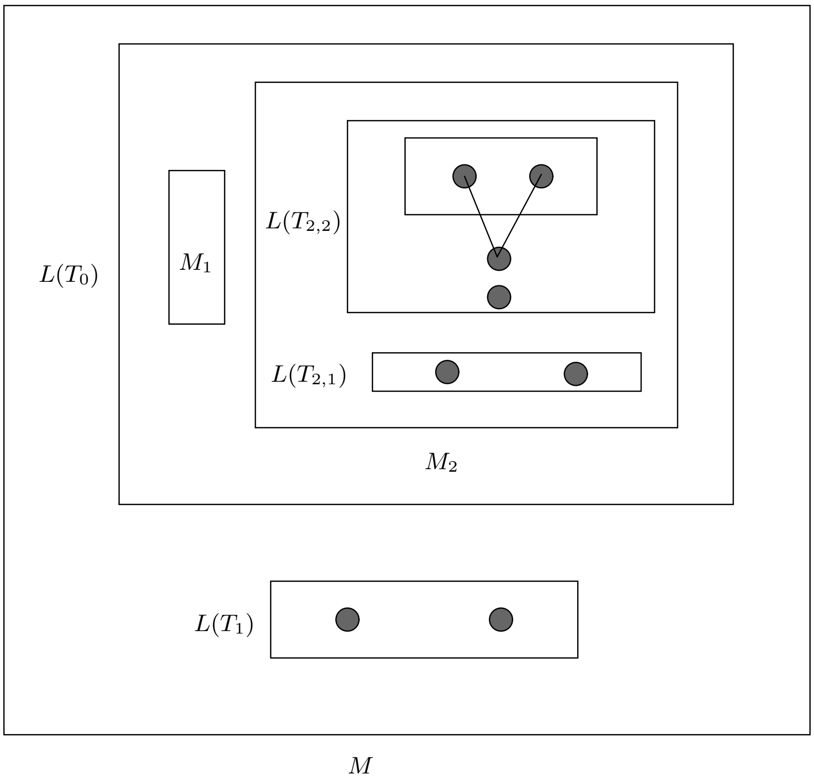

Figure 8: Subcase $1-1$

We have $|T_{2,2,1} \cap (V(M_2) - L(T_{2,2,1}))| = |T_{2,2,1} \cap (L(T_{2,2}) - L(T_{2,2,1}))| + |T_{2,2,1} \cap L(T_{2,1})| = 1 + |T_{2,2,1} \cap L(T_{2,1})|$. By Lemma 2.21, $T_{2,2,1} \subseteq (V(M_2) - L(T_{2,2,1})) \cup T_0$ and $T_0 \subseteq (M - L(T_0)) \cup T = L(T_1) \cup T$. Hence $T_{2,2,1} \subseteq (V(M_2) - L(T_{2,2,1})) \cup (L(T_1) \cup T)$. Consequently

$|T_{2,2,1}| = |T_{2,2,1} \cap (V(M_2) - L(T_{2,2,1}))| + |T_{2,2,1} \cap (L(T_1) \cup T)| = 1 + |T_{2,2,1} \cap (L(T_{2,1}) \cup L(T_1) \cup T)|$. Set $\mathscr{R} = \{T_{2,1}, T_1, R_1, \cdots, R_5\}$. Then $L(T_{2,1}) \cup L(T_1) \cup T = \cup_{R \in \mathscr{R}} L(R)$ by Lemma 3.5(I), and $|L(R)| = 2$ for all $R \in \mathscr{R}$ by Claim 4.13. By Lemma 2.19, we have $T_{2,2,1} \supseteq L(R)$ or $T_{2,2,1} \cap L(R) = \emptyset$ for each $R \in \mathscr{R}$. Therefore $|T_{2,2,1} \cap (L(T_{2,1}) \cup L(T_1) \cup T)|$ is even, which implies that $|T_{2,2,1}| = 1 + |T_{2,2,1} \cap (L(T_{2,1}) \cup L(T_1) \cup T)|$ is odd. This contradicts the fact that $|T_{2,2,1}| = 10$.

Subcase $1-2$. There exist $T_{2,1}, T_{2,2}, T_{2,3} \in \mathscr{T}_{M_2}$ such that $V(M_2) = L(T_{2,1}) \cup L(T_{2,2}) \cup L(T_{2,3})$.

Since $|V(M_2)| = 6$, $|L(T_{2,j})| = 2$ for each j. We divide this subcase further into two subcases.

Subcase $1-2-1$. $|V(M_1)| = 1$. By Lemma 2.8, T_1 meshes with T_0, and hence

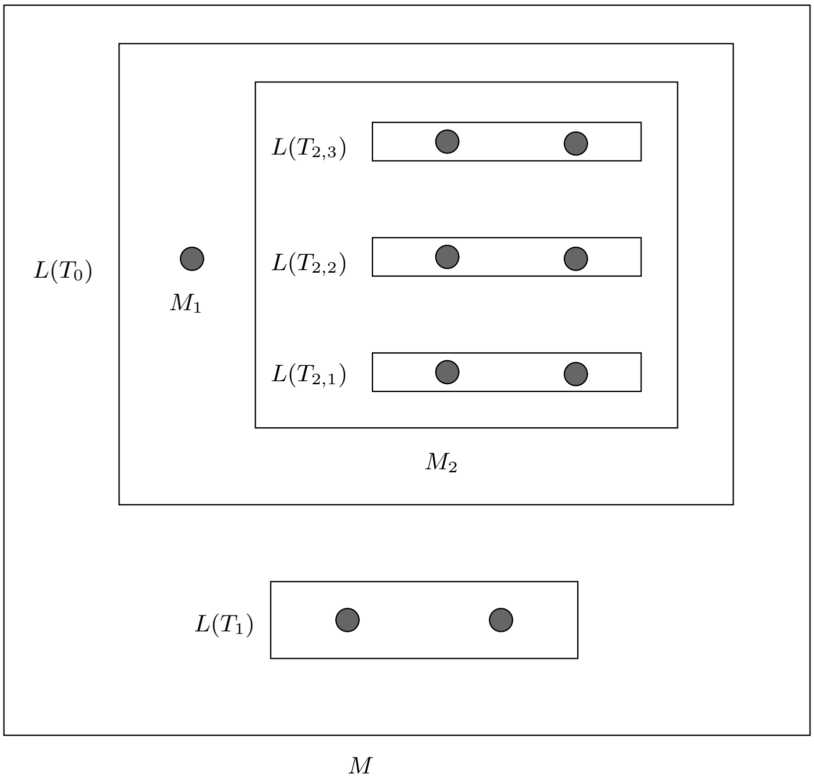

Figure 9: Subcase $1-2-1$

$T_1 \cap V(M_1) \neq \emptyset$. Since $|V(M_1)| = 1$, this implies $|T_1 \cap V(M_1)| = 1$. Hence $|T_1 \cap L(T_0)| = 1 + |T_1 \cap V(M_2)|$. By Lemma 2.21, $T_1 \subseteq (V(M) - L(T_1)) \cup T = L(T_0) \cup T$. Consequently $|T_1| = |T_1 \cap L(T_0)| + |T_1 \cap T| = 1 + |T_1 \cap (V(M_2) \cup T)|$. Set $\mathscr{R} = \{T_{2,1}, T_{2,2}, T_{2,3}, R_1, \cdots, R_5\}$. Then $V(M_2) \cup T = \cup_{R \in \mathscr{R}} L(R)$, and $|L(R)| = 2$ for all $R \in \mathscr{R}$ by Claim 4.13. By Lemma 2.19, we have $T_1 \supseteq L(R)$ or $T_1 \cap L(R) = \emptyset$ for each $R \in \mathscr{R}$. Therefore $|T_1| = 1 + |T_1 \cap (V(M_2) \cup T)|$ is odd, a contradiction.

Subcase $1-2-2$. $|V(M_1)| = 4$.

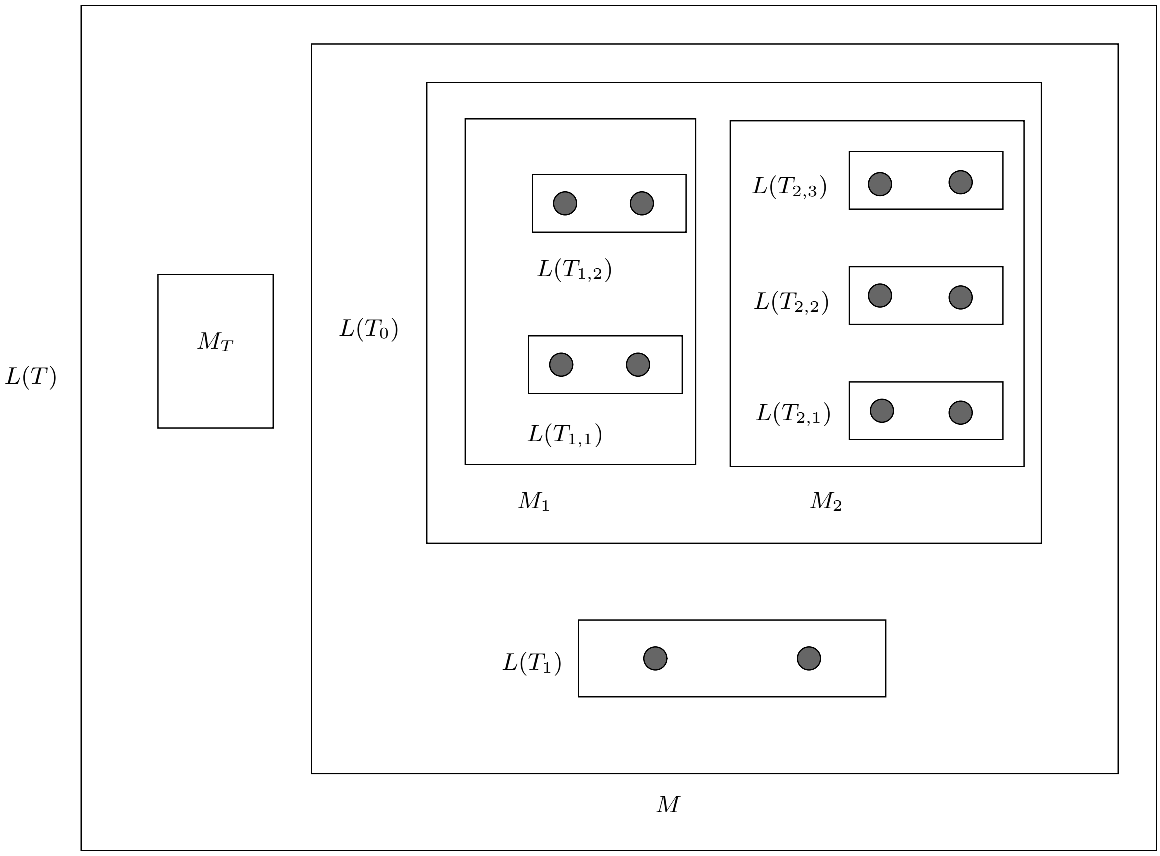

Figure 10: Subcase $1-2-2$

By Lemma 2.8, there exist $T_{1,1}, T_{1,2} \in \mathscr{T}_{M_1}$ with $|L(T_{1,1})| = |L(T_{1,2})| = 2$ such that $V(M_1) = L(T_{1,1}) \cup L(T_{1,2})$. Recall that $|\mathscr{L}(T)| = 2$. Write $\mathscr{L}(T) = \{M_T, M\}$. We now take T' and M' into consideration. Since $|\{S \in \mathscr{T}_{M'} | L(S) \subseteq L(T)\}| = (2|L(T)| - 2)/3$,(II)(ii) or (III)(i) of Lemma 2.8 cannot hold for M'. Since M' is saturated by Lemma 3.5(II), it follows that (III)(ii) of Lemma 2.8 holds for M'. Hence (II)(i) of Lemma 3.5 holds for M', which means that there exists $T_2 \in \mathscr{T}_{M'}$ with $|L(T_2)| = 2$ or 4 such that $V(M') = L(T) \cup L(T_2)$. Since $|\{S \in \mathscr{T}_{M'} | L(S) \subseteq L(T)\}| = (2|L(T)| - 2)/3$, it follows from Lemma 2.8(III)(iii) that $|\{S \in \mathscr{T}_{M'} | L(S) \subseteq L(T_2)\}| = (2|L(T_2)| - 1)/3$. Hence $|L(T_2)| = 2$. By Lemma 2.8, T_2 meshes with T. Suppose that $|V(M_T)| = 1$. Then arguing as in Subcase1 $-2-1$, we obtain $|T_2| = 1 + |T_2 \cap (V(M) \cup T')|$. Set $\mathscr{Q} = \{T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}, T_{2,3}, T_1, Q_1, \cdots Q_5\}$. Then $V(M) \cup T' = \cup_{Q \in \mathscr{Q}} L(Q)$, and $|L(Q)| = 2$ for all $Q \in \mathscr{Q}$ by Claim 4.14. Hence, arguing again as in Subcase $1-2-1$, we see that $|T_2|$ is odd, a contradiction.

Thus $|V(M_T)| \geq 2$. By Lemma 2.21, $T_0 \subseteq L(T_1) \cup T$ and $T \subseteq L(T_2) \cup T'$, which implies

$|T_0 \cap T \cap T'| \geq |T \cap T'| - 2 \geq |T'| - 4$. Hence at least three of $L(Q_1), \cdots, L(Q_5)$, say $L(Q_1), L(Q_2), L(Q_3)$, intersect with $T_0 \cap T \cap T'$. In view of Claim 4.14, it follows from Lemmas 2.19 and 2.4 that $L(Q_1), L(Q_2), L(Q_3)$ are contained in $T_0 \cap T \cap T'$, and Q_1, Q_2, Q_3 mesh with T_0, T, T'. Let $1 \leq j \leq 2$. By Lemma 2.21, $T_{1,j} \subseteq (V(M_1) - L(T_{1,j})) \cup T_0$, and hence $|T_{1,j} \cap T_0| \geq |T_0| - 2$, which implies $|T_{1,j} \cap T_0 \cap T \cap T'| = |T_0 \cap T \cap T'| - |(T_0 \cap T \cap T') - T_{1,j}| \geq |T_0 \cap T \cap T'| - |T_0 - T_{1,j}| = |T_0 \cap T \cap T'| - (|T_0| - |T_{1,j} \cap T_0|) \geq |T_0 \cap T \cap T'| - 2$. Hence at least two of $L(Q_1), L(Q_2), L(Q_3)$ intersect with $T_{1,j}$ which , by Lemma 2.4, implies that at least two of Q_1, Q_2, Q_3 mesh with $T_{1,j}$. Similarly, for each $1 \leq j \leq 3$, $|T_{2,j} \cap T_0| \geq |T_0| - 4$ by Lemma 2.21, and hence at least one of Q_1, Q_2, Q_3 meshes with $T_{2,j}$. Consequently the number of those pairs (S, Q) with $S \in \{T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}, T_{2,3}\}$ and $Q \in \{Q_1, Q_2, Q_3\}$ for which Q meshes with S is at least $2 \cdot 2 + 3 \cdot 1 = 7$. Hence one of Q_1, Q_2, Q_3, say Q_1, meshes with at least three of $T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}$ and $T_{2,3}$. In view of Lemmas 2.4 and 2.19, this implies $|Q_1 \cap L(T_0)| \geq 6$. Since Q_1 meshes with T and $|V(M_T)| \geq 2$, this together with Lemma 2.19 implies $|Q_1 \cap L(T)| \geq |Q_1 \cap V(M_T)| + |Q_1 \cap V(M)| \geq 2 + |Q_1 \cap L(T_0)| \geq 8$. Since Q_1 mesh with T', we now obtain $|Q_1 \cap L(T')| \geq 1 + |Q_1 \cap V(M')| \geq 1 + |Q_1 \cap L(T)| \geq 9$. Since $L(Q_1) \subseteq T'$, this contradicts Lemma 2.20. This is the final contradiction in Case 1.

Case 2. (iv) of Claim 4.10 holds.

We have $|\mathscr{T}_{M_1}| = (2|V(M_1)| - 2)/3$ or $(2|V(M_1)| - 3)/3$. Hence $|V(M_1)| = 1, 3$ or 4. By Lemma 2.8, there exist $T_{2,1}, T_{2,2} \in \mathscr{T}_{M_2}$ with $|L(T_{2,1})| = |L(T_{2,1})| = 2$ such that $V(M_2) = L(T_{2,1}) \cup L(T_{2,2})$.

Subcase $2-1$. $|V(M_1)| = 3$.

By Lemma 2.22, there exists $T_{1,1} \in \mathscr{T}_{M_1}$ such that $|T_{1,1} \cap (V(M_1) - L(T_{1,1}))| = 1$. By Lemma 2.21, $T_{1,1} \subseteq (V(M_1) - L(T_{1,1})) \cup T_0 \subseteq (V(M_1) - L(T_{1,1})) \cup L(T_1) \cup T$. Consequently $|T_{1,1}| = 1 + |T_{1,1} \cap (L(T_1) \cup T)|$. Set $\mathscr{R} = \{T_1, R_1, \cdots, R_5\}$. Then $L(T_1) \cup T = \cup_{R \in \mathscr{R}} L(R)$, and $|L(R)| = 2$ for all $R \in \mathscr{R}$ by Claim 4.13. By Lemma 2.19, we have $T_{1,1} \supseteq L(R)$ or $T_{1,1} \cap L(R) = \emptyset$ for all each $R \in \mathscr{R}$. Therefore $|T_{1,1} \cap (L(T_1) \cup T)|$ is even, which implies that $|T_{1,1}| = 1 + |T_{1,1} \cap (L(T_1) \cup T)|$ is odd. This contradicts the fact $|T_{1,1}| = 10$.

Subcase $2-2$. $|V(M_1)| = 1$.

By Lemma 2.8, T_1 meshes with T_0, and hence $T_1 \cap V(M_1) \neq \emptyset$. Since $|V(M_1)| = 1$, this implies $|T_1 \cap V(M_1)| = 1$. Hence $|T_1 \cap L(T_0)| = 1 + |T_1 \cap V(M_2)|$. By Lemma 2.21, $T_1 \subseteq (V(M) - L(T_1)) \cup T = L(T_0) \cup T$. Consequently $|T_1| = |T_1 \cap L(T_0)| + |T_1 \cap T| = 1 + |T_1 \cap (V(M_2) \cup T)|$ Set $\mathscr{R} = \{T_{2,1}, T_{2,2}, R_1, \cdots, R_5\}$. Then $V(M_2) \cup T = \cup_{R \in \mathscr{R}} L(R)$, and $|L(R)| = 2$ for all $R \in \mathscr{R}$ by Claim 4.13. By Lemma 2.19, we have $T_1 \supseteq L(R)$ or $T_1 \cap L(R) = \emptyset$ for each $R \in \mathscr{R}$. Therefore $|T_1| = 1 + |T_1 \cap (V(M_2) \cup T)|$ is odd, a contradiction.

Subcase $2-3$. $|V(M_1)| = 4$.

By Lemma 2.8, there exist $T_{1,1}, T_{1,2} \in \mathscr{T}_{M_1}$ with $|L(T_{1,1})| = |L(T_{1,2})| = 2$ such that $V(M_1) = L(T_{1,1}) \cup L(T_{1,2})$. Write $\mathscr{L}(T) = \{M_T, M\}$. By Lemma 3.5(II), either there exists $T_2 \in \mathscr{T}_{M'}$ with $|L(T_2)| = 2$ or 4 such that $V(M') = L(T) \cup L(T_2)$, or there exist

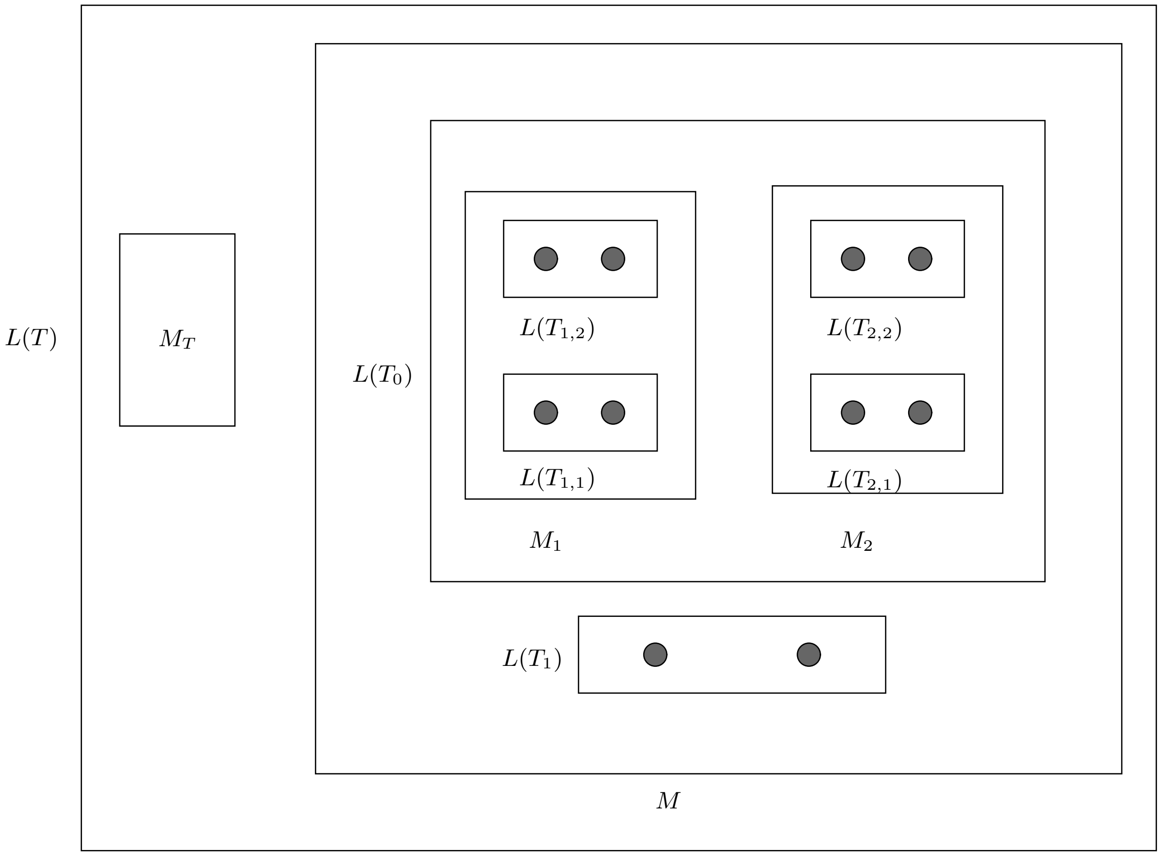

Figure 11: Subcase $2-3$

$T_2, T_2' \in \mathscr{T}_{M'}$ with $|L(T_2)| = |L(T_2')| = 2$ such that $V(M') = L(T) \cup L(T_2) \cup L(T_2')$. In the case where $V(M') = L(T) \cup L(T_2)$, T_2 meshes with T by Lemma 2.8; in the case where $V(M') = L(T) \cup L(T_2) \cup L(T_2')$, T_2 or T_2' meshes with T by Lemma 2.8 and, by symmetry, we may assume that T_2 meshes with T. Thus in either case, T_2 meshes with T.

Suppose that $|V(M_T)| = 1$. Since T_2 meshes with T, $T_2 \cap V(M_T) \neq \emptyset$. Since $|V(M_T)| = 1$, this implies $|T_2 \cap V(M_T)| = 1$. Hence $|T_2 \cap L(T)| = 1 + |T_2 \cap V(M)|$. By Lemma 2.21, $T_2 \subseteq (V(M') - L(T_2)) \cup T'$. Consequently $|T_2| = |T_2 \cap (V(M') - L(T_2))| + |T_2 \cap T'| = |T_2 \cap L(T)| + |T_2 \cap (V(M') - L(T) - L(T_2))| + |T_2 \cap T'| = 1 + |T_2 \cap (V(M) \cup (V(M') - L(T_0) - L(T_2)) \cup T')|$. Set $\mathscr{Q} = \{T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}, T_1, Q_1, \cdots, Q_5\}$ or $\{T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}, T_1, T_2', Q_1, \cdots, Q_5\}$ according as $V(M') = L(T) \cup L(T_2)$ or $L(T) \cup L(T_2) \cup L(T_2')$. Then $V(M) \cup (V(M') - L(T) - L(T_2)) \cup T' = \cup_{Q \in \mathscr{Q}} L(Q)$, and $|L(Q)| = 2$ for all $Q \in \mathscr{Q}$ by Claim 4.14. By Lemma 2.19, we have $T_2 \supseteq L(Q)$ or $T_2 \cap L(Q) = \emptyset$ for each $Q \in \mathscr{Q}$. Therefore $|T_2| = 1 + |T_2 \cap (V(M) \cup (V(M') - L(T) - L(T_2)) \cup T')|$ is odd, a contradiction.

Thus $|V(M_T)| \geq 2$. By Lemma 2.21, $|T_0 \cap T| \geq |T| - 2$. Hence at least four of

$L(R_1), \cdots, L(R_5)$, say $L(R_1), \cdots, L(R_4)$, intersect with $T_0 \cap T$. It follows from Lemmas 2.19 and 2.4 that $L(R_1), \cdots, L(R_4)$ are contained in $T_0 \cap T$, and $R_1, \cdots, R_4$ mesh with T_0 and T. Let $1 \leq i, j \leq 2$. Then $|T_{i,j} \cap T_0| \geq |T_0| - 2$ by Lemma 2.21. Since $L(R_1), \cdots, L(R_4)$ are contained in T_0, it follows that at least three of $L(R_1), \cdots, L(R_4)$ intersect with $T_{i,j}$. Hence by Lemma 2.4, at least three of $R_1, \cdots, R_4$ mesh with $T_{i,j}$. Since this holds for each i, j with $1 \leq i, j \leq 2$, the number of those pairs (S, R) with $S \in \{T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}\}$ and $R \in \{R_1, \cdots, R_4\}$ for which R meshes with S is at least 12. If there exists $R \in \{R_1, \cdots, R_4\}$ such that R meshes with all of $T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}$, then $|R \cap L(T_0)| = 8$ by Lemma 2.19, and hence $|R \cap L(T)| \geq 1 + |R \cap V(M)| \geq 9$, which contradicts Lemma 2.20. Thus there is no such R. Hence each R_i $(1 \leq i \leq 4)$ meshes with three of $T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}$. By Lemma 2.21, $|T \cap T'| \geq |T| - 4$, which implies $|T_0 \cap T \cap T'| = |T_0 \cap T| - |(T_0 \cap T) - T'| \geq |T_0 \cap T| - |T - T'| = |T_0 \cap T| - (|T| - |T \cap T'|) \geq |T_0 \cap T| - 4$. Since $L(R_1), \cdots, L(R_4)$ are contained in $T_0 \cap T$, it follows that at least two of $L(R_1), \cdots, L(R_4)$ intersect with $T_0 \cap T \cap T'$. We may assume $L(R_1) \cap (T_0 \cap T \cap T') \neq \emptyset$. Since $|L(R_1)| = 2$, it follows from Lemma 2.3 that $L(R_1) \cap L(T') = \emptyset$, and hence R_1 meshes with T' by Lemma 2.4, and $L(R_1) \subseteq T'$ by Lemma 2.19. Since R_1 meshes with three of $T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}$ and $|V(M_T)| \geq 2$, it now follows from Lemma 2.19 that $|R_1 \cap L(T')| \geq 1 + |R_1 \cap V(M')| \geq 1 + |R_1 \cap V(M_T)| + |R_1 \cap V(M)| \geq 1 + 2 + 3 \cdot 2 = 9$, which contradicts Lemma 2.20. This concludes the discussion for Case 2.

Case 3. (i) of Claim 4.10 holds.

We have $|\mathscr{T}_M| = (2|V(M)| - 3)/3$, and $|\{S \in \mathscr{T}_{M'} \, |L(S) \subseteq L(T)\}| = (2|L(T)| - 2)/3$ by Lemma 2.9(II). By Lemma 2.8, there exist $T_{2,1}, T_{2,2} \in \mathscr{T}_{M_2}$ with $|L(T_{2,1})| = |L(T_{2,2})| = 2$ such that $V(M_2) = L(T_{2,1}) \cup L(T_{2,2})$. By Lemma 2.8, we may assume that T_1 meshes with T_0. Suppose that $|V(M_1)| = 1$. Since T_1 meshes with T_0, $T_1 \cap V(M_1) \neq \emptyset$. Since $|V(M_1)| = 1$, this implies $|T_1 \cap V(M_1)| = 1$. Hence $|T_1 \cap L(T_0)| = 1 + |T_1 \cap V(M_2)|$. By Lemma 2.21, $T_1 \subseteq (V(M) - L(T_1)) \cup T = L(T_0) \cup L(T_1') \cup T$. Consequently $|T_1| = |T_1 \cap L(T_0)| + |T_1 \cap L(T_1')| + |T_1 \cap T| = 1 + |T_1 \cap (V(M_2) \cup L(T_1') \cup T)|$. Set $\mathscr{R} = \{T_{2,1}, T_{2,2}, T_1', R_1, \cdots, R_5\}$. Then $V(M_2) \cup L(T_1') \cup T = \cup_{R \in \mathscr{R}} L(R)$, and $|L(R)| = 2$ for all $R \in \mathscr{R}$ by Claim 4.13. By Lemma 2.19, we have $T_1 \supseteq L(R)$ or $T_1 \cap L(R) = \emptyset$ for each $R \in \mathscr{R}$. Therefore $|T_1| = 1 + |T_1 \cap (V(M_2) \cup L(T_1') \cup T)|$ is odd, a contradiction. Thus $|V(M_1)| = 4$. By Lemma 2.8, there exist $T_{1,1}, T_{1,2} \in \mathscr{T}_{M_1}$ with $|L(T_{1,1})| = |L(T_{1,2})| = 2$ such that $V(M_1) = L(T_{1,1}) \cup L(T_{1,2})$. Write $\mathscr{L}(T) = \{M_T, M\}$. Since $|\{S \in \mathscr{T}_{M'} \, |L(S) \subseteq L(T)\}| = (2|L(T)| - 2)/3$, it follows from Lemma 3.5(II) and Lemma 2.8 that there exists $T_2 \in \mathscr{T}_{M'}$ with $|L(T_2)| = 2$ such that $V(M') = L(T) \cup L(T_2)$. By Lemma 2.8, T_2 meshes with T.

Suppose that $|V(M_T)| = 1$, Since T_2 meshes with T, $T_2 \cap V(M_T) \neq \emptyset$. Since $|V(M_T)| = 1$, this implies $|T_2 \cap V(M_T)| = 1$. Hence $|T_2 \cap L(T)| = 1 + |T_2 \cap V(M)|$. By Lemma 2.21, $T_2 \subseteq (V(M') - L(T_2)) \cup T' = L(T) \cup T'$. Consequently $|T_2| = |T_2 \cap L(T)| + |T_2 \cap T'| = 1 + |T_2 \cap (V(M) \cup T')|$. Set $\mathscr{Q} = \{T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}, T_1, T_1', Q_1, \cdots, Q_5\}$. Then $V(M) \cup T' = \cup_{Q \in \mathscr{Q}} L(Q)$, and $|L(Q)| = 2$ for all $Q \in \mathscr{Q}$ by Claim 4.14. By Lemma 2.19, we have $T_2 \supseteq L(Q)$ or $T_2 \cap L(Q) = \emptyset$ for each $Q \in \mathscr{Q}$. Therefore $|T_2| = 1 + |T_2 \cap (V(M) \cup T')|$

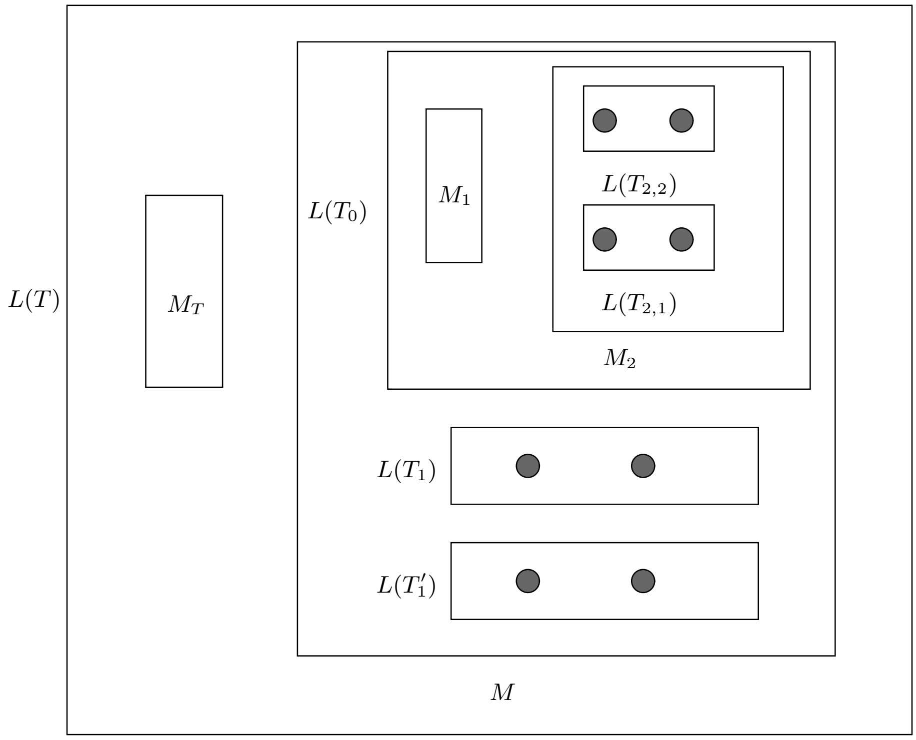

Figure 12: Case 3

is odd, a contradiction.

Thus $|V(M_T)| \geq 2$. By Lemma 2.21, $|T_0 \cap T| \geq |T| - 4$. Hence by Lemma 2.19, at least three of $L(R_1), \cdots, L(R_5)$, say $L(R_1), L(R_2), L(R_3)$, are contained in $T_0 \cap T$, and hence R_1, R_2, R_3 mesh with T_0 and T. Let $1 \leq i, j \leq 2$. Then $|T_{i,j} \cap T_0| \geq |T_0| - 2$ by Lemma 2.21. Since $L(R_1)$, $L(R_2)$ and $L(R_3)$ are contained in T_0, it follows that at least two of $L(R_1), L(R_2), L(R_3)$ intersect with $T_{i,j}$. Hence by Lemma 2.4, at least two of R_1, R_2, R_3 mesh with $T_{i,j}$. Since this holds for each i, j with $1 \leq i, j \leq 2$, the number of those pairs (S, R) with $S \in \{T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}\}$ and $R \in \{R_1, R_2, R_3\}$ for which R meshes with S is at least 8. If there exists $R \in \{R_1, R_2, R_3\}$ such that R meshes with all of $T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}$, then $|R \cap L(T_0)| = 8$ by Lemma 2.19, and hence $|R \cap L(T)| \geq 1 + |R \cap V(M)| \geq 9$, which contradicts Lemma 2.20. Thus there in no such R. Hence some two of R_1, R_2, R_3, say R_1 and R_2, mesh with three of $T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}$. By Lemma 2.21, $|T \cap T'| \geq |T| - 2$, which implies $|T_0 \cap T \cap T'| = |T_0 \cap T| - |(T_0 \cap T) - T'| \geq |T_0 \cap T| - |T - T'| = |T_0 \cap T| - (|T| - |T \cap T'|) \geq |T_0 \cap T| - 2$. Since $L(R_1)$ and $L(R_2)$ are contained in $T_0 \cap T$, it follows that at least one of $L(R_1)$ and $L(R_2)$, say $L(R_1)$ intersects with T'. Then $L(R_1) \cap L(T') = \emptyset$ by Lemma 2.3, and hence R_1 meshes with T' by Lemma 2.4, and $L(R_1) \subseteq T'$ by Lemma 2.19. Since R_1 meshes with three of $T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}$ and $|V(M_T)| \geq 2$, it now follows from Lemma 2.19 that $|R_1 \cap L(T')| \geq 1 + |R_1 \cap V(M')| \geq 1 + |R_1 \cap V(M_T)| + |R_1 \cap V(M)| \geq 1 + 2 + 3 \cdot 2 = 9$, which contradicts Lemma 2.20.

Case 4. (ii) of Claim 4.10 holds.

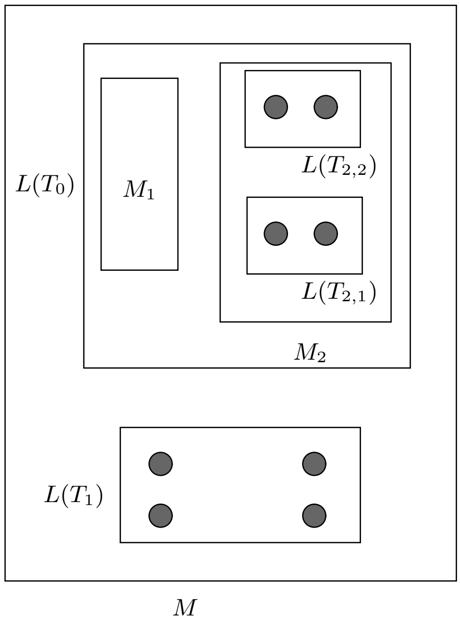

Figure 13: Case 4

By Lemma 2.8, T_1 meshes with T_0. By Lemma 2.8, there exist $T_{2,1}, T_{2,2} \in \mathscr{T}_{M_2}$ with $|L(T_{2,1})| = |L(T_{2,2})| = 2$ such that $V(M_2) = L(T_{2,1}) \cup L(T_{2,1})$. Suppose that $|V(M_1)| = 1$. Since T_1 meshes with T_0, $T_0 \cap V(M_1) \neq \emptyset$. Since $|V(M_1)| = 1$, this implies $|T_1 \cap V(M_1)| = 1$. Hence $|T_1 \cap L(T_0)| = 1 + |T_1 \cap V(M_2)|$. By Lemma 2.21, $T_1 \subseteq (V(M) - L(T_1)) \cup T = L(T_0) \cup T$. Consequently $|T_1| = |T_1 \cap L(T_0)| + |T_1 \cap T| = 1 + |T_1 \cap (V(M_2) \cup T)|$. Set $\mathscr{R} = \{T_{2,1}, T_{2,1}, R_1, \cdots, R_5\}$. Then $V(M_2) \cup T = \cup_{R \in \mathscr{R}} L(R)$, and $|L(R)| = 2$ for all $R \in \mathscr{R}$ by Claim 4.13. By Lemma 2.19, we have $T_1 \supseteq L(R)$ or $T_1 \cap L(R) = \emptyset$ for each $R \in \mathscr{R}$. Therefore $|T_1| = 1 + |T_1 \cap (V(M_2) \cup T)|$ is odd, a contradiction. Thus $|V(M_1)| = 4$. By Lemma 2.19, $T_1 \supseteq V(M_1)$ and $T_1 \supseteq V(M_2)$. Hence $T_1 \supseteq L(T_0)$. By Lemma 2.19, $T_0 \supseteq L(T_1)$. Since $|L(T_0)| + |L(T_1)| = 12$, this contradicts Lemma 2.13. This completes the proof of the Theorem.

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