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6-Factors in 2-Connected Star-free Graphs

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Abstract

Let $t \geq 3$ be an integer and G be a 2-connected $K_{1,t}$-free graph. We show that if $t \geq 4$ and minimum degree of G is at least $2t+1$, then G has a 6-factor. We also show that if $t = 3$ and minimum degree of G is at least 8, then G has a 6-factor.

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1. Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges.

Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, $\deg_G(x)$ denotes the degree of $x \in G$. We let $\delta(G)$ denote the minimum of $\deg_G(x)$ as x ranges over $V(G)$. For an integer $r \geq 1$, a subgraph F of G such that $V(F) = V(G)$ and $\deg_F(x) = r$ for all $x \in V(F)$ is called an r-factor of G. A complete bipartite graph $K_{1,t}$ with partite sets of cardinalities 1 and t is called a t-star. We say that G is $K_{1,t}$-free or t-star-free if G contains no $K_{1,t}$ as an induced subgraph.

The following theorem was proved by Tokuda and Ota in [4].

Theorem 1.1. *Let t, r be integers with $t \geq 3$ and $r \geq 2$. Let G be a connected $K_{1,t}$-free graph, and suppose that*

$$\delta(G) \geq \left(t + \frac{t-1}{r}\right)\left\lceil \frac{t}{2(t-1)}r \right\rceil - \frac{t-1}{r}\left\lceil \frac{t}{2(t-1)}r \right\rceil^2 + t - 3.$$

In the case where r is odd, suppose further that $r \geq t-1$ and $|V(G)|$ is even. Then G has an r-factor.

The cases where $r = 2$, $r = 4$ and $r = 6$ of Theorem 1.1 are particularly important because the minimum degree condition takes the following simple form.

Corollary 1.2. *Let $t \geq 3$ be an integer. Let G be a connected $K_{1,t}$-free graph and suppose that $\delta(G) \geq 2t - 2$. Then G has a 2-factor.*

Corollary 1.3. *Let $t \geq 3$ be an integer. Let G be a connected $K_{1,t}$-free graph and suppose that $\delta(G) \geq \frac{5t-3}{2}$. Then G has a 4-factor.*

Corollary 1.4. *Let $t \geq 3$ be an integer. Let G be a connected $K_{1,t}$-free graph. Then the followings hold.*

(i) If $t \geq 4$ and $\delta(G) \geq 3t - 1$, then G has a 6-factor.

(ii) If $t = 3$ and $\delta(G) \geq 9$, then G has a 6-factor.

The minimum degree condition in Theorem 1.1 is best possible and hence so are those in Corollaries 1.2, 1.3 and 1.4. On the other hand, as for Corollaries 1.2 and 1.3, it was shown by Aldred et al. in [1] and Egawa et al. in [2] that if we add the assumption that G is 2-connected, then we can relax the minimum degree condition as follows, respectively.

Theorem 1.5. [1] *Let $t \geq 3$ be an integer. Let G be a 2-connected $K_{1,t}$-free graph and suppose that $\delta(G) \geq t$. Then G has a 2-factor.*

Theorem 1.6. [2] *Let $t \geq 3$ be an integer. Let G be a 2-connected $K_{1,t}$-free graph and suppose that $\delta(G) \geq \frac{3t+1}{2}$. Then G has a 4-factor.*

The purpose of this paper is similarly to weaken the minimum degree condition in Corollary 1.4 under the condition that G is 2-connected.

Theorem 1.7. *Let $t \geq 3$ be an integer. Let G be a 2-connected $K_{1,t}$-free graph. Then the followings hold.*

(i) If $t \geq 4$ and $\delta(G) \geq 2t + 1$, then G has a 6-factor.

(ii) If $t = 3$ and $\delta(G) \geq 8$, then G has a 6-factor.

Our notation is standard possibly except the following.

Let G be a graph. For $x \in V(G)$, $N_G(x)$ denotes the set of vertices adjacent to x in G; thus $\deg_G(x) = |N_G(x)|$. For $A \subseteq V(G)$, we let $N_G(A)$ denote the union of $N_G(x)$ as x ranges over A. For A, $B \subseteq V(G)$ with $A \cap B = \emptyset$, $E(A, B)$ denotes the set of edges of G joining a vertex in A and a vertex in B. For $A \subseteq V(G)$, the graph obtained from G by deleting all vertices in A together with the edges incident with them is denoted by $G - A$. We often identify a subgraph H of G with its vertex set and, for example, write $N(H)$ for $N(V(H))$. Also a vertex x of G is often identified with the set $\{x\}$; for example if H is a subgraph with $x \notin V(H)$, we write $E(x, H)$ for $E(\{x\}, V(H))$.

2. Preliminary Results

In this section, we state preliminary lemmas, which we use in the proof of Theorem 1.7. Let G be a graph. For S, $T \subseteq V(G)$ with $S \cap T = \emptyset$, denote $\theta(S,T)$ by

$$\theta(S,T) = 6|S| + \sum_{y \in T} \big(\deg_{G-S}(y) - 6\big) - h(S,T),$$

where $h(S,T)$ denotes the number of components C of $G - S - T$ such that $|E(T,C)|$ is odd. The following lemma is a special case of the f-Factor Theorem of Tutte [5].

Lemma 2.1.

(i) G has a 6-factor if and only if $\theta(S,T) \geq 0$ for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$.

(ii) $\theta(S,T)$ is even for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$.

The following lemma is proved in [3], but we include its proof for the convenience of the reader.

Lemma 2.2. *Suppose that G has no 6-factor and choose $S, T \subseteq V(G)$ with $S \cap T = \emptyset$ and $\theta(S,T) < 0$ so that $|S \cup T|$ is as large as possible. Then $|V(C)| \geq 3$ for every component C of $G - S - T$.*

Proof. Note that we have $\theta(S,T) \leq -2$ by Lemma 2.1(ii). Suppose that there exists a component C of $G - S - T$ with $|V(C)| \leq 2$, and take $v \in V(C)$. If $|E(v,T)| \leq 5$, then $\sum_{y \in T \cup \{v\}}(\deg_{G-S}(y) - 6) = \sum_{y \in T}(\deg_{G-S}(y) - 6) + \deg_{G-S}(v) - 6 \leq \sum_{y \in T}(\deg_{G-S}(y) - 6) + (|E(v,T)| + 1) - 6 \leq \sum_{y \in T}(\deg_{G-S}(y) - 6)$ and $h(S, T \cup \{v\}) \geq h(S,T) - 1$, and hence $\theta(S, T \cup \{v\}) \leq \theta(S,T) + 1 \leq -1$. If $|E(v,T)| \geq 6$, then $\sum_{y \in T}(\deg_{G-(S \cup \{v\})}(y) - 6) = \sum_{y \in T}(\deg_{G-S}(y) - 6) - |E(v,T)| \leq \sum_{y \in T}(\deg_{G-S}(y) - 6) - 6$ and $h(S \cup \{v\}, T) \geq h(S,T) - 1$, and hence $\theta(S \cup \{v\}, T) \leq \theta(S,T) + 1 \leq -1$. In either case, we get a contradiction to the maximality of $|S \cup T|$. □

The purpose of the rest of this section is to prove inequalities (Lemmas 2.5 and 2.7), which play an important role in the proof of Theorem 1.7. Throughout the rest of this section, we let f, f' and g' denote the functions defined by

$$f(t,d;\alpha,\beta) = \frac{6(2t-d+1)}{t-1} + \left(d - \frac{t-7}{t-1}\alpha - \frac{\beta}{3} - 6\right)(d - \alpha - \beta + 1),$$

$$f'(d;\alpha,\beta) = 24 - 3d + \left(d + 2\alpha - \frac{\beta}{3} - 6\right)(d - \alpha - \beta + 1) \text{ and}$$

$$g'(d;\alpha,\beta,\gamma) = 24 - 3d + \left(d + 2\alpha - \frac{\beta}{3} - \frac{\gamma}{5} - 6\right)(d - \alpha - \beta - \gamma + 1)$$

(see Sections 3 and 4 for the role of the functions f, f' and g' in the proof of Theorem 1.7).

Lemma 2.3. *Let t, d, α be integers with $t \geq 3$ and $0 \leq \alpha \leq d < \frac{t-7}{t-1}\alpha + 6$. Then the following hold.*

(i) $f(t, d; \alpha, 0) \geq 0$ unless $t = 3, d = 4$ and $\alpha = 0$.

(ii) If $t = 3, d = 4$ and $\alpha = 0$, then $f(t, d; \alpha, 0) = -1$.

Proof. From the assumption that $0 \leq \alpha \leq d < (t-7)\alpha/(t-1) + 6$, it follows that $d - 6 < \alpha \leq d$, i.e.,

$$d - 5 \leq \alpha \leq d. \tag{2.1}$$

Since (2.1) implies $\alpha - d + 6 \geq 1$ and $d - \alpha + 1 \geq 1$ and since $(\alpha - d + 6) + (d - \alpha + 1) = 7$, we get

$$(\alpha - d + 6)(d - \alpha + 1) \leq 12. \tag{2.2}$$

If $d \leq 3$, then $6(2t - d + 1)/(t-1) \geq 12$, and hence it follows from (2.2) that

$$\begin{aligned} f(t, d; \alpha, 0) &= \frac{6(2t - d + 1)}{t - 1} - \left(\frac{t-7}{t-1}\alpha - d + 6\right)(d - \alpha + 1) \\ &\geq 12 - (\alpha - d + 6)(d - \alpha + 1) \geq 0. \end{aligned}$$

Thus we may assume $d \geq 4$.

If $\alpha = 0$, then it follows from (2.1) that $d = 4$ or 5. If $\alpha = 0$ and $d = 4$, then $f(t, 4; 0, 0) = 2 - 6/(t-1)$. Hence we see that if $t = 3$, then $f(3, 4; 0, 0) = -1$; if $t \geq 4$, then $f(t, 4; 0, 0) \geq 0$.

Also we see that if $\alpha = 0$ and $d = 5$, then $f(t, 5; 0, 0) = 6 - 12/(t-1) \geq 0$. Thus we may assume $\alpha \geq 1$. Then it follows from (2.1) that $\alpha(d - \alpha + 1) \geq d$. Hence

$$\left(\frac{t-7}{t-1}\alpha - d + 6\right)(d - \alpha + 1) = (\alpha - d + 6)(d - \alpha + 1) - \frac{6}{t-1}\alpha(d - \alpha + 1) \leq 12 - \frac{6d}{t-1}$$

by (2.2). Consequently

$$f(t, d; \alpha, 0) = 12 - \frac{6(d-3)}{t-1} - \left(\frac{t-7}{t-1}\alpha - d + 6\right)(d - \alpha + 1) \geq \frac{18}{t-1} > 0,$$

as desired. □

Lemma 2.4. *Let t, d, β be integers with $t \geq 4$ and $0 \leq \beta \leq d < \frac{\beta}{3} + 6$. Then $f(t, d; 0, \beta) \geq 0$.*

Proof. From $0 \le \beta \le d < \beta/3 + 6$, we get $\beta < 9$, and hence

$$0 \le \beta \le d \le 8. \tag{2.3}$$

In the case where $\beta \ge 1$, $f(t,d;0,\beta) \ge f(4,d;0,\beta) \ge f(4, 2\beta/3+7/2; 0, \beta) = -\beta^2/9 + \beta - 1/4 > 0$ by (2.3). In the case where $\beta = 0$, since d is an integer, $f(t,d;0,0) \ge f(4,d;0,0) \ge f(4,4;0,0) = 0$. Hence in either case, we obtain $f(t,d;0,\beta) \ge 0$. □

Lemma 2.5. *Let t, d, α, β be nonnegative integers with $t \ge 4$ and $\alpha + \beta \le d < \frac{t-7}{t-1}\alpha + \frac{\beta}{3} + 6$. Then $f(t,d;\alpha,\beta) \ge 0$.*

Proof. First assume $t \ge 10$. Then $(t-7)/(t-1) \ge 1/3$, which implies $d < (t-7)(\alpha+\beta)/(t-1) + 6$ and $f(t,d;\alpha,\beta) \ge f(t,d;\alpha+\beta,0)$. Hence $f(t,d;\alpha,\beta) \ge 0$ by Lemma 2.3.

Next assume $4 \le t \le 9$. Then $1/3 > (t-7)/(t-1)$, which implies $f(t,d;\alpha,\beta) > f(t,d;0,\alpha+\beta) \ge 0$. Hence $f(t,d;\alpha,\beta) \ge 0$ by Lemma 2.4. □

Lemma 2.6. *Let d, α, β be nonnegative integers with $\alpha + \beta \le d < -2\alpha + \frac{\beta}{3} + 6$. Then $f'(d;\alpha,\beta) \ge 0$ unless $d = 8, \alpha = 0$ and $\beta = 7$ or 8.*

Proof. Since $0 \le \alpha \le d$, $f'(d;\alpha,0) \ge \min\{f'(d;0,0), f'(d;d,0)\} = \min\{d^2 - 8d + 18, 18\} > 0$. Hence

$$f'(d;\alpha,0) \ge 0. \tag{2.4}$$

Since $0 \le \beta \le d < \frac{\beta}{3} + 6$, we get $\beta < 8$, and hence $0 \le \beta \le d \le 8$. If $d \le 4$, then $24 - 3d \ge 12 \ge (\beta/3 - d - 6)(d - \beta + 1)$; if $5 \le d \le 7$, then $f'(d;0,\beta) \ge f'(d;0,2d - 17/2) = -d^2/3 + 10d/3 - 73/12 > 0$. Now we may assume $d = 8$. If $\beta \le 6$, then $f'(8;0,\beta) = \beta^2/3 - 5\beta + 18 \ge 0$; if $\beta = 7$ or 8, $f'(8;0,\beta) = -2/3$. Consequently

$$f'(d;0,\beta) \ge 0 \text{ unless } d = 8 \text{ and } \beta = 7 \text{ or } 8, \tag{2.5}$$

$$\text{If } d = 8 \text{ and } \beta = 7 \text{ or } 8, \text{then } f'(d;0,\beta) = -\frac{2}{3}. \tag{2.6}$$

If $\alpha = 0$, the desired conclusion immediately follows from (2.5). Thus we may assume $\alpha \ge 1$. Then $f'(d;\alpha,\beta) = f'(d;0,\alpha+\beta) + 7\alpha(d - \alpha - \beta + 1)/3$. Hence $f'(d;\alpha,\beta) \ge -2/3 + 7/3 > 0$ by (2.5) and (2.6). □

Lemma 2.7. *Let d, α, β, γ be nonnegative integers with $\alpha + \beta + \gamma \le d < -2\alpha + \frac{\beta}{3} + \frac{\gamma}{5} + 6$. Then $g'(d;\alpha,\beta,\gamma) \ge 0$ unless $d = 8, \alpha = 0$ and $\beta + \gamma = 7$ or 8.*

Proof. Since $g'(d;\alpha,\beta,\gamma) \ge f'(d;\alpha,\beta+\gamma)$, Lemma 2.6 follows the desired conclusion immediately. □

3. Proof of Theorem 1.7(i)

Let t, G be as in Theorem 1.7(i); thus $t \geq 4$, and G is a 2-connected $K_{1,t}$-free graph with $\delta(G) \geq 2t+1$.

By way of contradiction, suppose that G does not have a 6-factor. Then by Lemma 2.1, there exist $S, T \subseteq V(G)$ such that $S \cap T = \emptyset$ and $\theta(S,T) < 0$. If $T = \emptyset$, then $h(S,T) = 0$, and hence $\theta(S,T) = 6|S| \geq 0$, a contradiction. Thus $T \neq \emptyset$. Suppose that $|S \cup T| \leq 1$. Then $S = \emptyset$ and $|T| = 1$. Since $\delta(G) \geq 2t+1 \geq 9$, we have $\sum_{y \in T}(\deg_{G-S}(y) - 6) \geq 3$. Since G is 2-connected and $|S \cup T| \leq 1$, $G-S-T$ is connected, which implies $h(S,T) \leq 1$. Hence $\theta(S,T) \geq 3-1 > 0$, a contradiction. Thus $|S \cup T| \geq 2$.

Now we may assume that we have chosen S and T so that $|S \cup T|$ is as large as possible. Then by Lemma 2.2, $|V(C)| \geq 3$ for every component C of $G-S-T$. We call a component C of $G-S-T$ an odd component or an even component according as $|E(T,C)|$ is odd or even. We proceed to estimate the cardinarity of S from below by using the assumption that G is 2-connected and $K_{1,t}$-free(Claim 3.4).

Let $C_1, ..., C_a, ..., C_{a+b}, ..., C_k$ be the components of $G-S-T$ such that $|E(T,C_i)| = 1$ for each $i = 1, ..., a$, $|E(T,C_j)| \geq 3$ is odd for each $a < j \leq a+b$ and $|E(T,C_k)|$ is even for each $k > a+b$. Then $h(S,T) = a+b$. For each $y \in T$, set $\alpha(y) = \sum_{1 \leq i \leq a} |E(y,C_i)|$ and $\beta(y) = \sum_{a+1 \leq i \leq k} |E(y,C_i)|$. Then

$$\alpha(y) + \beta(y) = |E(y, V(G)-S-T)|; \qquad (3.7)$$

in particular,

$$\alpha(y) + \beta(y) \leq \deg_{G-S}(y). \qquad (3.8)$$

Claim 3.1.

(i) $a = \sum_{y \in T} \alpha(y)$.

(ii) For each i *with* $a+1 \leq i \leq a+b, \sum_{y \in T} |E(y,C_i)| \geq 3$.

(iii) $b \leq (\sum_{y \in T} \beta(y))/3$.

Proof. We have $a = \sum_{1 \leq i \leq a} |E(T,C_i)| = \sum_{1 \leq i \leq a} \sum_{y \in T} |E(y,C_i)| = \sum_{y \in T} \alpha(y)$, which proves (i). Let $a+1 \leq i \leq a+b$.Then $|E(T,C_i)| \neq 1$ and $|E(T,C_i)|$ is odd. Hence $\sum_{y \in T} |E(y,C_i)| = |E(T,C_i)| \geq 3$. Thus (ii) is proved. By (ii), $b \leq (\sum_{a+1 \leq i \leq a+b} \sum_{y \in T} |E(y,C_i)|)/3 \leq (\sum_{a+1 \leq i \leq k} \sum_{y \in T} |E(y,C_i)|)/3 = (\sum_{y \in T} \beta(y))/3$, which proves (iii). □

Recall that $T \neq \emptyset$. We choose vertices $z_1, ..., z_m$ of T and define subsets $N_1, ..., N_m$ of T inductively by the following procedure. First let $z_1 \in T$ be a vertex such that $\deg_{G-S}(z_1) - \frac{(t-7)\alpha(z_1)}{t-1} - \frac{\beta(z_1)}{3} \leq \deg_{G-S}(y) - \frac{(t-7)\alpha(y)}{t-1} - \frac{\beta(y)}{3}$ for all $y \in T$, and set $N_1 = (N(z_1) \cap T) \cup \{z_1\}$. Now let $j \geq 2$, and assume that $z_1, ..., z_{j-1}$ and $N_1, ..., N_{j-1}$

have been defined. If $T - \left(\bigcup_{1\le i\le j-1} N_i\right) \neq \emptyset$, then let $z_j \in T - \left(\bigcup_{1\le i\le j-1} N_i\right)$ be a vertex such that $\deg_{G-S}(z_j) - \frac{(t-7)\alpha(z_j)}{t-1} - \frac{\beta(z_j)}{3} \le \deg_{G-S}(y) - \frac{(t-7)\alpha(y)}{t-1} - \frac{\beta(y)}{3}$ for all $y \in T - \left(\bigcup_{1\le i\le j-1} N_i\right)$, and set $N_j = \left(N(z_j) \cap \left(T - \left(\bigcup_{1\le i\le j-1} N_i\right)\right)\right) \cup \{z_j\}$; if $T - \left(\bigcup_{1\le i\le j-1} N_i\right) = \emptyset$, then let $m = j-1$ and terminate the procedure.

Claim 3.2.

(i) $\{z_1, ..., z_m\}$ *is independent.*

(ii) $T = \bigcup_{1\le j\le m} N_j$ *(disjoint union).*

(iii) For each $1 \le j \le m, \deg_{G-S}(z_j) - \frac{(t-7)\alpha(z_j)}{t-1} - \frac{\beta(z_j)}{3} \le \deg_{G-S}(y) - \frac{(t-7)\alpha(y)}{t-1} - \frac{\beta(y)}{3}$ *for all* $y \in N_j$.

(iv) For each $1 \le j \le m, |N_j| \le \deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) + 1$.

Proof. Statements (i) through (iii) follow from the definition of z_j and N_j. Let $1 \le j \le m$. Then $|N_j| \le |N(z_j) \cap T| + 1 = \deg_{G-S}(z_j) - |E(z_j, V(G) - S - T)| + 1 = \deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) + 1$ by (3.1). □

Let $1 \le i \le a$. Since $|V(C_i)| \ge 3$ and G is 2-connected, there exists an edge joining S and $V(C_i) - N(T)$. Let x_iu_i be such an edge ($x_i \in S$, $u_i \in V(C_i) - N(T)$). Set $L = \{u_i | 1 \le i \le a\}$ and thus $|L| = a$. For each $x \in S$, let $L(x) = \{u_i | 1 \le i \le a, x_i = x\}$. Clearly

$$L(x) \subseteq N(x). \tag{3.9}$$

Also

$$L = \bigcup_{x\in S} L(x) \text{ (disjoint union)},$$

and hence

$$\sum_{x\in S} |L(x)| = a. \tag{3.10}$$

Claim 3.3.

(i) L *is independent.*

(ii) $\{z_1, ..., z_m\} \cup L$ *is independent.*

(iii) For each $x \in S, (N(x) \cap \{z_1, ..., z_m\}) \cup L(x)$ *is independent.*

Proof. For every i, $i' \in \{1, ..., a\}$ with $i \neq i'$, u_i and $u_{i'}$ belong to distinct components of $G - S - T$. Hence (i) holds. Further for every $1 \leq i \leq a$, $u_i \notin N(T)$. Consequently (ii) follows from Claim 3.2(i). Also (iii) follows from (ii). □

Claim 3.4. $|S| \geq \dfrac{1}{t-1}\Bigg(\displaystyle\sum_{1\leq j\leq m}\Big(|E(z_j,S)| + \sum_{y\in N_j}\alpha(y)\Big)\Bigg).$

Proof. Since G is $K_{1,t}$-free, it follows from (3.3) and Claim 3.3(iii) that $|E(x, \{z_1, ..., z_m\})| + |L(x)| \leq t-1$ for every $x \in S$. Note that $\sum_{x\in S}|E(x,\{z_1,...,z_m\})| = |E(S,\{z_1,...,z_m\})| = \sum_{1\leq j\leq m}|E(z_j,S)|$, $\sum_{x\in S}|L(x)| = a = \sum_{y\in T}\alpha(y) = \sum_{1\leq j\leq m}\sum_{y\in N_j}\alpha(y)$ by (3.4), Claims 3.1(i) and 3.2(ii). Consequently $(t-1)|S| \geq \sum_{x\in S}\Big(|E(x,\{z_1,...,z_m\})| + |L(x)|\Big) = \sum_{1\leq j\leq m}\Big(|E(z_j,S)| + \sum_{y\in N_j}\alpha(y)\Big)$, as desired. □

We now estimate $\theta(S,T)$ from below by using the assumption that $\delta(G) \geq 2t+1$. For each $1 \leq j \leq m$, set

$$\begin{aligned} p_j &= \frac{6}{t-1}|E(z_j,S)| + \Bigg(\deg_{G-S}(z_j) - \frac{t-7}{t-1}\alpha(z_j) - \frac{\beta(z_j)}{3} - 6\Bigg)|N_j|, \\ r_j &= \frac{6(2t - \deg_{G-S}(z_j) + 1)}{t-1} \\ &\quad + \Bigg(\deg_{G-S}(z_j) - \frac{t-7}{t-1}\alpha(z_j) - \frac{\beta(z_j)}{3} - 6\Bigg)(\deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) + 1). \end{aligned}$$

Claim 3.5. $\theta(S,T) \geq \displaystyle\sum_{1\leq j\leq m} p_j.$

Proof. Note that $h(S,T) = a + b \leq \sum_{1\leq j\leq m}\sum_{y\in N_j}(\alpha(y) + \beta(y)/3)$ and $\sum_{y\in T}(\deg_{G-S}(y) - 6) = \sum_{1\leq j\leq m}\sum_{y\in N_j}(\deg_{G-S}(y) - 6)$ by Claims 3.1(i), (iii) and 3.2(ii). Therefore it follows from Claims 3.4 and 3.2(iii) that

$$\begin{aligned} \theta(S,T) &\geq \sum_{1\leq j\leq m}\Bigg(\frac{6}{t-1}|E(z_j,S)| + \sum_{y\in N_j}\big(\deg_{G-S}(y) - \Big(1 - \frac{6}{t-1}\Big)\alpha(y) - \frac{\beta(y)}{3} - 6\big)\Bigg) \\ &\geq \sum_{1\leq j\leq m}\Bigg(\frac{6}{t-1}|E(z_j,S)| + \big(\deg_{G-S}(z_j) - \frac{t-7}{t-1}\alpha(z_j) - \frac{\beta(z_j)}{3} - 6\big)|N_j|\Bigg), \end{aligned}$$

as desired. □

Claim 3.6. *Let* $1 \leq j \leq m$ *and suppose that* $\deg_{G-S}(z_j) - \frac{(t-7)\alpha(z_j)}{t-1} - \frac{\beta(z_j)}{3} - 6 < 0$. *Then* $p_j \geq r_j$.

Proof. Since $|N_j| \le \deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) + 1$ by Claim 3.2(iv) and $|E(z_j, S)| \ge 2t + 1 - \deg_{G-S}(z_j)$ by the assumption that $\delta(G) \ge 2t+1$, the desired inequality follows immediately. □

Claim 3.7. *For each* $1 \le j \le m, p_j \ge 0$.

Proof. We may assume $\deg_{G-S}(z_j) - \frac{(t-7)\alpha(z_j)}{t-1} - \frac{\beta(z_j)}{3} - 6 < 0$. Then $p_j \ge r_j$ by Claim 3.6. In view of (3.2), it follows from Lemma 2.5 that $r_j \ge 0$. Hence $p_j \ge 0$. □

Now it follows from Claims 3.5 and 3.7 that $\theta(S,T) \ge 0$, which contradicts the assumption that $\theta(S,T) < 0$. Thus Theorem 1.7(i) is proved.

4. Proof of Theorem 1.7(ii)

Now let t and G be as in Theorem 1.7(ii); thus $t = 3$ and G is a 2-connected $K_{1,3}$-free graph with $\delta(G) \ge 8$. Here we make a similar definition as in Section 3.

Let $C_1, ..., C_k$ be the components of $G - S - T$. We may assume that there exist a and b with $0 \le a + b \le k$ such that $|E(T, C_i)| = 1$ for each $1 \le i \le a$, $|E(T, C_i)| = 3$ for each $a+1 \le i \le a+b$, and $|E(T, C_i)| = 2$ or $|E(T, C_i)| \ge 4$ for each $a+b+1 \le i \le k$. Then the components $C_1, ..., C_{a+b}$ are odd components. We may further assume that there exists c with $0 \le c \le k - a - b$ such that C_i is an odd component for each $a+b+1 \le i \le a+b+c$ and C_i is an even component for each $a+b+c+1 \le i \le k$. Then $h(S,T) = a+b+c$. For each $y \in T$, set $\alpha(y) = \sum_{1 \le i \le a} |E(y, C_i)|$, $\beta(y) = \sum_{a+1 \le i \le a+b} |E(y, C_i)|$ and $\gamma(y) = \sum_{a+b+1 \le i \le k} |E(y, C_i)|$. Then $\alpha(y) + \beta(y) + \gamma(y) = |E(y, V(G) - S - T)|$; in particular,

$$\alpha(y) + \beta(y) + \gamma(y) \le \deg_{G-S}(y). \tag{4.11}$$

Moreover we obtain the following similar claim to Claim 3.1.

Claim 4.1.

(i) $a = \sum_{y \in T} \alpha(y)$.

(ii) $b = (\sum_{y \in T} \beta(y))/3$.

(iii) $c \le (\sum_{y \in T} \gamma(y))/5$.

Recall that $T \ne \emptyset$. We choose vertices $z_1, ..., z_m$ of T and define subsets $N_1, ..., N_m$ of T inductively by the following procedure in the similar way to the paragraph preceding the statement of Claim 3.2. First let $z_1 \in T$ be a vertex such that $\deg_{G-S}(z_1) + 2\alpha(z_1) - \frac{\beta(z_1)}{3} - \frac{\gamma(z_1)}{5} \le \deg_{G-S}(y) + 2\alpha(y) - \frac{\beta(y)}{3} - \frac{\gamma(y)}{5}$ for all $y \in T$, and set $N_1 = (N(z_1) \cap T) \cup \{z_1\}$. Now let $j \ge 2$, and assume that $z_1, ..., z_{j-1}$ and $N_1, ..., N_{j-1}$ have been defined. If $T - \left(\bigcup_{1 \le i \le j-1} N_i\right) \ne \emptyset$, then let $z_j \in T - \left(\bigcup_{1 \le i \le j-1} N_i\right)$ be a vertex such that $\deg_{G-S}(z_j) +$

$2\alpha(z_j) - \frac{\beta(z_j)}{3} - \frac{\gamma(z_j)}{5} \le \deg_{G-S}(y) + 2\alpha(y) - \frac{\beta(y)}{3} - \frac{\gamma(y)}{5}$ for all $y \in T - \left(\bigcup_{1\le i\le j-1} N_i\right)$, and set $N_j = \left(N(z_j) \cap \left(T - \left(\bigcup_{1\le i\le j-1} N_i\right)\right)\right) \cup \{z_j\}$; if $T - \left(\bigcup_{1\le i\le j-1} N_i\right) = \emptyset$, then let $m = j-1$ and terminate the procedure. Then, arguing as in the proof of Claims 3.2 and 3.4, we obtain the following two claims.

Claim 4.2.

(i) For each $1 \le j \le m, \deg_{G-S}(z_j) + 2\alpha(z_j) - \frac{\beta(z_j)}{3} - \frac{\gamma(z_j)}{5} \le \deg_{G-S}(y) + 2\alpha(y) - \frac{\beta(y)}{3} - \frac{\gamma(y)}{5}$ *for all* $y \in N_j$.

(ii) For each $1 \le j \le m,\ |N_j| \le \deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) - \gamma(z_j) + 1$.

Claim 4.3. $|S| \ge \dfrac{1}{t-1}\left(\displaystyle\sum_{1\le j\le m}\left(|E(z_j,S)| + \sum_{y\in N_j}\alpha(y)\right)\right)$.

We now estimate $\theta(S,T)$ from below by using the assumption that $\delta(G) \ge 8$. For each $1 \le j \le m$, set

$$\begin{aligned} p'_j &= 3|E(z_j,S)| + \left(\deg_{G-S}(z_j) + 2\alpha(z_j) - \frac{\beta(z_j)}{3} - \frac{\gamma(z_j)}{5} - 6\right)|N_j|, \\ r'_j &= 24 - 3\deg_{G-S}(z_j) \\ &\quad + \left(\deg_{G-S}(z_j) + 2\alpha(z_j) - \frac{\beta(z_j)}{3} - \frac{\gamma(z_j)}{5} - 6\right) \\ &\quad (\deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) - \gamma(z_j) + 1). \end{aligned}$$

Claim 4.4. $\theta(S,T) \ge \displaystyle\sum_{1\le j\le m} p'_j$.

Proof. Note that $h(S,T) = a+b+c \le \sum_{1\le j\le m}\sum_{y\in N_j}(\alpha(y) + \beta(y)/3 + \gamma(y)/5)$ by Claim 4.1. Then, by the similar argument in the proof of Claim 3.5, the desired inequality follows from Claim 4.2(ii) and Claim 4.3. □

Claim 4.5. *Let* $1 \le j \le m$, *and suppose that* $\deg_{G-S}(z_j) + 2\alpha(z_j) - \frac{\beta(z_j)}{3} - \frac{\gamma(z_j)}{5} - 6 < 0$. *Then* $p'_j \ge r'_j$.

Proof. Since $|N_j| \le \deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) - \gamma(z_j) + 1$ by Claim 4.2(ii) and $|E(z_j,S)| \ge 8 - \deg_{G-S}(z_j)$ by the assumption that $\delta(G) \ge 8$, the desired inequality follows immediately. □

Claim 4.6. *For each* $1 \le j \le m,\ p'_j \ge 0$.

Proof. We may assume that $\deg_{G-S}(z_j)+2\alpha(z_j)-\beta(z_j)/3-\gamma(z_j)/5-6<0$. Then $p'_j \geq r'_j$ by Claim 4.5. Thus we will show that $r'_j \geq 0$. In view of (4.1), it follows from Lemma 2.7 that $r'_j \geq 0$ unless $\deg_{G-S}(z_j)=8$, *and* $\beta(z_j)+\gamma(z_j)=7$ *or* $\beta(z_j)+\gamma(z_j)=8$. Thus we may assume that $\deg_{G-S}(z_j)=8, \alpha(z_j)=0$ and $\beta(z_j)+\gamma(z_j)=7$ or 8. Note that if $\deg_{G-S}(z_j)=8$ and $\alpha(z_j)=0$, then $r'_j=(8-\beta(z_j)/3-\gamma(z_j)/5-6)(8-\beta(z_j)-\gamma(z_j)+1)$.

Since G is $K_{1,3}$-free, the number of components $C_i(a+1\leq i\leq k)$ with $|E(z_j,C_i)|\neq\emptyset$ is at most 2. Thus from the definition of β, we have $\beta(z_j)\leq 6$, and hence $\gamma(z_j)\neq 0$. Once more taking into consideration the assumption that G is $K_{1,3}$-free, we have $\beta(z_j)\leq 3$. Therefore it suffices to examine separately the following eight cases;

(a) $\beta(z_j)=0$ *and* $\gamma(z_j)=7$,
(b) $\beta(z_j)=0$ *and* $\gamma(z_j)=8$,
(c) $\beta(z_j)=1$ *and* $\gamma(z_j)=6$,
(d) $\beta(z_j)=1$ *and* $\gamma(z_j)=7$,
(e) $\beta(z_j)=2$ *and* $\gamma(z_j)=5$,
(f) $\beta(z_j)=2$ *and* $\gamma(z_j)=6$,
(g) $\beta(z_j)=3$ *and* $\gamma(z_j)=4$,
(h) $\beta(z_j)=3$ *and* $\gamma(z_j)=5$.

In either case, it is immediate that $r'_j \geq 0$. Consequently we obtain the desired inequality. □

By Claims 4.4 and 4.6, $\theta(S,T)\geq 0$. This contradicts the assumption that $\theta(S,T)<0$. Therefore Theorem 1.7(ii) is proved.

5. Sharpness

In this section, we give the examples which show that Theorem 1.7 is best possible.

Example 5.1. *We construct examples which show that in Theorem 1.7(i), the lower bound $2t+1$ on $\delta(G)$ is best possible. Let $t\geq 4$ and set $r=2t-3$. We first define a graph I of order $r+4(t-1)$ by*

$$V(I)=\{v_i|1\leq i\leq r\}\cup\{w_j,x_j,y_j,z_j|1\leq j\leq t-1\},$$
$$E(I)=\{v_iv_j|1\leq i<j\leq\lfloor r/2\rfloor\}\cup\{v_iv_j|\lfloor r/2\rfloor+1\leq i<j\leq r\}$$
$$\cup\{w_jx_j,w_jy_j,w_jz_j,x_jy_j,x_jz_j,y_jz_j|1\leq j\leq t-1\}.$$

Thus I is the union of a complete graph of order $\lfloor r/2\rfloor$, a complete graph of order $\lceil r/2\rceil$, and $t-1$ complete graphs of order 4. Set $S_1=\{v_i|1\leq i\leq\lfloor r/2\rfloor\}$, $S_2=\{v_i|\lfloor r/2\rfloor+1\leq i\leq r\}$, $T_1=\{w_j,x_j,y_j,z_j|1\leq j\leq\lfloor(t-1)/2\rfloor\}$, $T_2=\{w_j,x_j,y_j,z_j|\lfloor(t-1)/2\rfloor+1\leq j\leq t-1\}$. Let H be the graph obtained from I by joining each vertex in S_1 to all vertices in $T_1\cup T_2$ and joining each vertex in S_2 to all vertices in T_2. Let $n\geq 1$, and let $H_1,\ldots,H_n$ be disjoint *copies of H. For each k ($1 \le k \le n$), let $S_{k,1}$, $S_{k,2}$, $T_{k,1}$ and $T_{k,2}$ denote the subsets of $V(H_k)$ which correspond to S_1, S_2, T_1 and T_2, respectively. Now let G be the graph obtained from the union of $H_1, ..., H_n$ by joining each vertex in $S_{k,2}$ to all vertices in $T_{k+1,1}$ for each $1 \le k \le n$ (we take $T_{n+1,1} = T_{1,1}$). Then G is 2-connected and $K_{1,t}$-free, and $\delta(G) = 2t$. However, we easily see that G has no 6-factor (for example, if we apply Lemma 2.1 in Section 2 with $S = \bigcup_{1 \le k \le n}(S_{k,1} \cup S_{k,2})$ and $T = \bigcup_{1 \le k \le n}(T_{k,1} \cup T_{k,2})$,then we get $\theta(S,T) = -6n$).*

Example 5.2. *We construct examples which show that in Theorem 1.7(ii), the lower bound 8 on $\delta(G)$ is best possible. We first define I of order 13 by*

$$V(I) = \{v_1, v_2, v_3\} \cup \{x_1, x_2, x_3, x_4, x_5\} \cup \{y_1, y_2, y_3, y_4, y_5\},$$
$$E(I) = \{v_2v_3\} \cup \{x_ix_j | 1 \le i < j \le 5\} \cup \{y_iy_j | 1 \le i < j \le 5\}.$$

Thus I is the union of a complete graph of order 1, a complete graph of order 2, and 2 complete graphs of order 5. Set $S_1 = \{v_1\}$, $S_2 = \{v_2, v_3\}$, $T_1 = \{x_1, x_2, x_3, x_4, x_5\}$, $T_2 = \{y_1, y_2, y_3, y_4, y_5\}$. Let H be the graph obtained from I by joining each vertex in S_1 to all vertices in $T_1 \cup T_2$ and joining each vertex in S_2 to all vertices in T_2. Let $n \ge 1$, and let $H_1, ..., H_n$ be disjoint copies of H. For each k ($1 \le k \le n$), let $S_{k,1}$, $S_{k,2}$, $T_{k,1}$ and $T_{k,2}$ denote the subsets of $V(H_k)$ which correspond to S_1, S_2, T_1 and T_2, respectively. Now let G be the graph obtained from the union of $H_1, ..., H_n$ by joining each vertex in $S_{k,2}$ to all vertices in $T_{k+1,1}$ for each $1 \le k \le n$ (we take $T_{n+1,1} = T_{1,1}$). Then G is 2-connected and $K_{1,3}$-free, and $\delta(G) = 7$. However, arguing as in Example(i), we easily see that G has no 6-factor.

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