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Mean Graphs

Christian Barrientos and Elliot Krop
Department of Mathematics
Clayton State University
2000 Clayton State Blvd, Morrow, GA 30260, USA
e-mail: *ChristianBarrientos@clayton.edu, ElliotKrop@clayton.edu*

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Abstract

Mean labelings were introduced in the early 2000's, in the context of additive vertex labelings. In this case, non-negative integers are assigned to the vertices of a graph in such a way that all edge-weights are different, where the weight of an edge is defined as the mean of the end-vertex labels rounded up to the nearest integer. Among our results, we give some conditions for the existence of such a labeling, investigate which regular graphs are mean graphs, connect α-labelings to mean graphs, and introduce α-mean labelings. We also show that all quadrilateral snakes are α-mean. Finally, we state some open questions.

Keywords: graph labeling, graceful labeling, mean labeling.

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1. Introduction

In 2003, Somasundaram and Ponraj [4] introduced the concept of mean labelings of graphs. A graph G with m vertices and n edges is called a *mean graph* if there is an injective function $f : V(G) \to \{0, 1, 2, \ldots, n\}$ such that when each edge uv is labeled with $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$, then the resulting edge-labels (or *weights*) are distinct. The above authors have proven in a series of papers ([4], [5], [6], [7], and [3]), that the following graphs are mean graphs: the path P_n, the cycle C_n, the complete bipartite graph $K_{2,n}$, the disjoint union $C_m \cup P_n$, the Cartesian products $P_m \times P_n$ and $P_m \times C_n$, the coronas $C_m \odot K_1$ and $P_m \odot K_1$, as well as triangular and some quadrilateral snakes; they also showed that the wheel W_n is not a mean graph for $n > 3$. The interested reader can find more about this subject in J.Gallian's survey [1].

In this paper we present several results on mean graphs. We start by counting the number of mean labeled graphs followed by two necessary conditions for the existence of a mean labeling. Section 3 is devoted to the study of mean labelings of some regular graphs; including the disjoint union of n triangles as well as the non-existence of a mean labeling when G is a r-regular graph of order n and $r \in \{n-1, n-2\}$. In section 4 we present a relationship between α-labelings and mean labelings. One of the reasons this section is relevant is because we connect two labelings that induce edge labelings associated with two distinct operations as addition (mean labeling) and subtraction (α-labeling). The main result of this section shows how to transform an α-labeling into a mean labeling. The following section comes naturally after the introduction of α-labelings. In that section we introduce the concept of α-mean labelings, present some families of α-mean graphs, but mainly use α-mean graphs to construct new α-mean graphs using two distinct methods. In section 6 we completely solve the problem of α-mean labelings of quadrilateral snakes.

2. Basic Properties

Let G be a graph of size n. Let (i, j) denote the corresponding labels of the end vertices of the edge xy of G under the mean labeling f, that is, $f(x) = i$ and $f(y) = j$. Without loss of generality, we may assume that $i > j$. Thus, $1 \leq i \leq n$ and $0 \leq j \leq n-1$.

For any number $k \in [1, n]$ we want to count the number of possible pairs (i, j) such that $\left\lceil \frac{i+j}{2} \right\rceil = k$. When $k \leq \left\lfloor \frac{n}{2} \right\rfloor$, we claim that there are $2k$ pairs (i, j) such that $\left\lceil \frac{i+j}{2} \right\rceil = k$. In fact, these pairs form the set

$$\{(k+r, k-s) : 0 \leq r \leq k, 1 \leq s \leq k, \text{ and } s - r = 0, 1\}.$$

Observe that when $r = 0$, $s = 1$, and when $r = k$, $s = k$. Otherwise, for each value of r, $s = r$ or $s = r + 1$. Therefore, there are $2k$ pairs (i, j) such that $\left\lceil \frac{i+j}{2} \right\rceil = k$.

If $k > \left\lfloor \frac{n}{2} \right\rfloor$, we claim that there are $2(n-k)+1$ pairs (i, j) such that $\left\lceil \frac{i+j}{2} \right\rceil = k$. In fact, in this case, the pairs form the set

$$\{(k+r, k-s) : 0 \leq r \leq n-k, 1 \leq s \leq k-1, \text{ and } s - r = 0, 1\}.$$

Thus, when $r = 0$, $s = 1$, otherwise, for each value of r, $s = r$ or $s = r + 1$. Then, there are $2(n-k)+1$ pairs (i, j) such that $\left\lceil \frac{i+j}{2} \right\rceil = k$.

Given that under a mean labeling, for each value of $k \in [1, n]$, there are $2k$ or $2(n-k)+1$ pairs (i, j) and

$$\{2k : 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor\} \cup \{2(n-k)+1 : \left\lceil \frac{n+1}{2} \right\rceil \leq k \leq n\} = [1, n],$$

we conclude that there are $n!$ graphs of size n which are mean labeled. Thus, we have proven the following proposition:

Proposition 2.1. *There exist $n!$ mean labeled graphs of size n.*

Now we show two necessary conditions for a mean graph. The first one establishes the relationship between the order and the size of the graph. The second one presents a constraint on the maximum degree.

Proposition 2.2. *Let G be a graph of order m and size n. If G is a mean graph, then $n+1 \geq m$.*

Proof. From the definition of mean labeling, we know that vertex labels are taken from the set $[0, n]$ in and an injective form, that is, without repetition. Therefore, $|V(G)| \leq m+1$. □

As a consequence of this condition we know that non-trivial forests are not mean graphs.

Proposition 2.3. *If G is a mean graph of size n, then $\Delta(G) \leq \frac{n+3}{2}$ when n is odd and $\Delta(G) \leq \frac{n+2}{2}$ when n is even.*

Proof. Let f be a mean labeling of G. Let r be a fixed integer in $[1, n]$, such that $f(v) = r$ for some $v \in V(G)$. If $uv \in E(G)$ and $f(u) = i$, then the weight of uv is in the set

$$W = \left\{ \left\lceil \frac{r+i}{2} \right\rceil : i \in [0, n], r \neq i \right\} = \left[\left\lceil \frac{r}{2} \right\rceil, \left\lceil \frac{r+n}{2} \right\rceil \right].$$

Since

$$|W| = \begin{cases} \frac{n+3}{2}, & \text{if } n \text{ is odd and } r \text{ is even,} \\ \left\lceil \frac{n+1}{2} \right\rceil, & \text{otherwise,} \end{cases}$$

the number of vertices adjacent to v is bounded by $|W|$. Therefore, $\Delta(G) \leq \frac{n+3}{2}$ when n is odd and $\Delta(G) \leq \left\lceil \frac{n+1}{2} \right\rceil$ when n is even. □

Using this proposition, we can prove that the friendship graph (or Dutch t-windmill) C_3^t, that is, the one-point union of t cycles of length 3, is a mean graph if and only if $t \leq 2$.

Conjecture 2.4. *All trees of size n with $\Delta \leq \left\lceil \frac{n+1}{2} \right\rceil$ are mean graphs.*

3. Regular Graphs

We devote this section to the study of mean labelings of regular graphs, beginning with a subclass of 2-regular graphs, and finish with the analysis of $(n-1)$ and $(n-2)$-regular graphs of order n.

Proposition 3.1. *The disjoint union of n triangles, nC_3, is a mean graph.*

Proof. Let $v_{i,1}$, $v_{i,2}$, and $v_{i,3}$ be the vertices of the i^{th} copy of C_3. For $1 \le i \le n$, consider the labeling f of the vertices of nC_3 defined by

$$f(v_{i,j}) = \begin{cases} 3i-3, & \text{if } 1 \le i \le n \text{ and } j = 1 \\ 3i-1, & \text{if } 1 \le i \le n \text{ and } j = 2 \\ 3i+1, & \text{if } 1 \le i \le n-1 \text{ and } j = 3 \\ 3i, & \text{if } i = n \text{ and } j = 3 \end{cases}$$

We claim that f is a mean labeling of nC_3. Clearly, all labels used are different and taken from the set $[0, 3n]$. Hence, we just need to verify that the weights induced by f are also different.

For $1 \le i \le n$, the edge $v_{i,1}v_{i,2}$ has weight $3i-2$. For $1 \le i \le n-1$, the edge $v_{i,1}v_{i,3}$ has weight $3i-1$, while the edge $v_{i,2}v_{i,3}$ has weight $3i$. The edge $v_{n,1}v_{n,3}$ has weight $3n$. Thus, the weights induced by f are $1, 2, 3, \ldots, 3n$. Therefore, nC_3 is a mean graph. □

In the following figure we apply the previous proposition for $4C_3$.

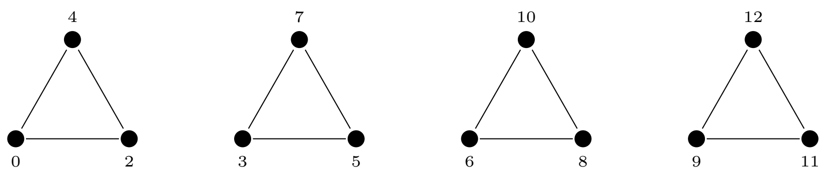


Figure 1: Mean labeling of $4C_3$

Question 1. *Are all* 2*-regular graphs mean graphs?*

In the following figure we show some 2-regular mean graphs of the form $C_3 \cup C_n$.

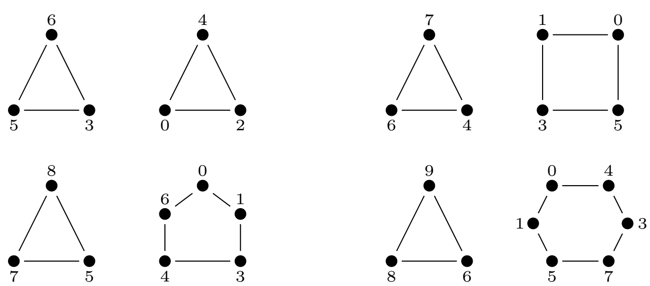


Figure 2: 2-regular mean graphs

We also show a mean labeling for $C_4 \cup C_4$.

Figure 3: Mean labeling of $2C_4$

Next we consider the following question:

Question 2. *For any integers $n > 0$, is there a largest integer r, $1 \leq r \leq n-1$, so that all r-regular graphs of order n are mean graphs?*

If the answer is in the affirmative, then we are obliged to ask:

Question 3. *What is this integer r in terms of n?*

We begin with the largest regularity. Although it is known that complete graphs are not mean, we provide the proof for completeness, and to introduce the more general argument.

For ease of notation, we refer to a vertex labeled with label l by x_l or an l-vertex. An edge labeled by label l will be called e_l or an l-edge. We denote adjacency between vertices u and v by $u \sim v$.

Proposition 3.2. *For any positive integer $n \geq 4$, K_n is not a mean graph.*

Proof. Suppose K_n is a mean graph with labeled vertex set V and labeled edge set E. Notice that e_1 can only be obtained by the vertex-label pairs $(0, 1)$ and $(0, 2)$ which means that $x_0 \in V$. Let $m = \binom{n}{2}$ be the size of G and suppose m is even. Note that e_m can only be obtained by the vertex-label pairs $(m, m-1)$ which means that $x_m, x_{m-1} \in V$. However, $x_0 x_m = e_{\lceil \frac{m}{2} \rceil}$ and $x_0 x_{m-1} = e_{\lceil \frac{m-1}{2} \rceil}$ which are identical edge-labels and a contradiction.

Suppose next that m is odd. If $x_1 \in V$, then $x_1 x_m = e_{\lceil \frac{m+1}{2} \rceil}$ and $x_1 x_{m-1} = e_{\lceil \frac{m}{2} \rceil}$ which is a contradiction. To form e_1 we must have $x_0, x_2 \in V$. However, $x_0 x_m = e_{\lceil \frac{m}{2} \rceil}$ and $x_2 x_{m-1} = e_{\lceil \frac{m+1}{2} \rceil}$ which is a contradiction. This exhausts all possibilities and therefore K_n cannot be a mean graph. □

To answer questions 2 and 3 we consider $(n-2)$-regular graphs.

Theorem 3.3. *If G is a $(n-2)$-regular graph of order n and size m, then G is not a mean graph.*

Proof. Suppose the proposed graph G is a mean graph. We notice that since $m = \frac{(n-2)n}{2}$, m is even. Since the edge e_m is only obtained from the vertex-label pair $(m, m-1)$ we conclude that $x_m, x_{m-1} \in V$. Let $S = N[x_m] \cap N[x_{m-1}]$. If $v \in S$ and the label $f(v)$ is even, then vx_m has the same label as vx_{m-1} which is a contradiction. Thus, the label on every $v \in V$ is odd. If S contains x_{j-1}, x_j for any $j \in [1, m]$, then G is not mean. Furthermore, for such j, $f(x_m x_j) = f(x_{m-1} x_{j+2})$, so S does not contain terms x_j, x_{j+2} for any $j \in [0, m-2]$. We summarize this as follows:

$$\text{For any } u, v \in S, |f(u) - f(v)| \geq 3 \tag{3.1}$$

Note that $|S| = n-3$ or $n-4$. In the first case, notice that $x_1 \in S$ since $x_0 x_1$ is the only way to make e_1. Likewise, e_2 is obtained from the vertex-label pairs $(1, 3)$ and $(1, 2)$. If $x_3 \in S$, then notice $f(x_m x_1) = f(x_{m-1} x_3)$. Since $x_2 \notin S$, this leads to a contradiction.

Suppose $|S| = n-4$. Let $u \sim x_{m-1}, u \not\sim x_m$ and $v \sim x_m, v \not\sim x_{m-1}$. Notice that one of u or v is x_0. If $x_1 \in S$, then to form e_2, $x_3 \in S$, which hs a contradiction by 3.1. If $\{u, v\} = \{x_0, x_1\}$, then to form e_2, $x_3 \in S$. Note that e_3 can be obtained from the vertex-label pairs $(0, 6), (1, 5), (1, 4), (2, 4), (2, 3)$. Since S contains only odd-labeled vertices, only $(1, 5)$ can be realized, and so $x_5 \in S$. This contradicts 3.1.

We have shown that $x_1 \notin V$, so to form e_1, $x_2 \in V$. Thus, $\{u, v\} = \{x_0, x_2\}$. To form e_2, $x_3 \in S$. However, $x_{m-1} x_3 = e_{\lceil \frac{m+2}{2} \rceil}$ and $x_{m-1} x_2 = e_{\lceil \frac{m+1}{2} \rceil}$ which are identical edge-labels, so $u = x_0$ and $v = x_2$. Again, $x_{m-1} x_3 = e_{\lceil \frac{m+2}{2} \rceil}$ and $x_m x_2 = e_{\lceil \frac{m+2}{2} \rceil}$, which contradicts that G is a mean graph. □

4. Alpha-labelings and mean graphs

In this section we present a connection between mean labelings and the special kind of graceful labeling known as α-labeling. This result connects two labelings of different natures. The first type assigns a weight obtained by adding the end-vertex labels to each edge while the other one takes the absolute difference between them. Given that graceful labelings have been studied for more than 40 years, many families of α-graphs are known, which enlarges the number of mean graph families.

Recall that a graph G of order m and size n is a *graceful* graph if there exists an injective function $f : V(G) \to \{0, 1, \dots, n\}$ such that when the edge $e = uv$ is assigned the weight $|f(u) - f(v)|$, the set of all weights is $\{1, 2, \dots, n\}$. In addition, if there is an integer λ such that $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$ for every edge uv in G, then f is said to be and α-labeling and G is an α-graph. The number λ is named the *boundary value* of f. Notice that in order to be an α-graph, G must be a bipartite graph.

Let f be an α-labeling of a bipartite graph G of size n with bipartition $\{A, B\}$ and boundary value λ. Suppose that $|A| \geq |B|$. There exists a vertex $v \in G$ such that $f(v) = \lambda$. If $v \in B$, then the complementary labeling of f, defined as $\bar{f}(u) = n - f(u)$ for every $u \in V(G)$, assigns its boundary value on a vertex of A. Therefore, there always exists an α-labeling of G such that its boundary value is assigned to a vertex of the largest of the bipartite sets.

Proposition 4.1. *Every α-tree is a mean tree when the absolute difference of the cardinalities of its bipartite sets is at most* 1*.*

Proof. Let T be an α-tree of size n with bipartite sets A and B, where $\big||A|-|B|\big| \leq 1$. Without loss of generality, we may assume that $|A| \geq |B|$. Let f be an α-labeling of T that assigns its boundary value λ on a vertex of A. Let g be a new labeling of T defined as:

$$g(v) = \begin{cases} 2f(v), & \text{if } f(v) \leq \lambda \\ 2n+1-2f(v), & \text{if } f(v) > \lambda. \end{cases}$$

We claim that g is a mean labeling of T. In fact, the labels assigned on the vertices of A are $0, 2, 4, \ldots, 2\lambda$, and on the vertices of B are $1, 3, 5, \ldots, 2(n-\lambda)-1$. When n is even, $\lambda = \frac{n}{2}$, thus the labels assigned on A and B are $0, 2, 4, \ldots, n-1$ and $1, 3, 5, \ldots, n$, respectively.

Now we want to prove that the induced weights are $1, 2, \ldots, n$. Let $e = uv \in E(T)$ such that $u \in A$ and $v \in B$. Thus,

$$\begin{aligned} \left\lceil \frac{g(u)+g(v)}{2} \right\rceil &= \left\lceil \frac{2f(u)+2n+1-2f(v)}{2} \right\rceil \\ &= \left\lceil \frac{2n-2(f(v)-f(u))+1}{2} \right\rceil \\ &= n+1-(f(v)-f(u)). \end{aligned}$$

Since f is an α-labeling, $\{f(v)-f(u) : uv \in E(T)\} = \{1, 2, \ldots, n\}$. Whence, the set of weights induced on the edges of T is $\{1, 2, \ldots, n\}$. Therefore, g is a mean labeling of T. □

In the following figure we show an example of this transformation, from α to mean labeling, on a tree of size 10 with boundary value $\lambda = 5$.

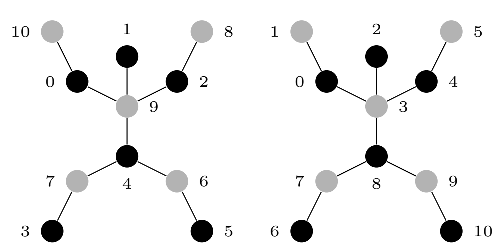


Figure 4: α and α-mean labelings of a tree

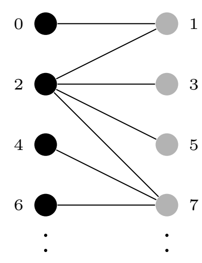
As a consequence of this result, we conclude that all caterpillars which satisfy the above condition on their bipartite sets are mean graphs. An example would be the path P_n and the corona $P_n \odot K_1$. Moreover, the corona $P_n \odot mK_1$ is a mean graph when P_n has odd length. Also in this category, we can find the path-like trees.

The following result about caterpillars is a little bit stronger because it includes the case where $|A| - |B| = 2$.

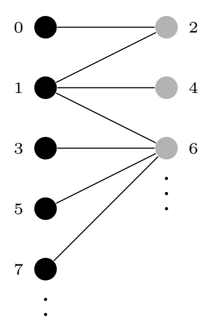
Proposition 4.2. *Let T be a caterpillar of size n. If the absolute difference of the cardinalities of its bipartite sets is at most 2, then T is a mean graph.*

Proof. We show two cases and exhibit the labelings in both.

Case 1. Suppose the cardinalities of the bipartite sets are the same.



Case 2. Suppose the cardinalities of the bipartite sets are not the same.



□

5. Alpha-mean labelings

In this section we introduce the concept of α-mean labeling. We study some properties of these labelings and use them to construct new mean graphs starting with some α-mean graphs.

Suppose that f is a mean labeling of a bipartite graph with the property that the labels assigned to vertices of the same color have the same parity. In this case, we say that f is an *α-mean labeling.* The following figure shows an α-mean labeling of the complete bipartite graph $K_{2,n}$.

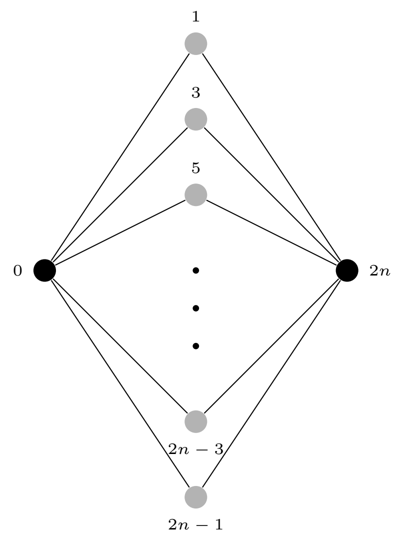


Figure 5: α-mean labeling of $K_{2,n}$

Notice that the labelings obtained in Proposition 4.1 are α-mean labelings. Thus, all caterpillars of size n, with bipartition $\{A, B\}$ where $||A| - |B|| \leq 1$, are α-mean graphs.

Let f be a labeling of a graph G of size n. The *complementary* labeling of f is defined as

$$\bar{f}(v) = n - f(v)$$

for every $v \in V(G)$. In the next proposition we explore the properties of $\bar{f}$ when f is an α-mean labeling. First, notice that if f is a mean labeling that is not α-mean, its complementary labeling is not a mean labeling. We show an example of this fact in the next figure. The graph on the left has a mean labeling, and the graph on the right has its complementary labeling which is not a mean labeling. Edges $1-2$ and $0-4$ have the same weight.

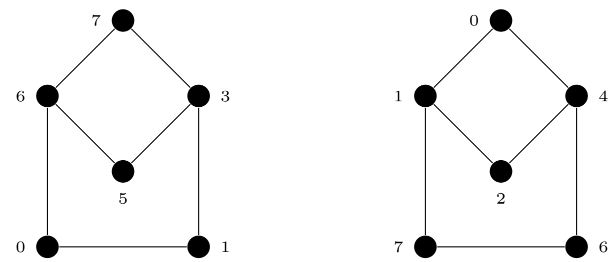


Figure 6: A complementary labeling that is not a mean labeling

Proposition 5.1. *If f is an α-mean labeling of a bipartite graph G, then its complementary labeling is also an α-mean labeling of G.*

Proof. Let f be an α-mean labeling of a bipartite graph G of size n, and let $\bar{f}$ be its complementary labeling. Given that $f(v) \in [0, n]$ for every $v \in V(G)$, we know that $\{\bar{f}(v) : v \in V(G)\} = [0, n]$.

Let $uv \in E(G)$ such that $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil = k$. Since $f$is an α-mean labeling, $f(u)$ and $f(v)$ have different parity, which implies that $\bar{f}(u)$ and $\bar{f}(v)$ also have different parity.

Thus,

$$\left\lceil \frac{\bar{f}(u)+\bar{f}(v)}{2} \right\rceil = \frac{\bar{f}(u)+\bar{f}(v)+1}{2} = \frac{n-f(u)+n-f(v)+1}{2} = n+1-k.$$

Given that k takes all the values in the set $[1, n]$, $n+1-k$ also takes all the values in the set $[1, n]$. Therefore, $\bar{f}$ is an α-mean labeling of G. □

Now we use graphs that accept an α-mean labeling to make new mean graphs.

Proposition 5.2. *Let H be a graph obtained from two copies of a graph G by connecting a new edge to any pair of corresponding vertices. If G has an α-mean labeling, then H also has an α-mean labeling.*

Proof. Let G_1 and G_2 be two copies of an α-mean graph G of size n. Let f be an α-mean labeling of G. Apply this labeling to the copy G_1; thus the labels assigned are in the set $[0, n]$ and the induced weights form the set $[1, n]$. On the copy G_2 apply the complementary labeling $\bar{f}$, and add to each vertex label the constant $n+1$. Thus, the labels assigned are in the set $[n+1, 2n+1]$ and the induced weights form the set $[n+2, 2n+1]$.

Let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ be a pair of corresponding vertices. Their labels are $f(v_1)$ and $\bar{f}(v_2)+n+1 = 2n+1-f(v_1)$ because $\bar{f}(v_2) = n - f(v_2)$ and $f(v_2) = f(v_1)$.

To create the graph H we connect these two vertices with an edge that has weight

$$\left\lceil \frac{f(v_1) + 2n + 1 - f(v_1)}{2} \right\rceil = n + 1.$$

Therefore, the labeling of H assigns labels injectively, from $[0, 2n + 1]$ and induces the weights $1, 2, \ldots, 2n + 1$. Hence, H is a mean graph. Notice that in fact, H is an α-mean graph because vertices v_1 and v_2 have different parity. This concludes the proof. □

In the next figure we show an example of this construction. Any of the dashed lines can be used to create the edge v_1v_2 that produces the α-mean graph H.

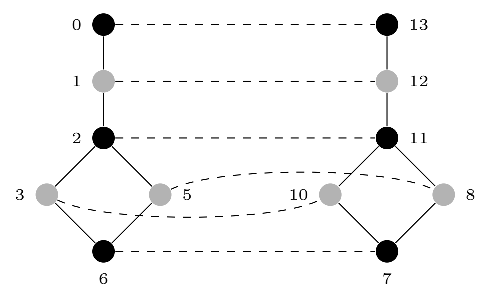


Figure 7: An α-mean graph H

Let G_1 and G_2 be two graphs, we say that the graph G is an *amalgamation* of G_1 and G_2 if G is obtained by identifying one vertex of G_1 with one vertex of G_2.

Proposition 5.3. *If G_1 and G_2 are two mean graphs, then there exists a mean graph H that is obtained by an amalgamation of G_1 and G_2.*

Proof. For $i = 1, 2$, let f_i be a mean labeling of a graph G_i of size n_i. If f_2 is shifted by adding the constant n_1 to each vertex label, then the labels assigned on G_2 are in the set $[n_1, n_1 + n_2]$ and the induced weights form the set $[n_1 + 1, n_1 + n_2]$. Notice that the vertex v of G_2 that was originally labeled 0 is now labeled n_1. Thus, the label n_1 has been used twice, once on a vertex of G_1 and once on a vertex of G_2. Identifying these vertices we obtain a mean graph H. □

Suppose that f_1 and f_2 are α-mean labelings. Since the shift of f_2 does not modify the fact that vertices of different colors have labels with different parity, we can say that the labeling of H is an α-mean labeling. Thus, we have proven the following corollary:

Corollary 5.4. *If G_1 and G_2 are two α-mean graphs, then there exists an α-mean graph H that is obtained by an amalgamation of G_1 and G_2.*

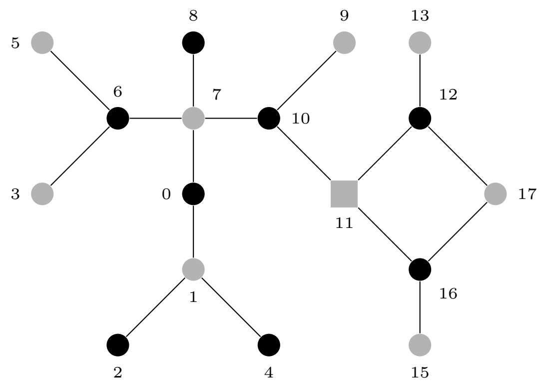


Figure 8: amalgamation of α-mean graphs

In the next figure we show an example of this construction. The vertex used for the amalgamation is highlighted.

6. Quadrilateral Snakes

Quadrilateral snakes are obtained by the amalgamation of copies of the cycle C_4 in such a way that the block-cut point graph is a path. In this paper, we denote the family of quadrilateral snakes with k blocks by kC_4-snakes. Notice that the graphs in this family are bipartite of order $4 + 3(k + 1)$ and size $4k$. If we substitute cycles C_n for C_4 in the above definition, one can consider the mean labeling of kC_n snakes. This problem was completely solved in [2]. We consider the problem of α-mean labeling. When k is 1 or 2 there exists only one quadrilateral snake and the mean labelings are shown below.

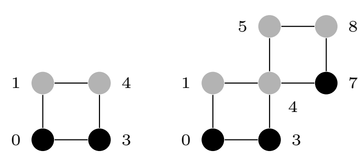


Figure 9: amalgamation of α-mean graphs

For a fixed value of $k > 2$, there are $2^{k-3} + 2^{\lfloor \frac{k-1}{2} \rfloor} - 2^{\lfloor \frac{k-3}{2} \rfloor}$ non-isomorphic quadrilateral snakes. In order to distinguish between them, we introduce the string of numbers $(d_1, d_2, \ldots, d_{k-2})$, where $d_i = 1, 2$ and d_i represents the distance between the cut-vertices of the $(i+2)^{\text{nd}}$ block. In the figure below, we show all non-isomorphic quadrilateral snakes with three blocks, together with their corresponding strings.

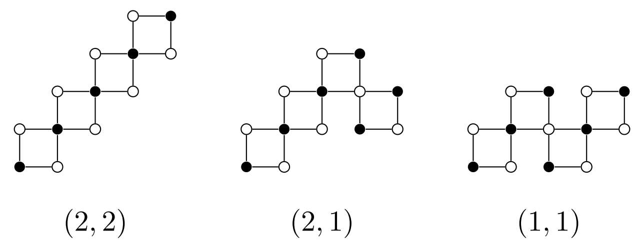


Figure 10: All quadrilateral snakes with four blocks

The mean labeling of the quadrilateral snake with string $(2, 2, \ldots, 2)$ can be obtained using the result of the last corollary. The general case is solved in the next proposition.

Proposition 6.1. *All quadrialteral snakes are α-mean graphs.*

Proof. First, consider the following labeling models of C_4. In both cases, the induced weights are consecutive integers.

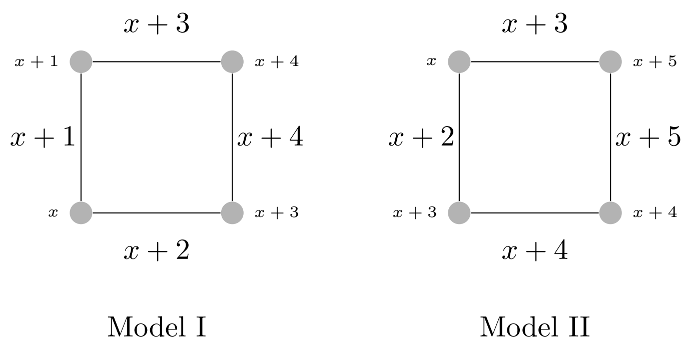


Figure 11: Models I and II of C_4

Let G be the quadrilateral snake with k blocks and string $(d_1, d_2, \ldots, d_{k-2})$. We label the first two blocks as shown before. To label the third block we proceed as follows. If $d_1 = 2$, use Model I with $x = 8$. If $d_1 = 1$, use Model II with $x = 7$. In both cases the induced weights are $9, 10, 11$, and 12. So far, we have obtained the weights $1, 2, \ldots, 12$ using the labels from $\{0, 1, 2, \ldots, 12\}$.

Let $2 \le i \le k-2$. If $d_i = d_{i-1}$, the $(i+2)^{\text{nd}}$ block is labeled using the same model used in the previous block with $x = 4i+4$ when Model I is used and $x = 4i+3$ when Model II is used. If $d_i \ne d_{i-1}$, the $(i+2)^{\text{nd}}$ block is labeled using the model different from the one used in the previous block. Once again, $x = 4i+4$ when Model I is used and $x = 4i+3$, otherwise.

Independently of model used, the induced weights are $4i+5, 4i+6, 4i+7$, and $4i+8$. Since $2 \le i \le k-2$, the induced weights, from the fourth block until the last one, are

13, 14, ..., $4k$. Thus, G has been labeled using numbers from the set $[0, 4k]$, with induced weights $1, 2, \ldots, 4k$. Moreover, since vertices of different color have labels with different parity, the labeling on G is an α-mean labeling. □

In the following figure we show the α-mean labeling of the quadrilateral snake with seven blocks and string $(2, 1, 2, 1, 2)$ obtained by this method.

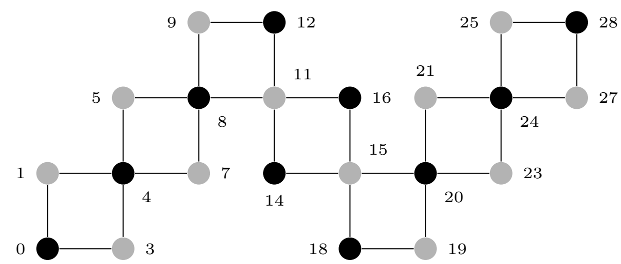


Figure 12: α-mean quadrilateral snake

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