AKCE Int. J. Graphs Comb., 11, No. 1 (2014), pp. 27-49

Even Harmonious Graphs

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Communicated by: S. Arumugam
Received 30 April 2013; accepted 28 May 2013

Abstract

A graph G with q edges is said to be *harmonious* if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices.

Recently two variants of harmonious labelings have been defined. A function f is said to be an *odd harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q-1$ such that the induced mapping $f^*(uv) = f(u) + f(v)$ from the edges of G to the odd integers between 1 to $2q-1$ is a bijection. A function f is said to be an *even harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q$ and the induced function f^* from the edges of G to $\{0, 2, \ldots, 2(q-1)\}$ defined by $f^*(uv) = f(u) + f(v) \pmod{2q}$ is bijective.

In this paper we investigate the existence of even harmonious labelings for a number of common graph families. Special attention is given to disconnected graphs. We state some conjectures and open problems.

Keywords: even harmonious labelings, harmonious labelings, graph labelings.

2010 Mathematics Subject Classification: 05C78.

1. Introduction

A vertex *labeling* of a graph G is an assignment f of labels to the vertices of G that induces for each edge xy label depending on the vertex labels $f(x)$ and $f(y)$. Many graph labeling methods can be traced back to Rosa [8] in 1967 and to Graham and Sloane [3] in 1980. Harmonious graphs naturally arose in the study of error-correcting codes and channel assignment problems. Since then more than 50 papers have been published on harmonious labelings and a half dozen or so on odd harmonious labelings have appeared (see [2]). Only one paper has been published on even harmonious labelings [9]. We introduce two special kinds of even harmonious labelings, discuss the relationship between harmonious labelings and even harmonious labelings, state some conjectures, and identify several specific families as good candidates for further investigation. An extensive survey of graph labeling methods is available online at [2].

2. Preliminaries

Definition 2.1. *A graph G with q edges is said to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. When G is a tree exactly one label may be used on two vertices.*

Definition 2.2. *A function f is said to be an odd harmonious labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q - 1$ such that the induced mapping $f^*(uv) = f(u) + f(v)$ from the edges of G to the odd integers between 1 to $2q - 1$ is a bijection.*

Liang and Bai [6] have shown the following: odd harmonious graphs are bipartite; P_n $(n > 1)$ is odd harmonious; C_n is odd harmonious if and only if $n \equiv 0 \bmod 4$; K_n is odd harmonious if and only if $n = 2$; $K_{n_1,n_2,\ldots,n_k}$ is odd harmonious if and only if $k = 2$; K_n^t is odd harmonious if and only if $n = 2$; $P_m \times P_n$ is odd harmonious; the graph obtained by identifying the endpoint of a path with a vertex of an n-cycle is odd harmonious if $n \equiv 0 \bmod 4$; the graph obtained by appending two or more pendant edges to each vertex of C_{4n} is odd harmonious; the graph obtained by subdividing every edge of the cycle of a wheel is odd harmonious; and caterpillars and lobsters are odd harmonious. Yan [11] proved that $P_m \times P_n$ is odd harmonious. Li, Li, and Yan [5] proved that $K_{m,n}$ is odd harmonious.

Definition 2.3. *A function f is said to be an even harmonious labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q$ and the induced function f^* from the edges of G to $\{0, 2, \ldots, 2(q-1)\}$ defined by $f^*(uv) = f(u) + f(v) \pmod{2q}$ is bijective. See Figure 1.*

Sarasija and Binthiya [9] proved the following graphs are even harmonious: non-trivial paths; complete bipartite graphs; odd cycles; bistars $B_{m,n}$; $K_2+\overline{K_n}$; P_n^2; and the friendship graphs F_{2n+1}. López, Muntaner-Batle and Rius-Font [7] proved that every graph with p vertices and q edges where $q \geq p - 1$ that has a super edge-magic total labeling also has an even harmonious labeling. (A *super edge-magic total labeling* of a graph $G(V, E)$ is a bijection f from $V \cup E$ to $\{1, 2, \ldots, |V \cup E|\}$ such that for all edges xy, $f(x)+f(y)+f(xy)$ is constant and the vertex labels are 1 to $|V|$).

Because $2q$ is 0 modulo $2q$ we find it more convenient to introduce the following equivalent definition of an even harmonious labeling (see Theorem 2.7).

Definition 2.4. *A function f is said to be an even harmonious labeling of a graph G with q edges if f is a function from the vertices of G to the integers from 0 to $2q-2$ and the induced function f^* from the edges of G to $\{0, 2, \ldots, 2(q-1)\}$ defined by $f^*(uv) = f(u) + f(v) \pmod{2q}$ has at most one label used twice.*

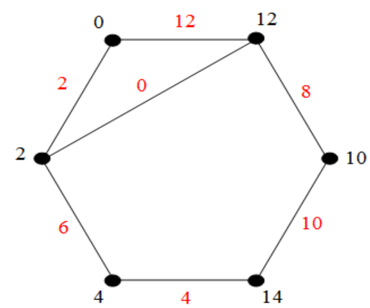

Figure 1: Even harmonious graph C_6 with a cord mod 14

Definition 2.5. *An even harmonious labeling of a graph with q edges is called a properly even harmonious labeling if no vertex label is duplicated.*

Definition 2.6. *A graph that has a (properly) even harmonious labeling is called a (properly) even harmonious graph.*

The following theorem will be used frequently. It follows from the fact that in a cyclic group of odd order the sum of the elements is 0 whereas in a non-trivial cyclic group of even order the sum of the elements is the unique element of order 2.

Theorem 2.7. *For any even harmonious labeling of a graph with q edges, the sum of the edge labels* mod $2q$ *is* 0 *when q is odd and the sum is q when q is even.*

Proof. When q is odd the nonzero edge labels are $2, 4, \ldots, 2q-4, 2q-2$. Rearrange them as $2, 2q-2, 4, 2q-4, \ldots, q-1, q+1$. Observing that each successive pair sums to $0 \mod 2q$, we have the sum 0.

When q is even the nonzero edge labels are $2, 4, \ldots, q-2, q, q+2, \ldots, 2q-4, 2q-2$. Rearranging them as $2, 2q-2, 4, 2q-4, \ldots, q-2, q+2, q$. Notice the consecutive pairs up to the last term sum to $0 \mod 2q$. So, the sum is q when q is even. □

The following theorem shows that in an even harmonious labeling with duplicate vertices the duplicate label can be modified as we wish. It also shows that we can change the parity of the vertex labels from even to odd and vice versa.

Theorem 2.8. *If f is a (properly) even harmonious labeling for a graph G with q edges then for any unit a in Z_{2q} and any b in Z_{2q} the labeling $f^*(v) = af(v) + b$ (viewed as an element of Z_{2q}) is also a (properly) even harmonious labeling of G.*

Proof. Let $v_1, v_2, \ldots, v_m$ be the vertices of G. Then the vertex labels of f^* are $af(v_1) + b, af(v_2) + b, \ldots, af(v_m) + b$. Observe that $f^*(v_i) = af(v_i) + b = af(v_j) + b = f^*(v_j)$ if and only if $f(v_i) = f(v_j)$. To see that the edge labels induced by f^* are distinct, observe that because a is a unit $f^*(v_i) + f^*(v_j) = af(v_i) + b + af(v_j) + b = a(f(v_i) + f(v_j)) + 2b$ are distinct when the terms $f(v_i) + f(v_j)$ are distinct. □

Theorem 2.8 gives the following useful corollaries.

Corollary 2.9. *In any even harmonious graph we may assume that the duplicate label is* 0.

Corollary 2.10. *In any connected even harmonious label we may assume the vertex labels are even.*

Jared Bass [1] has observed that for connected graphs any harmonious labeling of a graph with q edges yields an even harmonious labeling by simply multiplying each vertex label by 2 and adding the vertex labels modulo $2q$. Conversely, by Corollary 2.10 of Theorem 2.8, a properly even harmonious labeling of a connected graph with q edges yields a harmonious labeling of the graph by dividing each vertex label by 2 and adding the vertex labels modulo q. (See Figure 2.) Thus we know that every connected harmonious graph is an even harmonious graph and every connected graph that has a harmonious labeling and is not a tree also has a properly even harmonious labeling.

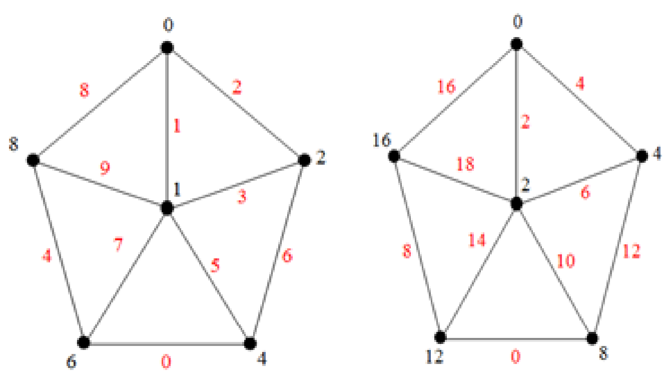

Figure 2: A harmonious graph mod 10 equivalent to a properly even harmonious graph mod 20

3. Connected Graphs

Theorem 3.1. *A tree (or forest) cannot have a properly even harmonious labeling.*

Proof. Observe that a tree with n vertices has $n-1$ edges and is connected. In order for this to be a properly even harmonious labeling there cannot be a repeat of any vertex label. However, this is impossible since there are n vertices with only $n-1$ edges. Therefore, a tree (or forest) cannot have a properly even harmonious labeling since we will need to use a duplicate of some number. □

Theorem 3.2. *The wheel, $W_n = C_n + K_1$, is properly even harmonious when n is odd.*

Proof. Since W_n has $2n$ edges the modulus is $4n$. Label the center vertex v_0 and label the consecutive cycle vertices $v_1, v_2, ..., v_n$. Label the vertex $v_0 = 2$. For $i \geq 1$, let $v_i = 4(i-1)$. See Figure 2. Since these labels are strictly increasing and less than $4n$ they are distinct.

To verify that the edge labels are distinct observe that the cycle edges for W_n have the form $4(i-1)+4i = 8i-4$. So if the edges $v_i v_{i+1}$ and $v_j v_{j+1}$ are equal we have $8i-4 = 8j-4$ mod $4n$. Thus $8(i-j) = k4n$ or $2(i-j) = kn$ for some k. Without loss of generality, we may assume $i > j$. Then $0 < i-j < n$ and $0 < kn = 2(i-j) < 2n$. Then $k = 1$ and $2(i-j) = n$. Since n is odd we have a contradiction.

Since the spoke labels have the form $2 + v_i$ we have $2 + v_i \neq 2 + v_j$ whenever $v_i \neq v_j$. So the spoke labels are distinct.

Lastly, assume some cycle edge label say $8i-4$ is the same as the spoke label $2 + v_j = 4j-2$. Then we have $8i-4 = 4j-2 \pmod{4n}$, which is equal to $8i-4j-2 = 0 \mod 4n$. This equation simplifies to $-2 = 0 \pmod{4n}$, which is a contradiction. □

Theorem 3.3. *The helm graph H_n obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle is properly even harmonious when n is odd.*

Proof. Since H_n has $3n$ edges the modulus is $6n$. Denote the vertex of degree n (the "center") by v_0, the consecutive cycle vertices by $v_1, v_2, \ldots, v_n$, and the vertex of degree 1 adjacent to v_i by w_i. First, label the center $v_0 = 0$. Next, label the cycle vertices $v_i = 6i-2, i = 1, \ldots, n$. Lastly, label the outer most vertices with $w_i = 6i-6, i = 1, \ldots, n$.

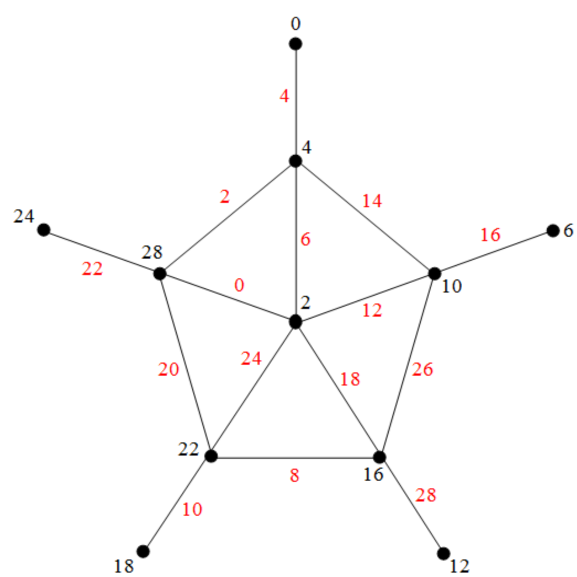

Figure 3: Properly even harmonious labeling of the helm H_5 mod 30

Note that $v_0 = 2$; the v_i's are $4, 10, 16, \ldots, 6n-2$; and the w_i's are $0, 6, 12, \ldots, 6n-6$. So all vertex labels are distinct.

Next note that the spoke v_0v_i has the label $6i$; the cycle edge v_iv_{i+1} has the label $12i-10$; and the pendant edge v_iw_i has the label $12i-8$. Before reducing modulo $6n$, this yields the following edge labels: spokes; $6, 12, 18, \ldots, 6i, \ldots, 6n$; cycle edges: $14, 26, 38, \ldots, 12i-10, \ldots, 12n-10$; pendant edges: $4, 16, 28, \ldots, 12i-8, \ldots, 12n-8$. By observation, these are distinct mod $6n$ (each type has a different remainder modulo 6). See Figure 3. □

Graham and Sloane [3] proved that C_n is harmonious if and only if n is odd. So we will consider even harmonious labelings for the case when n is even.

Although we are not able to prove all cycles of the form C_{4n} are even harmonious we can prove the following.

Theorem 3.4. *The graph, C_{2n}, is not even harmonious when n is odd.*

Proof. Suppose n is odd. Rosa [8] proved that when n is odd C_{2n} does not have a properly even harmonious labeling.

If C_{2n} has an even harmonious labeling, by Corollary 2.9, we may assume that two of the vertex labels are 0 and 0 and the edge labels are $0, 2, 4, \ldots, 4n-2$. When we add these mod $4n$ we get $2n$ by Theorem 2.7. But we know that the sum of the edge labels is just the sum of the vertices where each vertex appears exactly two times. Say that x is the missing nonzero label in C_{2n}. Now look at the sum of all the edges of the labels we use. It must be of the form $2(0+0+2+\cdots+4n-2)-2x$ since every entry in the sum appears exactly twice except x. But notice that $2(0+0+2+\cdots+2n-2+2n+2n+2+\cdots+4n-2)-2x = 0 \mod 4n$. Moreover, since each pair $2, 4n-2; 4, 4n-4; \ldots, 2n-2, 2n+2$ sum to $0 \mod 4n$ we have $2n-2x = 0 \mod 4n$. This reduces to $n - x = 0 \mod 2n$ but n is odd and x is even, which is a contradiction. □

In contrast to Theorem 3.4, the labeling $0, 0, 2, 4$ is an even harmonious labeling for C_4. Moreover, a computer search done by Adam Hesterberg shows that $C_8, C_{12}, C_{16}, C_{20}$ and C_{24} are even harmonious. Here are the even harmonious labels for these graphs.
C_8: $6, 2, 14, 0, 10, 8, 12, 0$ (See Figure 4)
C_{12}: $20, 0, 8, 16, 22, 18, 10, 12, 0, 2, 4, 14$
C_{16}: $20, 2, 16, 4, 22, 12, 24, 6, 0, 14, 18, 26, 30, 10, 0, 28$
C_{20}: $36, 4, 28, 14, 16, 32, 12, 26, 34, 2, 8, 20, 6, 0, 22, 30, 24, 0, 18, 38$
C_{24}: $36, 42, 40, 2, 22, 44, 30, 28, 20, 0, 4, 24, 14, 0, 32, 8, 38, 46, 10, 6, 16, 34, 26, 18$

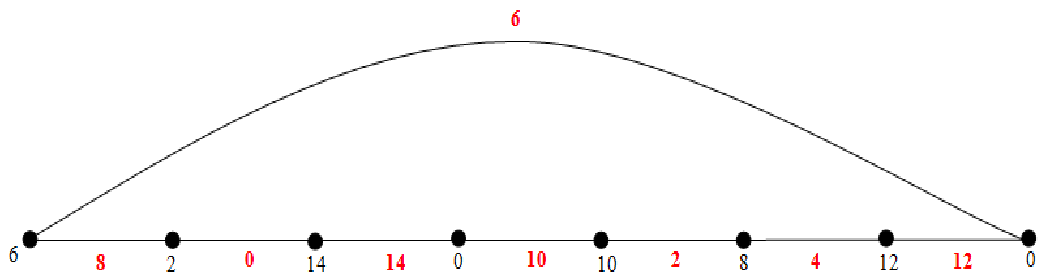

Figure 4: Even harmonious labeling of C_8 mod 16 (notice that 12 is not used in the vertex labeling)

Notice that for each of the six cycles above the missing label is $2n$.

On the basis of these examples we make the following conjecture.

Conjecture 3.5. *C_{4n} is harmonious for all n.*

Although we can not prove C_{4n} is even harmonious, we can say the following.

Theorem 3.6. *For any even harmonious labeling of C_{4n} the label not used is $2n$ or $6n$.*

Proof. The modulus is $8n$. Recall that C_{4n} does not have a proper even harmonious labeling. So we may assume that two of the labels are 0 and 0. Now if we have an even harmonious labeling we know the edges are $0, 2, 4, \ldots, 8n-2$. When we add these mod $8n$ we get $4n$ (see Theorem 2.7). But we know that the edges are just the sum of the vertices where each vertex appears exactly two times. Say that x is the missing nonzero label in C_{4n}. Now look at the sum of all the edges of the labels we used. It must be of the form $2(0+0+2+4+\cdots+8n-2)-2x$ since every entry in the sum appears exactly twice except x. But notice that $2(0+0+2+4+\cdots+8n-2) = 2(4n)-2x = -2x \mod 8n$. So, we have that $-2x = 4n \bmod 8n$. Solving for x we get $x = 2n$ or $x = 6n$. □

The result of Graham and Sloane that K_m is harmonious if and only if for $m \leq 4$ (see the next to last paragraph of Section 2 in [3]) also settles the question of which complete graphs are even harmonious.

Theorem 3.7. *K_n is even harmonious if and only if $n \leq 4$.*

Proof. By the result of Graham and Sloane, any even harmonious labeling of K_n would have $n > 4$ and have a duplicate vertex label x. But then for any vertex label y other than x the edge label $x+y$ is used twice. □

Theorem 3.8. *A graph obtained by identifying exactly one vertex from any finite number of complete graphs (the one-point union) where each has order at least 3 (they need not have the same order) is even harmonious if and only if it is harmonious.*

Proof. Since the graph is connected we need only consider the case where the graph is not harmonious but is even harmonious. By Corollary 2.9, we may assume the duplicate label is 0. If the two vertices labeled 0 are on the same complete graph then for any nonzero label x on that graph the edge x appears twice. If the two 0's are on different cycles and the vertex where the cycles are joined is labeled x, then x appears twice on edge label. □

Since Graham and Sloane [3] have proved that the one-point union two copies of K_n for $n \geq 3$ and odd is not harmonious we have the following corollary.

Corollary 3.9. *The one-point union of two copies of K_n, where $n \geq 3$ and n is odd is not even harmonious.*

4 Disconnected Graphs

Theorem 4.1. *The matching nP_2 is properly even harmonious if and only if n is even.*

Proof. First suppose that n is even. Drawing the graph as shown in Figure 5, label the vertices starting in the top left corner to the bottom left corner with $v_1, v_2, ..., v_n$ then start at the top right corner to the bottom right corner with $v_{n+1}, v_{n+2}, ..., v_{2n}$. To label the vertices, let $v_i = (i-1) \bmod 2n$. By observation, the edge labels are $n, n+2, \ldots, 2n-2; 0, 2, \ldots, n-2$. See Figures 5.

We now suppose n is odd and show nP_2 is not properly even harmonious. The $n-1$ even integers can be used on at most $(n-1)/2$ edges. Likewise, the odds can be used on at most $(n-1)/2$ edges. This leaves one edge that uses the remaining two labels, one of which is even and odd, but then one edge label is odd. □

Theorem 4.2. *The graph nP_2 is even harmonious if n is odd.*

Proof. Drawing the graph as shown in Figure 5, label the vertices starting in the top left corner to the bottom left corner with $0, 1, \ldots, n-1$ then start at the top right corner to the bottom right corner with $n-1, n, \ldots, 2n-2$. Observe that all of the vertex labels are distinct except for the one repeat. The edge labels are $n-1, n+1, \ldots, 2n-2; 0, 2, \ldots, n-3$. Obviously the edge labels are distinct. □

Many of the graphs we consider have a special kind of properly even harmonious labeling defined as follows.

Definition 4.3. *An even harmonious labeling of a graph with q edges is called strongly even harmonious if it satisfies the additional condition that for any two adjacent vertices with labels u and v, $0 < u + v \leq 2q$.*

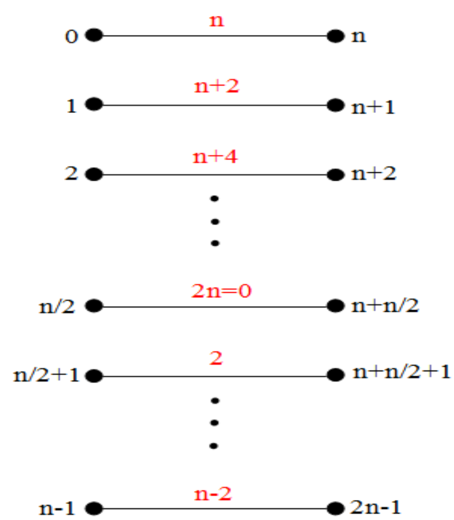

Figure 5: Properly even harmonious labeling of nP_2 mod $2n$ where n is even

So, in the case of a strongly even harmonious labeling of a graph with q edges modular arithmetic is not done except for the case when the sum of two vertex labels is $2q$. See Figure 6.

Theorem 4.4. *The graph $S_m \cup P_n$ (S_m is the star $K_{1,m}$) is strongly even harmonious if $n \geq 2$.*

Proof. The modulus is $2m + 2n - 2$.

Step 1. Label the center of the star 0. Label the end point vertices of the star $2, 4, 6, \ldots$.

Step 2. Starting with the first vertex on P_n label the vertices $1, 3, 5, \ldots$ skipping a vertex each time. Starting with the second vertex label the vertices $2m + 1, 2m + 3, \ldots$ skipping a vertex each time.

The only modular arithmetic used is for the last edge of P_n, therefore all vertex labels and edge labels are distinct. See Figure 6. □

Theorem 4.5. *The graph $S_{m_1} \cup S_{m_2} \cup P_n$ is properly even harmonious if $4 \leq n < 2m_1 + 2m_2 + 1$.*

Proof. The modulus is $2m_1 + 2m_2 + 2n - 2$. We may assume $m_1 \geq m_2$.

Step 1. Label the center of S_{m_1} with 0. Then label the outside vertices with $0, 2, 4, \ldots, 2m_1$.

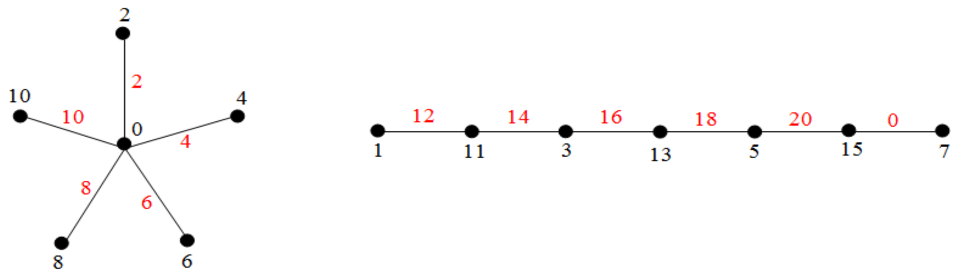

Figure 6: Strongly even harmonious labeling of $S_5 \cup P_7 \mod 22$

Step 2. Label the center of S_{m_2} with $2m_1 + 2m_2 + 2n - 4$. Then label the outside vertices with $2m_1 + 4, 2m_1 + 6, 2m_1 + 8, \ldots, 2m_1 + 2m_2 + 2$.

Step 3. Starting with the first vertex of P_n label the vertices $1, 3, 5, \ldots$ skipping a vertex each time. Starting with the second vertex label the vertices $2m_1 + 2m_2 + 1, 2m_1 + 2m_2 + 3, 2m_1 + 2m_2 + 5, \ldots$ skipping a vertex each time. See Figure 7.

□

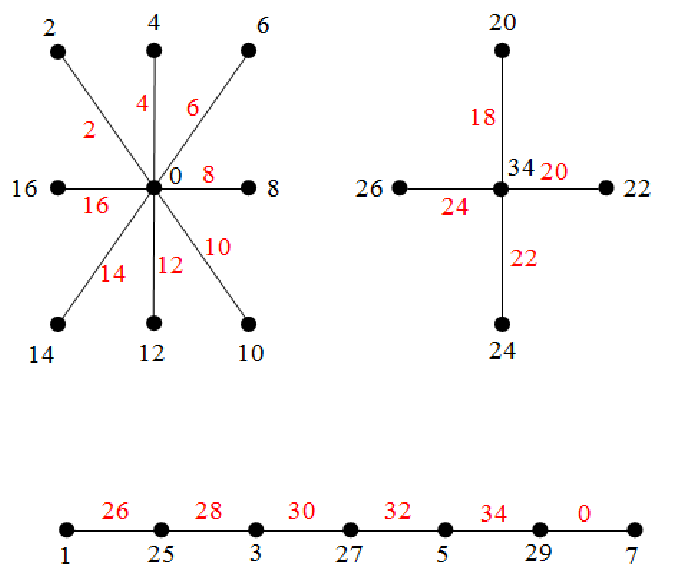

Figure 7: Properly even harmonious labeling of $S_8 \cup S_4 \cup P_7 \mod 36$

Theorem 4.6. *The graph $C_n \cup P_3$ where $n \geq 3$ and n is odd is properly even harmonious.*

Proof. The modulus is $2n + 4$.

Case i. $n = 3$.

Label C_3 with 0, 2, 4 and P_3 with 9, 1, 7 in order.

Case ii. $n > 3$.

Step 1. Label the consecutive labels of C_n with $v_i = 2i, i = 0, 1, \ldots, n-1$. Clearly, the vertex labels are distinct. The edge labels for C_n are $2, 6, 10, \ldots, 2n+2, 0, \ldots, 2n-10, 2n-2$.

Step 2. To label P_3, let x and y be the edge labels not appearing in C_n. Label the middle vertex of P_3 with a 1. Label one of the remaining vertices of P_3 with a $x-1$ and the other vertex $y-1$. See Figure 8.

□

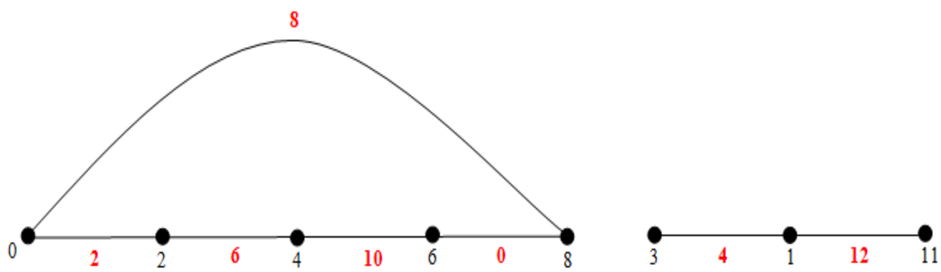

Figure 8: Properly even harmonious labeling of $C_5 \cup P_3$ mod 14

Theorem 4.7. *The graph $C_4 \cup P_n$ is properly even harmonious for all $n \geq 2$.*

Proof. The modulus is $2n + 6$.

Case i. $n = 2, 3, 4$.

Step 1. Label the vertices of C_4 with $0, 2, 2n+4$, and 6 in order. Because the mod is $2n+6$ this gives us the edges $2, 0, 6, 4$ respectively.

Step 2. Starting with the left end point of P_n label every other vertex with $1, 3, 5, \ldots$ (up to $\lceil \frac{n}{2} \rceil$ terms). Then start with the second vertex from the left and label every other vertex with $7, 9, 11, \ldots$ (up to $\lfloor \frac{n}{2} \rfloor$ terms).

Case ii. $n \geq 5$.

Step 1. Label the vertices of C_4 with $1, 2n-5, 3$, and $2n-1$ in order.

Step 2. Starting with the left end point of P_n label every other vertex with $6, 8, 10, \ldots$. Then start with the second vertex from the left and label every other vertex with $2n-2, 2n, 2n+2, \ldots$. The edge labels on P_n are $2n+4, 0, 2, \ldots, 4n-2, 4n$ and the edge labels on C_4 are $4n+2, 4n+4, 4n+6, 4n+8$. It is easy to see that the edge labels are distinct. See Figure 9.

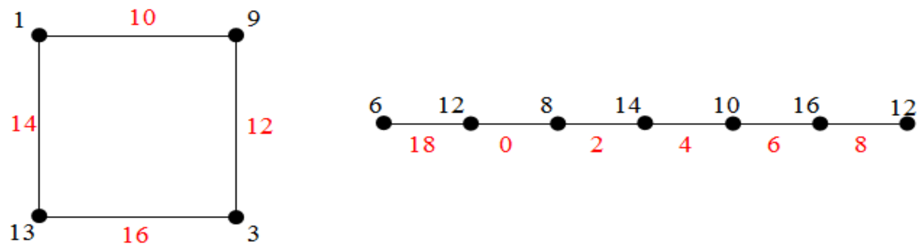

Figure 9: Properly even harmonious labeling of $C_4 \cup P_7$ mod 20

□

Theorem 4.8. *The graph $C_4 \cup F_n$ (F_n is the fan $P_n + K_1$) is properly even harmonious for $n > 1$.*

Proof. The modulus is $4n + 6$.

Case i. n is odd.

Step 1. Start at the first vertex on F_n with 1 then alternate by increments of 2. Start at the second vertex with $n+2$ and alternate with increments of 2. Label the vertex of degree n with $3n$.

Step 2. Label C_4 with $4n + 4, n - 3, 2, n - 1$. See Figure 10.

Case ii. $n = 2$ and $n = 4$ are left to the reader.

Case iii. n is even, $n > 4$.

Step 1. Start at the first vertex on F_n with 1 then alternate by increments of 2. Start at the second vertex with $n+1$ and alternate with increments of 2. Label the vertex of degree n with $3n - 1$.

Step 2. Label C_4 with $0, n - 4, 4n + 4, n$ in order. See Figure 11.

□

Theorem 4.9. *The graph $K_{m,2} \cup P_n$ is properly even harmonious for $1 < n < 4m + 3$.*

Proof. The modulus is $4m + 2n - 2$.

Case i. $2 \le n < 2m + 1$.

Step 1. Denote the partite set of $K_{m,2}$ with 2 vertices by A and the partite set of $K_{m,2}$ with m vertices by B. Label one vertex of A with 0 and the other vertex of A with $2m + 2$. Then label the vertices of B with $2, 4, 6, \ldots, 2m$.

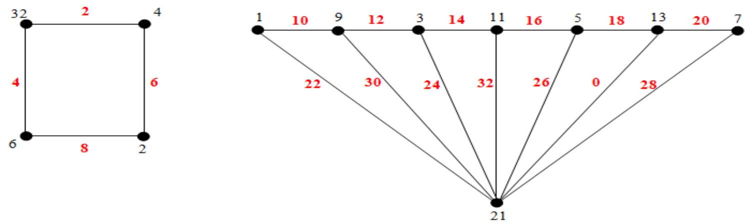

Figure 10: Properly even harmonious labeling of $C_4 \cup F_7$ mod 34

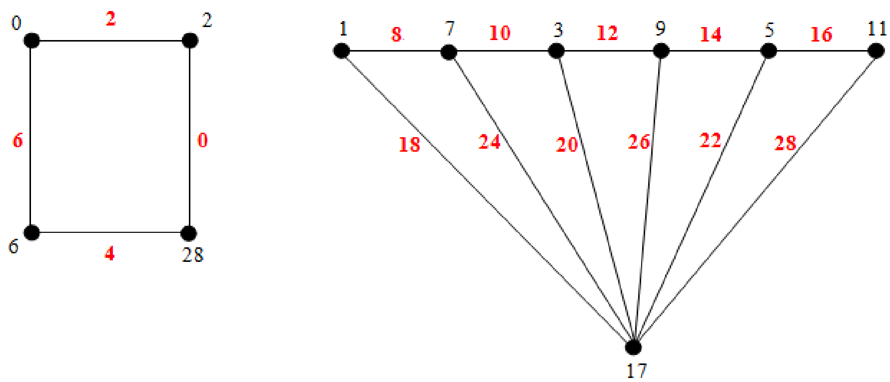

Figure 11: Properly even harmonious labeling of $C_4 \cup F_6$ mod 30

Step 2. Label the first vertex of P_n with $2m+1$. Starting with the second vertex of P_n label the vertices $1, 3, 5, \ldots, n-1$ or $n-2$ depending on whether n is even or odd skipping a vertex each time. Starting with the third vertex of P_n label the vertices $4m+3, 4m+5, 4m+7, \ldots$.

Our missing edge label on $K_{m,2}$ is $2m+2$. On P_n the first and second vertices are $2m+1$ and 1 respectively. To avoid using the vertex label $2m+1$ a second time, the vertex labels of P_n in the even numbered positions must stop before we reach the value $2m+1$. That is, $n < 2m+1$. See Figure 12.

Case ii. $2m+2 < n < 4m+3$.

Step 1. Label one vertex of A with 0 and the other vertex of A with 2. Then label the vertices of B with $2n-4m-2, 2n-4m+2, \ldots, 2n-6$. The smallest edge label of $K_{m,2}$ is $2n-4m-2$. The largest edge label of $K_{m,2}$ is $2n-4$.

Step 2. Label the first vertex of P_n with $2n-3$ then skipping a vertex each time increase by increments of 2. Starting with the second vertex of P_n label the vertices

$1, 3, 5, \ldots, n-1$ or $n-2$ (depending on whether n is even or odd) skipping a vertex each time the first edge label of P_n is $2n-2$. The last edge label of P_n is $2n-4m-4$. On P_n the first and second vertices are $2n-3$ and 1 respectively. To avoid using the vertex label $1 = 4m+2n-1 \bmod (4m+2n-2)$ a second time, the vertex labels of P_n in the even numbered positions must stop before we reach the value $4m+2n-1$. Observing that when n is even the last vertex label in the odd numbered positions is $2n-3+n-2 = 3n-5$, so we must have $3n-5 < 4m+2n-1$ or $n < 4m+4$. Likewise, when n is odd, the last vertex label in the odd numbered positions is $2n-3+n-1 = 3n-4$ so we must have $3n-4 < 4m+2n-1$ or $n < 4m+3$. To handle both cases simultaneously it suffices to take $n < 4m+3$. See Figure 13.

Case iii. $n = 2m+1$ or $n = 2m+2$.

Step 1. Label one vertex of A with 0 and the other vertex of A with $4m+2n-4$. Then label the vertices of B with $2, 6, \ldots, 4m-2$.

Step 2. Starting with the first vertex of P_n, label the vertices $1, 3, 5, \ldots, n-1$ or n, depending on whether n is even or odd. Label the second vertex of P_n with $4m-1$ then skipping a vertex each time increase by increments of two. See Figure 14.

□

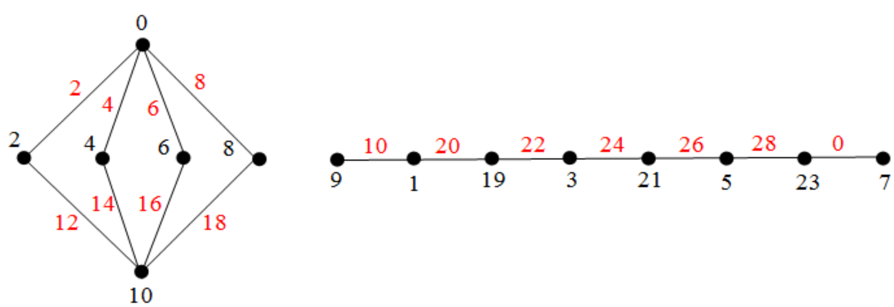

Figure 12: Properly even harmonious labeling of $K_{4,2} \cup P_8$ mod 30

Theorem 4.10. *The graph $P_m \cup P_n$ is properly even harmonious for all $m \geq 2$ and $n \geq 2$.*

Proof. We may assume that $m \geq n$. The modulus is $2m+2n-4$.

Step 1. Label every other vertex of P_m with $1, 3, 5, \ldots$ wrapping around. By construction the edge labels are distinct. For the case that $n = 2$ there will be a single edge label missing, call it x. If $x \neq 0$, label P_2 with 0 and x. If $x = 0$ and $m > 2$, label P_2 with $2m-2$ and 2. If $m = 2$, label P_2 with 0 and 2.

Step 2. For $n > 2$, let $r = 2\lceil \frac{m}{4} \rceil$.

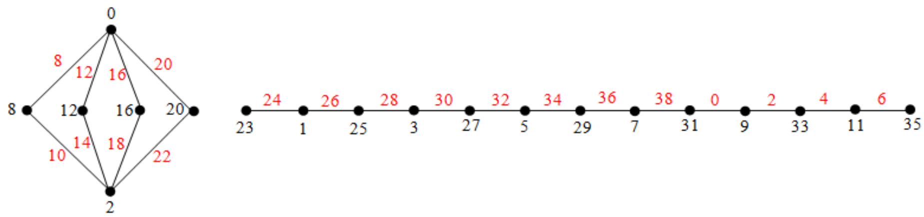

Figure 13: Properly even harmonious labeling of $K_{4,2} \cup P_{13}$ mod 40

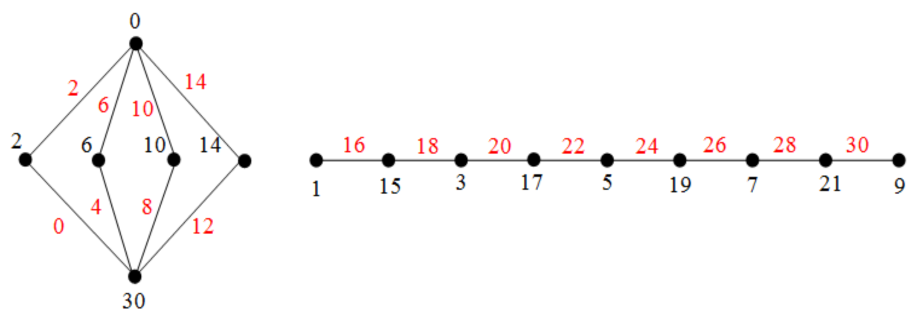

Figure 14: Properly even harmonious labeling of $K_{4,2} \cup P_9$ mod 32

Case i. m is odd.

Note that the last edge label of P_m is $3m-1 \mod (2m+2n-4)$. Label the first vertex of P_n with r. Label the second vertex of P_n with $3m+1-r$. Thus the first edge label of P_n is $3m+1 \mod (2m+2n-4)$. Then label every other vertex in an odd numbered position by increments of 2 and every other vertex in an even numbered position by increments of 2. See Figure 15.

To verify that this is a properly even harmonious labeling observe that for a duplicate vertex label of P_n to occur it is necessary that $3m+1-r+n-3=r$ which converts

$$3m+1+n-3 = 2r = \begin{cases} m+3 & \text{when } m \bmod 4 = 1 \\ m+1 & \text{when } m \bmod 4 = 3. \end{cases}$$

When $2r = m+3$ we have $3m+1+n-3 = m+3$ or $2m+n-5 = 0 \mod (2m+2n-4)$. But for $n \geq 3$, we have $2m+n-5 < 2m+n-1 \leq 2m+2n-4$, which is a contradiction since $2m+2n-5 > 1$. When $2r = m+1$ we have $3m+1+n-3 = m+1$ or $2m+n-3 = 0 \mod (2m+2n-4)$. But for $n \geq 1$, we have $2m+n-3 < 2m+n-1 \leq 2m+2n-4$, which is a contradiction.

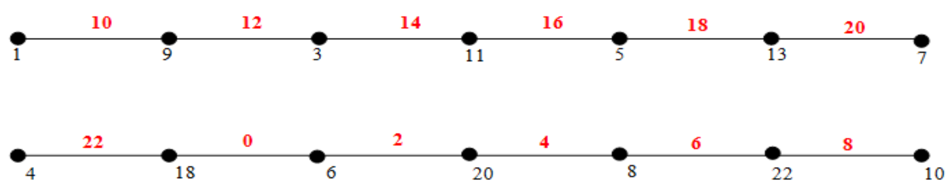

Figure 15: Properly even harmonious labeling of $P_7 \cup P_7$ mod 24 with $r = 4$

Case ii. m is even.

Note that the last edge of P_m is $3m - 2 \bmod (2m + 2n - 4)$. Label the first vertex of P_n with r. Label the second vertex of P_n with $3m - r \bmod (2m + 2n - 4)$. Label the remaining vertices of P_n as in the case that m is odd. For both cases of m the edge labels are distinct since there is no wrap around modulo $2m+2n-4$. By construction the vertex labels on P_m are distinct. To verify that the vertex labels for P_n are distinct we consider two cases. See Figure 16.

To verify that this is a properly even harmonious labeling observe that for a duplicate vertex for P_n to occur it is necessary that $3m - r + n - 2 = r$ which converts to

$$3m + n - 2 = 2r = \begin{cases} m + 2 & \text{when } m \bmod 4 = 1 \\ m + 1 & \text{when } m \bmod 4 = 3. \end{cases}$$

When $2r = m+3$ we have $3m+1+n-3 = m+3$ or $2m+n-5 = 0 \bmod (2m+2n-4)$. But for $n \geq 3$, we have $2m+n-5 < 2m+n-1 \leq 2m+2n-4$, which is a contradiction. When $2r = m+1$ we have $3m+1+n-3 = m+1$ or $2m+n-3 = 0 \bmod (2m+2n-4)$. But for $n \geq 1$, we have $2m+n-3 < 2m+n-1 \leq 2m+2n-4$, which is a contradiction. □

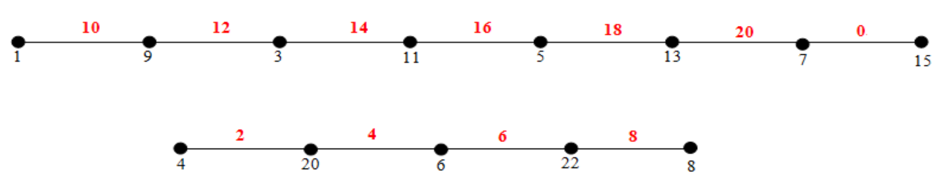

Figure 16: Properly even harmonious labeling of $P_8 \cup P_5$ mod 22 with $r = 4$

Theorem 4.11. *$S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ is strongly even harmonious for $n_1 \geq n_2 \geq \cdots \geq n_t$ and $t < \frac{n_1}{2} + 2$.*

Proof. Let $n = n_1 + n_2 + \cdots + n_t$. The modulus is mod $2n$.

Step 1. Label the center vertex of S_{n_1} with 0 and the centers of $S_{n_t}, \ldots, S_{n_3}, S_{n_2}$ with $1, 3, \ldots, 2t - 3$, respectively.

Step 2. Label the end vertices of S_{n_1} with $2, 4, \ldots, 2n_1$.

To label the remaining vertices, begin with S_{n_t} and recursively label the other stars from right to left as described below. See Figure 17.

Step 3. Label the end vertices of S_{n_t} with $2n-1, 2n-3, \ldots, 2n-1-2(n_t-1)$.

Step 4. Label one end vertex of $S_{n_{t-1}}$ with 4 less than the smallest end vertex label of S_{n_t}. (That is, $2n-1-2(n_t-1)-4$). Label the remaining end vertices of $S_{n_{t-1}}$ successively with increments of -2. The smallest end vertex label of $S_{n_{t-1}}$ will be $2n-1-2(n_t-1)-4-2(n_{t-1}-1)$.

Step 5. Step 4: Continue to move to each star to the left by labeling one end vertex with 4 less than the smallest end vertex label of the previous star and labeling the remaining end vertices of that star using increments of -2. The smallest label of the end vertices of S_{n_2} will be $2n-1-2(n_2+n_3+\cdots+n_t)+2(t-1)-4(t-2) = 2n_1-2t+5$. Since the label of the center of S_2 is $2t-3$ we may avoid a duplicating a vertex label by taking $2n_1-2t+5 > 2t-3$, which simplifies to $t < \frac{n_1}{2}+2$.

□

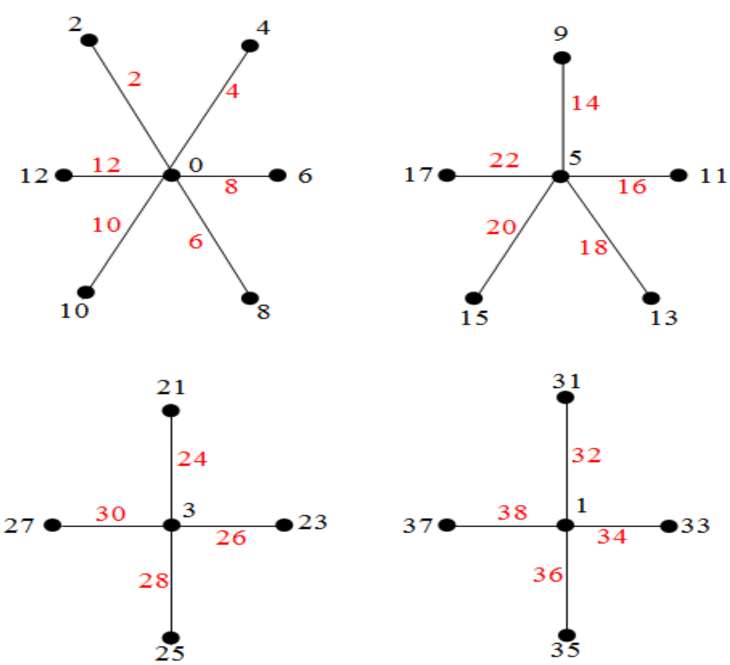

Figure 17: Strongly even harmonious labeling of $S_6 \cup S_5 \cup S_4 \cup S_4$ mod 38 $n = 19$

Although the argument in the proof of Theorem 4.11 does not handle the case when $t \geq \frac{n_1}{2}+2$, other methods might work. Figure 18 shows one such example.

Conjecture 4.12. *$S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ is strongly even harmonious if at least one star has more than 2 edges.*

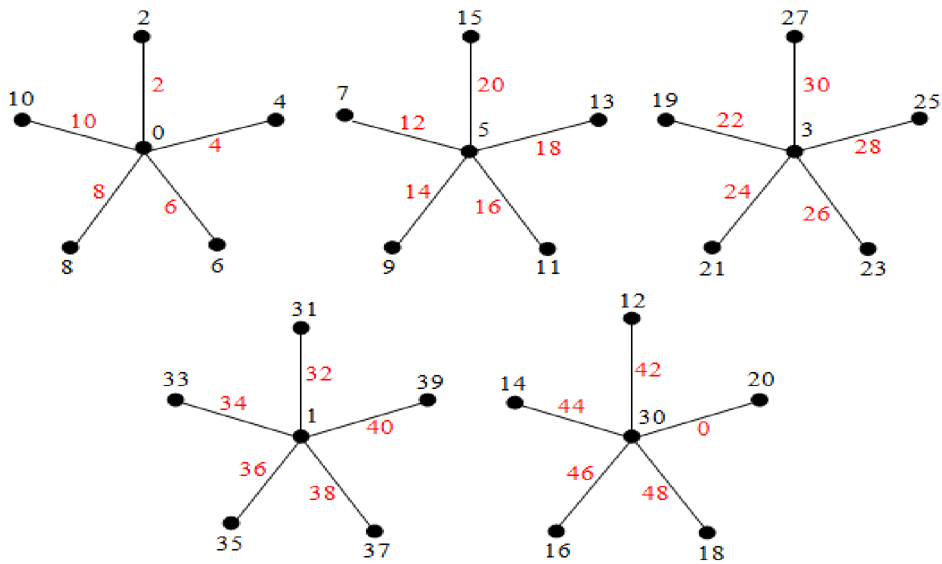

Figure 18: Strongly even harmonious labeling of $S_5 \cup S_5 \cup S_5 \cup S_5 \cup S_5$ mod 50

Theorem 4.13. *For $1 < n < 21$, $W_4 \cup P_n$ is even harmonious.*

Proof. The modulus is $2n + 14$.

Case i. $n = 2$.

Label P_2 with 3 and 17. On W_4 label the cycle 0, 6, 10 , 8 and label the center 4.

Case ii. $n \geq 3$.

Step 1. On W_4 label the cycle 0, 6, 10 , 8 and label the center 4.

Step 2. On P_n start with the first vertex and label $1, 3, 5, \ldots$ skipping a vertex each time. Now starting at the second vertex and skipping a vertex each time use $19, 21, 23, \ldots$. See Figure 19.

□

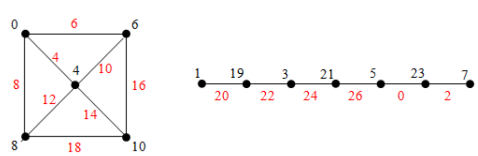

Figure 19: Even harmonious labeling of $W_4 \cup P_7$ mod 28

Theorem 4.14. *The graph $K_4 \cup P_n$ is properly even harmonious when $2 \leq n \leq 12$.*

Proof. The modulus is $2n + 10$.

Step 1. Label the vertices of K_4 in order $0, 2, 4, 8$.

Step 2. Starting with the first vertex of P_n then skipping a vertex each time use the labels $1, 3, 5, \ldots$. Starting at the second vertex and skipping a vertex each time use $13, 15, 17, \ldots$.

The edge labels on the outside of K_4 are $2, 6, 8, 12$ and on the inside the labels are 4 and 10. The edges on P_n are $14, 16, 18, \ldots, 2n + 10$. See Figure 20. □

We remark that the method used in Theorem 4.14 provides even harmonious labeling for $K_4 \cup P_n$ where the vertex label 13 is used twice.

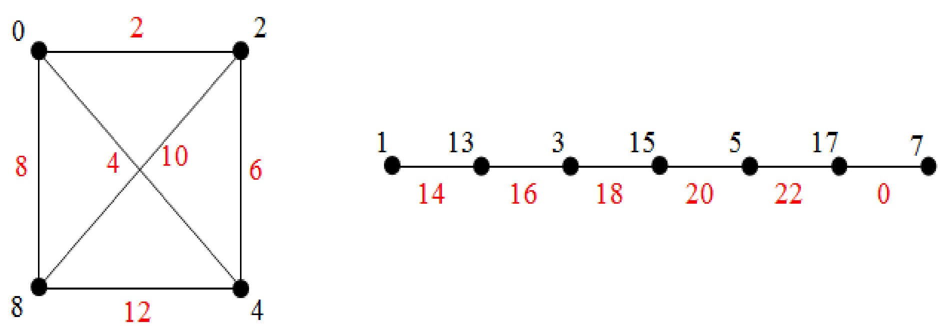

Figure 20: Properly even harmonious labeling of $K_4 \cup P_7$ mod 24

Theorem 4.15. *The graph $C_3 \cup P_n^2$ is properly even harmonious when $n \geq 2$.*

Proof. The modulus is $4n$.

Step 1. Label the vertices on C_3 with $0, 2, 4$ in order.

Step 2. Starting with the left endpoint of P_n^2 label the vertices with $3, 5, 7, \ldots$ in order. See Figure 21.

□

Theorem 4.16. *The graph $C_4 \cup P_n^2$ is properly even harmonious when $n \geq 2$.*

Proof. The modulus is $4n + 2$.

Step 1. Label the vertices on C_4 with $0, 2, 4n, 6$ in order.

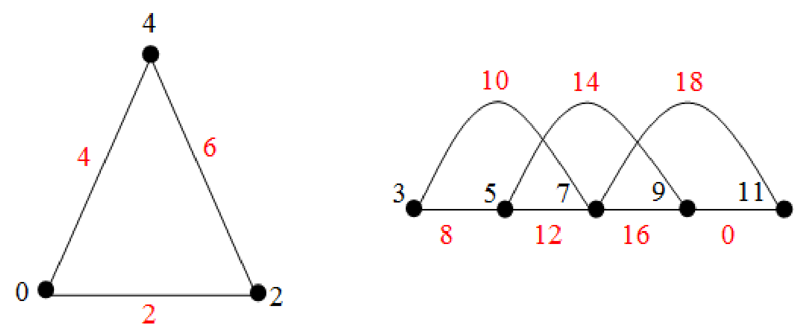

Figure 21: Properly even harmonious labeling of $C_3 \cup P_5^2$ mod 20

Step 2. Starting with the left endpoint of P_n^2 label the vertices with $3, 5, 7, \ldots$ in this order. See Figure 22.

□

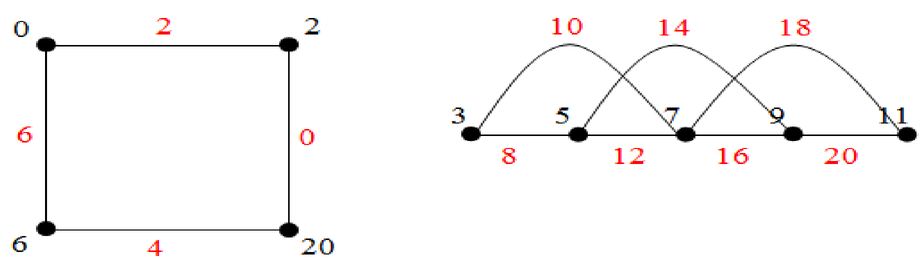

Figure 22: Properly even harmonious labeling of $C_4 \cup P_5^2$ mod 22

Theorem 4.17. *The graph $P_m^2 \cup P_n$ is even harmonious when $2 \leq n < 4m-5$ and $m \geq 2$.*

Proof. The modulus is $2(m+n-1)$.

Step 1. Label P_m^2 with $0, 2, 4, \ldots, 2m-2$ in order.

Step 2. Label P_n starting with the first vertex and skipping a vertex each time with $1, 3, 5, \ldots$. Starting at the second vertex and skipping a vertex each time use $4m-5, 4m-3, 4m-1, 4m+1, \ldots$ in order. To avoid using the vertex label $4m-5$ a second time, the vertex labels of P_n in the odd numbered position must stop before we reach the value $4m-5$. That is, $n < 4m-5$. See Figure 23.

□

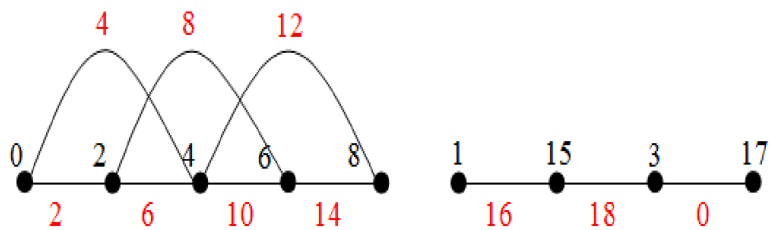

Figure 23: Even harmonious labeling of $P_5^2 \cup P_4$ mod 20

We conclude this section with a few results on the disjoint union of cycles. Recall that a connected graph that is not a tree has a properly even harmonious labeling if and only if it has a harmonious labeling. Although Seoud [10] has shown that both $C_3 \cup C_4$ and $C_4 \cup C_4$ are not harmonious, labeling C_3 with 3, 5, 7 and C_4 with 0, 2, 12, 6 in order and $C_4 \cup C_4$ with labels 0, 2, 14, 6 in order and 3, 7, 1, 11 in order shows that these graphs are properly even harmonious.

For completeness we mention the follow result due to Hillesheim, Kocina, Mall, and Schmit [4].

Theorem 4.18. *nC_{2m+1} is a properly even harmonious graph for all n and m.*

Proof. Label the first copy of C_{2m+1} with $0, 2, 4, \ldots, 4m$, skipping a vertex each time. To label copy $i+1$ of C_{2m+1}, add $2m+1$ to the corresponding vertex of the copy i.

□

5. Application

Government agencies in charge of public safety in most large metropolitan areas employ closed radio transmission networks to communicate among themselves. Commonly, police departments, fire departments, and first responders in emergencies use a central dispatcher for routing calls. In circumstances such as terrorist attacks or major disruptive events that require rapid large scale response from such agencies the usual means of communication is not able handle the demand. In such cases, harmonious labelings and their odd and even variations provide a way to efficiently utilize bandwidth and guaranty that certain parties in a network have a private channel by which they can directly communicate. For example, a city mayor could have a private line connecting to the police chief, the fire chief, and to the person who oversees the first responders. If such a network involves p people (or offices) with up to q channels that are to be dedicated as private communication links between certain entities, we create a graph where the vertices represent the people involved and where there is an edge between two people if and only if these two people are to have a channel of their own to communicate directly with each other and no one else. If this network can be harmoniously labeled then person with label x has a private link to the person with label y using channel $x + y$ mod q. The same idea works for networks whose graphs can be evenly harmoniously labeled.

Similarly, a graph of the form shown in Figure 17 could be useful for air traffic control in areas with multiple airports such as New York City. Each component star represents one airport and all flights coming or going to the airport within a given time frame. The air traffic control tower for each airport is assigned the central vertex label and each plane is assigned an end vertex label. If the network utilized an even harmonious labeling to assign labels each flight is guaranteed its own exclusive channel.

6. Further Research

The following families are good candidates for investigation: $K_4 \cup P_m \cup P_n, P_n^2 \cup K_5, C_m \cup P_n$ when m is odd and $m \geq n, C_m \cup S_n, S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$, and $P_s \cup P_t \cup P_u$.

Acknowledgements

The authors are grateful to Dalibor Froncek and the referee for their helpful comments. This paper is a modified version of a masters degree project done by the second author at the University of Minnesota Duluth done under the supervision of the first author.

Added in Proof: Conjecture 3.5 has been proved by M. Z. Youssef [12].

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