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A NOTE ON POWER DOMINATION PROBLEM IN DIAMETER TWO GRAPHS

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Abstract

The power domination problem aims to find the minimum number of phase measurement units (PMUs) required in order to observe the entire electric power system. Zhao and Kang [6] remarked that there is no known nonplanar graph of diameter two with a power domination number that is arbitrarily large. In this note, we show that the power domination number of such graphs can be arbitrarily large.

Keywords: power domination, diameter two graph, nonplanar graph, Rado graph, Cartesian product of graphs.

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1. Introduction

The notion of power domination in graphs was originated from an optimization problem faced by the electrical power system industry. Electrical power companies need to continually monitor their system's state as defined by a set of variables, for example, the voltage magnitude at loads and the machine phase angle at generators [2, 4]. These variables can be monitored by placing phase measurement units (PMUs) at selected locations in the system. Due to the high cost of a PMU, it is desirable to monitor (observe) the entire system using the least number of PMUs.

To model this optimization problem, we use a graph to represent an electrical network. A vertex denotes a possible location where PMU can be placed, and an edge denotes a current carrying wire. A PMU measures the state variable (voltage and phase angle) for the vertex at which it is placed and its incident edges and their ends. These vertices and edges are said to be observed by the PMU. We can apply Ohm's law and Kirchhoff's current law to deduce the other three observation rules:

1. Any vertex that is incident to an observed edge is observed.
2. Any edge joining two observed vertices is observed.
3. For $k \geq 2$, if a vertex is incident to k edges such that $k-1$ of these edges are observed, then all k of these edges are observed.

We consider only graphs without loops or multiple edges. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* $N(v)$ of a vertex v is the set of all vertices adjacent to v. We denote $N[v]$ for the set $N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, we write $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. The *distance* between two vertices u and v, which is denoted as $d(u, v)$, is the smallest length of any $u - v$ path in the graph. The *diameter* of a connected graph G is defined by $diam(G) = max\{d(u, v) | u, v \in V(G)\}$. If G can be drawn in the plane such that no two of its edges intersect each other, then G is called a *planar graph*. A graph that is not planar is called *nonplanar*. A graph is called a *complete graph* if every two of its vertices are adjacent. We denote a complete graph as K_n.

A set $S \subseteq V(G)$ is said to be a *dominating set* if every vertex in $V(G) \backslash S$ has at least one neighbour in S. A dominating set of minimum cardinality is called a *minimum dominating set*. The *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set of G. For the power system monitoring problem, a set S is defined to be a *power dominating set* (PDS) if every vertex and every edge in G are observed by S after applying the observation rules. The *power domination number* $\gamma_p(G)$ is the minimum cardinality of a power dominating set of G. We will call a power dominating set with minimum cardinality a $\gamma_p(G)$-set. The following algorithm is an alternative approach to the observation rules.

Algorithm 1.1. [1] *Let $S \subseteq V(G)$ be the set of vertices where the PMUs are placed.*

1. *(Domination)*
   Set $M(S) \leftarrow S \cup N(S)$
2. *(Propagation)*
   As long as there exists $v \in M(S)$ such that $N(v) \cap (V(G) - M(S)) = \{w\}$, set $M(S) \leftarrow M(S) \cup \{w\}$.

It is easy to see that for any PDS, applying the three observation rules to determine the set of all observed vertices yields the same result as invoking Algorithm 1.1.

Zhao and Kang [6] showed that any planar graph of diameter two has power dominating number at most two. For nonplanar graphs of diameter two, they noted that there is no known nonplanar graph of diameter two that has an arbitrarily large power domination number. Although MacGillivray and Seyffarth [3] gave a family of diameter two graphs with arbitrarily large domination numbers, the power domination numbers of these graphs are not arbitrarily large. In this paper, we show that the Rado graph and the Cartesian product of two complete graphs are diameter two graphs that can have large power domination numbers.

2. Rado graph

A binary representation of a number is a sequence of bits such that each bit is either 0 or 1. Denote R to be the Rado graph [5]. The Rado graph can be constructed by labeling the vertices of R with the natural numbers $0, 1, 2....$ For $x < y$, vertices x and y are adjacent if the x^{th} bit of the binary representation of y is nonzero. For example, vertex 2 is adjacent to vertex 5 ($0\underline{1}01_2$), but is not adjacent to vertex 3 ($0\underline{0}11_2$) or 8 ($1\underline{0}00_2$) (see Figure 1).

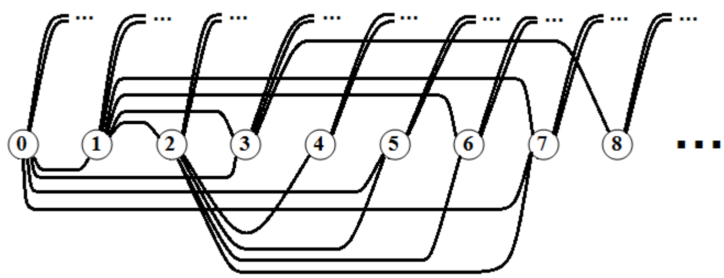

Figure 1: The Rado graph, R.

To verify that $diam(R) = 2$, we choose any two vertices a and b. Now consider vertex c, where $c = 2^a + 2^b$. Since there is a path $a - c - b$, we have $d(a, b) \leq 2$.

We now show that the power domination number of the Rado graph is infinite.

Theorem 2.1. *An infinite number of PMUs is required to observe the Rado graph.*

Proof. We label the vertices as $v_0, v_1, v_2...$, where the subscripts denote the vertex numbers. Let S be a $\gamma_p(R)$-set. Suppose on the contrary that S is a finite set with $v_k \in S$, where k is the largest integer. Applying Algorithm 1.1, we see that while the domination step admits new vertices to $M(S)$, the propagation step does not increase $|M(S)|$. This is because any vertex $v \in M(S)$ is adjacent to infinitely many vertices that are not in $M(S)$. Hence $|N(v) \cap \big(V(G) - M(S)\big)| > 1$.

The output from Algorithm 1.1 gives $M(S) = S \cup N(S)$. It suffices to show that there exists at least one vertex v_z that is not in $M(S)$. Consider $z = 2^{k+1} > k$. Vertex v_z is not in S; for otherwise, it will contradict the maximality of k. Also v_z is not in $N(S)$, as v_z is not adjacent to vertices $v_0, v_1, ..., v_k$. Since $M(S) \neq V(R)$, the Rado graph is not observed by S. This contradicts our assumption of S being a finite $\gamma_p(R)$-set. $\square$

3. Cartesian product $K_n \times K_m$

The Cartesian product of two graphs G_1 and G_2, denoted by $G = G_1 \times G_2$, has $V(G) = V(G_1) \times V(G_2) = \{(x_1, x_2) \mid x_i \in V(G_i) \text{ for } i = 1, 2\}$, and two vertices (u_1, u_2) and

(v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. The Cartesian product of two complete graphs is denoted as $K_n \times K_m$. We label the vertices of $K_n \times K_m$ as (i, j), where $1 \leq i \leq n$ and $1 \leq j \leq m$. The label may be interpreted as vertex i of the j^{th} copy of K_n, or as vertex j of the i^{th} copy of K_m.

To verify that the diameter of G is two, we consider any two distinct vertices (i_1, j_1) and (i_2, j_2), where $1 \leq i_1, i_2 \leq n$ and $1 \leq j_1, j_2 \leq m$. If either $i_1 = i_2$ or $j_1 = j_2$, then the two vertices are adjacent. Otherwise the two vertices are adjacent to a common neighbor (i_1, j_2). This implies that the distance between any two vertices of $K_n \times K_m$ is at most two.

Lemma 3.1. *For $m \geq n \geq 2$, $\gamma_p(K_n \times K_m) \leq n - 1$.*

Proof. Let $S = \{(1, 1), (2, 2), ..., (n - 1, n - 1)\}$. It can be verified that S observes the graph $K_n \times K_m$. □

Lemma 3.2. *For $m \geq n \geq 3$, $\gamma_p(K_{n-1} \times K_{m-1}) \leq \gamma_p(K_n \times K_m) - 1$.*

Proof. Let S be a PDS for $K_n \times K_m$. By symmetry of the graph, without loss of generality let vertex $(n, n) \in S$. We claim that there exists S such that $N\big((n, n)\big) \cap S = \emptyset$. Assume on the contrary that for some $k \neq n$, either vertex $(n, k) \in S$ or vertex $(k, n) \in S$. For the first case, since $|S| < n$ from Lemma 3.1, we let $(2, k) \notin S$. Observe that the set $S \cup \{(2, k)\} \setminus \{(n, k)\}$ is also a PDS. The proof is identical in the second case where $(k, n) \in S$.

We consider the induced subgraph $K_{n-1} \times K_{m-1}$ obtained by removing all vertices in $N\big[(n, n)\big]$. To prove this lemma, it suffices to show that there exists a set S' containing at most $|S| - 1$ vertices such that S' observes the induced subgraph $K_{n-1} \times K_{m-1}$. Suppose instead each and every possible S' does not observe the subgraph. Then there exists at least one vertex that is not observed by S'. Without loss of generality, we let $(1, 1) \in S'$, and let the unobserved vertex be $(n - 1, n - 1)$.

It is obvious that no PMU is placed in the $(n - 1)^{th}$ copy of K_{n-1}. We claim that at least two vertices in the $(n - 1)^{th}$ copy of K_{n-1} are not observed by S'; for otherwise, vertex $(n - 1, 1)$, which is adjacent to $(1, 1) \in S'$, will have exactly one unobserved vertex as its neighbor. The propagation step in Algorithm 1.1 results in vertex $(n - 1, n - 1)$ being observed by S'. This contradicts our earlier assumption.

Now since vertex $(n, n) \in S$, we consider $S = S' \cup \{(n, n)\}$ in the original graph $K_n \times K_m$ for all possible S'. Since all vertices in a copy of K_n are pairwise adjacent, the propagation step cannot be applied to vertices $(n - 1, n)$ or $(n, n - 1)$. The reason is that each of them is adjacent to vertex $(n - 1, n - 1)$ and at least one more unobserved vertex. Thus S does not observe $K_n \times K_m$. As this contradicts our earlier assumption, there must exist S' that observes $K_{n-1} \times K_{m-1}$. It follows that $\gamma_p(K_{n-1} \times K_{m-1}) \leq |S'| \leq |S| - 1$. □

Theorem 3.3. *For $m \geq n \geq 2$, $\gamma_p(K_n \times K_m) = n - 1$.*

Proof. We will show that $\gamma_p(K_n \times K_m) \geq n-1$. When $n=2$, any vertex observes the graph $K_2 \times K_m$, and thus $\gamma_p(K_2 \times K_m) = 1$. When $n \geq 3$, we apply Lemma 3.2 repeatedly for a finite number of times to obtain

$$\gamma_p(K_n \times K_m) \geq \gamma_p(K_{n-1} \times K_{m-1}) + 1 \geq ... \geq n-1.$$

Combining the above inequality and the inequality in Lemma 3.1 gives the required result. $\square$

References

[1] P. Dorbec, M. Mollard, S. Klavzar and S. Spacapan, Power Domination in product graphs, *SIAM J. Discrete Math.*, **22** (2) (2008), 554–567.

[2] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi and M. A. Henning, Domination in graphs applied to electric power networks, *SSIAM J. Discrete Math.*, **15** (4) (2002), 519–529.

[3] G. MacGillivray and K. Seyffarth, Domination numbers of planar graphs, *J. Graph Theory*, **22** (1996), 213–229.

[4] L. Mili, T. L. Baldwin and A. G. Phadke, *Phasor Measurements for Voltage and Transient Stability Monitoring and Control*, Proceedings of the EPRI-NSF Workshop on Application of Advanced Mathematics to Power Systems, San Francisco, CA, 1991.

[5] R. Rado, Universal graphs and universal functions, *Acta Arith.*, **9** (1964), 331–340.

[6] M. Zhao and L. Y. Kang, Power domination in planar graphs with small diameter, *J. Shanghai Univ., Engl. Ed.,*, **11** (3) (2007), 218–222.