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EDGE-MAXIMAL GRAPHS WITHOUT θ_{2k+1}-GRAPHS

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Abstract

Let $\mathcal{G}(n;\theta_{2k+1})$ denote the class of non-bipartite graphs on n vertices having no θ_{2k+1}-graph and $f(n;\theta_{2k+1}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;\theta_{2k+1})\}$. In this paper we determine $f(n;\theta_{2k+1})$, by proving that $f(n;\theta_{2k+1}) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ for $k \geq 4$ and $n \geq 36k$. Further, the bound is best possible. Our result confirms the conjecture made by Bataineh in his Ph.D. thesis "Some extremal problems in graph theory", Curtin University of Technology, Australia (2007), for large n.

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1. Introduction and preliminaries

For our purposes a graph G is finite, undirected and has no loops or multiple edges. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. The cardinalities of these sets are denoted by $v(G)$ and $\mathcal{E}(G)$, respectively. The cycle on n vertices is denoted by C_n. A theta graph θ_n is defined to be a cycle C_n to which we add a new edge that joins two non-adjacent vertices. The circumference of a graph G is denoted by $c(G)$ and defined to be the length of a longest cycle. The neighbor set of a vertex u of G in a subgraph H of G, denoted by $N_H(u)$, consists of the vertices of H adjacent to u. For vertex disjoint subgraphs H_1 and H_2 of G, we let $E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $\mathcal{E}(H_1, H_2) = |E(H_1, H_2)|$.

For a proper subgraph H of G we write $G[V(H)]$ and $G - V(H)$ simply as $G[H]$ and $G - H$, respectively. In this paper, we consider the Turán-type extremal problem with the

θ-graph being the forbidden subgraph. Since a bipartite graph contains no odd θ-graph, we consider non-bipartite graphs. First, we recall some notation and terminology. For a positive integer n and a set of graphs $\mathcal{F}$, let $\mathcal{G}(n;\mathcal{F})$ denote the class of non-bipartite $\mathcal{F}$-free graphs on n vertices, and

$$f(n;\mathcal{F}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;\mathcal{F})\}.$$

An important problem in extremal graph theory is that of determining the values of the function $f(n;\mathcal{F})$. Further, characterize the extremal graphs of $\mathcal{G}(n;\mathcal{F})$ where $f(n;\mathcal{F})$ is attained. For a given C_r , the edge maximal graphs of $\mathcal{G}(n;C_r)$ have been studied by a number of authors [5, 6, 7, 9]. Bondy [4] proved that a Hamiltonian graph G on n vertices without a cycle of length r has at most $\frac{1}{2}n^2$ edges with equality holding if and only if n is even and r is odd.

Höggkvist, Faudree and Schelp [8] proved that $f(n;C_r) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$ for all r. This result is sharp only for $r = 3$. Jia [10] proved that for $n \geq 9$, $f(n;C_5) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ and he characterized the extremal graphs as well. In the same work, Jia conjectured that $f(n;C_{2k+1}) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ for $n \geq 4k+2$. Recently, Bataineh [1] confirmed positively the above conjecture for large n. Moreover, Bataineh conjectured that for $k \geq 3$,

$$f(n;\theta_{2k+1}) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.$$

More recently, Bataineh, Jaradat and Al-Shboul [2], proved that for $n \geq 9$,

$$f(n;\theta_5) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

Furthermore, the bound is best possible. Also, in [3], Bataineh, Jaradat and Al-Shbou confirmed the above conjecture in the case $k = 3$ for graphs with minimum degree ≥ 25. In this paper, we prove that the above conjecture is true for $k \geq 4$ and $n \geq 36k$. Further, this bound is best possible. We now state a number of results which we make use of in our work.

Lemma 1.1. [11] *Let G be a graph on n vertices with no cycles of length greater than k. Then*

$$\mathcal{E}(G) \leq \frac{1}{2}k(n-1) - \frac{1}{2}r(k-r-1)$$

where $r = (n-1) - (k-1)\left\lfloor \frac{(n-1)}{(k-1)} \right\rfloor$.

Theorem 1.2. [1] *Let $G \in \mathcal{G}(n;C_{2k+1})$. Then for $k \geq 2$ and $n \geq 36k$,*

$$f(n;C_{2k+1}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.$$

Furthermore, equality holds if and only if $G \in \mathcal{G}^*(n)$ where $\mathcal{G}^*(n)$ is the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor (n-2)/2 \rfloor, \lceil (n-2)/2 \rceil}$.

Lemma 1.3. [4] *Let G be a graph on n vertices with $\mathcal{E}(G) > \left\lfloor \frac{n^2}{4} \right\rfloor$. Then $c(G) \geq \lfloor \frac{n+3}{2} \rfloor$ and G contains cycles of every length l for $3 \leq l \leq c(G)$.*

2. Edge-Maximal θ_{2k+1}-Free Graphs

In this section we determine $f(n; \theta_{2k+1})$. Observe that for a graph $G \in \mathcal{G}^*(n)$, we have that $\mathcal{E}(G) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ and has no θ_{2k+1} as a subgraph. Consequently, we have established that

$$f(n; \theta_{2k+1}) \geq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.$$

In the following theorem we determine the maximum number of edges of a graph with n vertices that contains no θ_{2k+1}-graph as a subgraph.

Theorem 2.1. *Let G be a graph on $n \geq 6k+3$ vertices that contains no θ_{2k+1} graph as a subgraph. Then*

$$\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. If G contains no cycle of length $2k+1$, then by Lemma 1.2 $\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. So we need to consider that G contains a cycle of length $2k+1$. Let t be the maximum number of vertex disjoint cycles of length $2k+1$, say $C^{(1)}, C^{(2)}, \ldots, C^{(t)}$ are vertex disjoint cycles of length $2k+1$. Define $R = G - \bigcup_{i=1}^{t} C^{(i)}$. Note that R has no cycle of length $2k+1$ and $\mathcal{V}(R) = n - t(2k+1)$. Observe that, if $c(R) \geq (2k+1)$, then by Lemma 1.2, $\mathcal{E}(R) \leq \left\lfloor \frac{(n-(2k+1)t)^2}{4} \right\rfloor$, as otherwise R contains a cycle of length $2k+1$. On the other hand, if $c(R) \leq 2k$, then by Lemma 1.1, $\mathcal{E}(R) \leq k(n-(2k+1)t-1)$. Therefore,

$$\mathcal{E}(R) \leq \begin{cases} \left\lfloor \frac{(n-(2k+1)t)^2}{4} \right\rfloor, & \text{if } c(R) \geq (2k+1), \\ k(n-(2k+1)t-1), & \text{if } c(R) \leq 2k. \end{cases} \tag{1}$$

Now, for $1 \leq i \leq t$, we find $\mathcal{E}(R, C^{(i)})$. Let $C^{(i)} = x_1^{(i)} x_2^{(i)} \ldots x_{2k+1}^{(i)} x_1^{(i)}$. For any $x \in V(R), \mathcal{E}(x, C^{(i)}) \leq k$ (to see that, assume that $x \in V(R)$ with $\mathcal{E}(x, C^{(i)}) > k$. Then x must be adjacent to $x_j^{(i)}$ and $x_{j+2}^{(i)}$ in $C^{(i)}$ for some j, and hence θ_{2k+1}-graph is produced, a contradiction). Let $x \in V(R)$, such that $\mathcal{E}(x, C^{(i)}) = k$. Then x has no neighbors of the form $x_j^{(i)}$ and $x_{j+2}^{(i)}$ in $C^{(i)}$, otherwise θ_{2k+1}-graph is produced. Therefore, $N_{C^{(i)}}(x)$ must be of the form:

(I) $N_{C^{(i)}}(x) = \{x_j^{(i)}, x_{j+1}^{(i)}, x_{j+4}^{(i)}, x_{j+5}^{(i)}, \ldots, x_{j-5}^{(i)}, x_{j-4}^{(i)}\}$ for some $1 \leq j \leq 2k+1$, if k is even.

(II) $N_{C^{(i)}}(x) = \{x_j^{(i)}, x_{j+1}^{(i)}, x_{j+4}^{(i)}, x_{j+5}^{(i)}, \ldots, x_{j-7}^{(i)}, x_{j-6}^{(i)}, x_{j-3}^{(i)}\}$ for some $1 \leq j \leq 2k+1$, if k is odd.

In (I) and (II), $x_0^{(i)} = x_{2k+1}^{(i)}, x_{-1}^{(i)} = x_{2k}^{(i)}, x_{-2}^{(i)} = x_{2k-1}^{(i)}, \ldots, x_{-2k}^{(i)} = x_1^{(i)}$. Define $B = \{x \in V(R) : |N_{C^{(i)}}(x)| = k\}$. We have the following claim:

Claim 1. $|B| \leq 1$.

Proof of Claim 1. Suppose that $|B| > 1$. Let $\{x, y\} \subseteq B$. Then we have two cases according to the parity of k:

Case I. k is even.

Then, without loss of generality, we may assume that $N_{C^{(i)}}(x) = \{x_1^{(i)}, x_2^{(i)}, x_5^{(i)}, x_6^{(i)}, \ldots, x_{2k-3}^{(i)}, x_{2k-2}^{(i)}\}$. Let s be the number of common neighbors of x and y in $C^{(i)}$. We use induction on s to show that G contains θ_{2k+1}-graph as a subgraph. For $s = 0$, we have that $N_{C^{(i)}}(x) = \{x_3^{(i)}, x_4^{(i)}, x_7^{(i)}, x_8^{(i)}, \ldots, x_{2k-1}^{(i)}, x_{2k}^{(i)}\}$ or $N_{C^{(i)}}(x) = \{x_3^{(i)}, x_4^{(i)}, x_7^{(i)}, x_8^{(i)}, \ldots, x_{2k-5}^{(i)}, x_{2k-4}^{(i)}, , x_{2k}^{(i)}, x_{2k+1}^{(i)}\}$. Note that the trail $x_1^{(i)} x_2^{(i)} x_3^{(i)} y x x_6^{(i)} x_7^{(i)} \ldots\ x_{2k+1}^{(i)} x_1^{(i)}$ forms C_{2k+1}-graph and y is adjacent to $x_{2k+1}^{(i)}$. So θ_{2k+1}-graph is produced. Now suppose that G has θ_{2k+1}-graph as a subgraph for values less than s and greater than 0. Now, we prove it for s. Let x and y have s common neighbors, then there is $(s-1)$ common neighbors and so by the induction step G has θ_{2k+1}-graph as a subgraph. This is a contradiction.

Case II. k is odd.

Then without loss of generality, we may assume that $N_{C^{(i)}}(x) = \{x_1, x_2, x_5, x_6, \ldots, x_{2k-5}, x_{2k-4}, x_{2k-1}\}$. As in Case I we use induction on s to show that G contains θ_{2k+1}-graph as a subgraph. For $s = 0$, we have that $N_{C^{(i)}}(x) = \{x_3^{(i)}, x_4^{(i)}, x_7^{(i)}, x_8^{(i)}, \ldots,\ x_{2k-3}^{(i)}, x_{2k-2}^{(i)}, x_{2k+1}^{(i)}\}$ or $N_{C^{(i)}}(x) = \{x_3^{(i)}, x_4^{(i)}, x_7^{(i)}, x_8^{(i)}, \ldots, x_{2k-5}^{(i)}, x_{2k-6}^{(i)}, x_{2k-3}^{(i)}, x_{2k}^{(i)}, x_{2k+1}^{(i)}\}$. Note that the trail $x_1^{(i)} x_2^{(i)} x_3^{(i)} y x x_6^{(i)} x_7^{(i)} \ldots x_{2k+1}^{(i)} x_1^{(i)}$ forms C_{2k+1}-graph and x is adjacent to $x_1^{(i)}$. So θ_{2k+1}-graph is produced. Now suppose that G has θ_{2k+1}-graph as a subgraph for values less than s and greater than 0. Now, for s, we follow word by word the argument as in Case I to get the same contradiction. The proof of Claim is complete.

Therefore, for any vertex $y \notin B$, we have $\mathcal{E}(y, C^{(i)}) \leq k-1$. Thus,

$$\begin{aligned} \mathcal{E}(R, C^{(i)}) &\leq (k-1)\,|R| + 1 \\ &= (k-1)(n-(2k+1)) + 1. \end{aligned} \tag{2}$$

Let $x \in C^{(j)}$ for some $1 \leq j \leq t$. Then by following, word by word, the same argument as in the above after replacing R by $C^{(j)}$ one can see that $\mathcal{E}(C^{(i)}, x) \leq k$. Moreover, if $B = \{x \in C^{(j)} : \mathcal{E}(C^{(i)}, x) = k\}$, then $|B| \leq 1$. Now we claim the following:

Claim 2. *For $x, y \in V(C^{(j)})$, if $\mathcal{E}(C^{(i)}, x) = k$ and $yx \in E(C^{(j)})$, then $\mathcal{E}(C^{(i)}, y) \leq k-2$.*

Proof of Claim 2. Assume that $\mathcal{E}(C^{(i)}, y) = k-1$. Then we consider the following cases:

Case 1. k is even.

As in the above, without loss of generality, we assume that $N_{C^{(i)}}(x) = \{x_1^{(i)}, x_2^{(i)}, x_5^{(i)}, x_6^{(i)}, \ldots, x_{2k-3}^{(i)}, x_{2k-2}^{(i)}\}$. Now we split our work into two subcases:

Subcase 1.1. $N_{C^{(i)}}(y) \subseteq N_{C^{(i)}}(x)$.

Then $N_{C^{(i)}}(y)$ contains two consecutive vertices, say $x_l^{(i)}, x_{l+1}^{(i)}$ for some $0 \leq l \leq 2k-1$. Thus, either $x_{l+4}^{(i)}, x_{l+5}^{(i)} \in N_{C^{(i)}}(x)$ or $x_{l-3}^{(i)}, x_{l-2}^{(i)} \in N_{C^{(i)}}(x)$. If $x_{l+4}^{(i)}, x_{l+5}^{(i)} \in N_{C^{(i)}}(x)$, then $C = x_{l+1}^{(i)} y x x_{l+4}^{(i)} x_{l+5}^{(i)} \ldots x_{2k+1}^{(i)} x_1^{(i)} x_2^{(i)} \ldots x_l^{(i)} x_{l+1}^{(i)}$ is a $(2k+1)$-cycle in G. Since $xx_l^{(i)} \in E(G)$, as a result G contains θ_{2k+1}-graph as a subgraph, a contradiction. If $x_{l-3}^{(i)}, x_{l-2}^{(i)} \in N_{C^{(i)}}(x)$, then $C = x_l^{(i)} y x_{l+1}^{(i)} x_{l+2}^{(i)} \ldots x_{2k+1}^{(i)} x_1^{(i)} x_2^{(i)} \ldots x_{l-3}^{(i)} x_{l-2}^{(i)} x x_l^{(i)}$ is a $(2k+1)$-cycle in G. Since $xy \in E(G)$, as a result G contains θ_{2k+1}-graph as a subgraph, a contradiction.

Subcase 1.2. $N_{C^{(i)}}(y) \nsubseteq N_{C^{(i)}}(x)$.

Let $x_l^{(i)} \in N_{C^{(i)}}(y)$-$N_{C^{(i)}}(x)$. Note that $3 \leq l \leq 2k+1$. Then at least one of $x_{l+3}^{(i)}$ and $x_{l-3}^{(i)}$ belongs to $N_{C^{(i)}}(x)$. If $x_{l+3}^{(i)} \in N_{C^{(i)}}(x)$, then $C = xyx_l^{(i)} x_{l-1}^{(i)} x_{l-2}^{(i)} \ldots x_1^{(i)} x_{2k+1}^{(i)} x_{2k}^{(i)} \ldots x_{l+3}^{(i)} x$ is a cycle of length $2k+1$. Since $xx_1^{(i)}$ and $xx_2^{(i)}$ belong to G, as a result G contains a θ_{2k+1}-graph as a subgraph, a contradiction. Now, if $x_{l-3}^{(i)} \in N_{C^{(i)}}(x)$, then $C = xyx_l^{(i)} x_{l+1}^{(i)} x_{l+2}^{(i)} \ldots x_{2k+1}^{(i)} x_1^{(i)} \ldots x_{l-3}^{(i)} x$ is a cycle of length $2k+1$. Since $xx_1^{(i)}$ and $xx_2^{(i)}$ belong to G, as a result G contains a θ_{2k+1}-graph as a subgraph, a contradiction.

Case 2. k is odd.

As in the above, without loss of generality, we assume that $N_{C^{(i)}}(x) = \{x_1, x_2, x_5, x_6, \ldots, x_{2k-5}, x_{2k-4}, x_{2k-1}\}$. Then we consider the following two subcases:

Subcase 2.1. $N_{C^{(i)}}(y) \subseteq N_{C^{(i)}}(x)$.

Then follow, word by word, the same argument as in Subcase 1.1, we get the same contradiction.

Subcase 2.2. $N_{C^{(i)}}(y) \nsubseteq N_{C^{(i)}}(x)$.

Then we consider two subsubcases:

Subsubcases 2.2.1. There is $x_l^{(i)} \in N_{C^{(i)}}(y)$ such that $x_{l+3}^{(i)}$ or $x_{l-3}^{(i)}$ belongs to $N_{C^{(i)}}(x)$. Then as in Subcase 1.2, G contains θ_{2k+1}-graph as a subgraph, a contradiction.

Subsubcases 2.2.2. For every $x_l^{(i)} \in N_{C^{(i)}}(y)$, neither $x_{l+3}^{(i)}$ nor $x_{l-3}^{(i)}$ belong to $N_{C^{(i)}}(x)$. Then $N_{C^{(i)}}(y) \subseteq \{x_{2k+1}^{(i)}, x_{2k-3}^{(i)}\}$, and so $|N_{C^{(i)}}(y)| \leq 2$. Thus, $k = 3$, which contradict the fact that $k \geq 5$. The proof of the claim is complete.

Hence,

$$\begin{aligned} \mathcal{E}(C^{(i)}, C^{(j)}) &\leq (k-1)\left|C^{(j)}\right| \\ &= (k-1)(2k+1) \quad \text{for } i \neq j. \end{aligned} \tag{3}$$

From (1), if $c(R) \geq 2k+1$, then $\mathcal{E}(R) \leq \left\lfloor \frac{(n-(2k+1)t)^2}{4} \right\rfloor$. Therefore, by (2) and (3)

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(R) + t\mathcal{E}(R, C^{(i)}) + \binom{t}{2}\mathcal{E}(C^{(i)}, C^{(j)}) + t\mathcal{E}(C^{(i)}) \\ &\leq \frac{(n-(2k+1)t)^2}{4} + t(k-1)(n-(2k+1)t) + t \\ &\quad + \frac{1}{2}t(t-1)(k-1)(2k+1) + t(2k+1) \\ &= \frac{1}{4}n^2 + (-\frac{3}{2}n + \frac{5}{2} - k^2 + \frac{5}{2}k)t + (\frac{3}{2}k + \frac{3}{4})t^2. \end{aligned}$$

$$\text{Let } g(t) = (-\frac{3}{2}n + \frac{5}{2} - k^2 + \frac{5}{2}k)t + (\frac{3}{2}k + \frac{3}{4})t^2, \quad t \in \left[1, \left\lfloor \frac{n}{2k+1} \right\rfloor\right].$$

Note that g is a parabola with positive coefficient of t^2. Thus, g is concaving up and so its maximum occurs at $t = 1$ and $t = \left\lfloor \frac{n}{2k+1} \right\rfloor$. Observe that

$$g\left(\left\lfloor \frac{n}{2k+1} \right\rfloor\right) = -\frac{1}{4}\frac{n}{2k+1}\left(4k^2 - 10k + 3n - 10\right) < 0$$

and

$$g(1) = -k^2 + 4k - \frac{3}{2}n + \frac{13}{4} < 0.$$

Thus,

$$\mathcal{E}(G) < \frac{n^2}{4}.$$

From (1) if $c(R) \leq 2k$, then $\mathcal{E}(R) \leq k(n-(2k+1)t-1)$. So, by (2) and (3)

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(R) + t\mathcal{E}(R, C^{(i)}) + \binom{t}{2}\mathcal{E}(C^{(i)}, C^{(j)}) + t\mathcal{E}(C^{(i)}) \\ &\leq k(n-(2k+1)t-1) + t(k-1)(n-(2k+1)t) + t \\ &\quad + \frac{1}{2}t(t-1)(k-1)(2k+1) + t(2k+1) \end{aligned}$$

$$= \frac{5}{2}t - k + \frac{1}{2}kt^2 - 3k^2t - k^2t^2 + kn + \frac{3}{2}kt - nt + \frac{1}{2}t^2 + knt$$
$$= (n-1)k + ((k-1)n + \frac{5}{2} + \frac{3}{2}k - 3k^2)t - (\frac{-1}{2} - \frac{k}{2} + k^2)t^2.$$
$$\text{Let } g(t) = (n-1)k + ((k-1)n + \frac{5}{2} + \frac{3}{2}k - 3k^2)t - (\frac{-1}{2} - \frac{k}{2} + k^2)t^2.$$
$$\text{Then } g'(t) = ((k-1)n + \frac{5}{2} + \frac{3}{2}k - 3k^2) - (-1 - k + 2k^2)t,$$

which implies that $g^{'}(t) = 0$ at

$$t = \frac{-(6k^2 - 3k - 2kn - 5 + 2n)}{2(2k^2 - k - 1)} \in \left[1, \left\lfloor \frac{n}{2k+1} \right\rfloor\right]$$

Since $g^{''}(t) = k + 1 - 2k^2 < 0$, $g(t)$ is concave down. Therefore $g(t)$ has its maximum at $t = \frac{-(6k^2-3k-2kn-5+2n)}{2(2k^2-k-1)}$. Thus,

$$\begin{aligned}
\mathcal{E}(G) &\leq g(t) \\
&= g\left(\frac{-(6k^2 - 3k - 2kn - 5 + 2n)}{2(2k^2 - k - 1)}\right) \\
&= -\frac{1}{8\left(-2k^2 + k + 1\right)}(36k^4 - 8k^3n - 52k^3 + 4k^2n^2 + 28k^2n - 43k^2 - 8kn^2 \\
&\quad +38k + 4n^2 - 20n + 25) \\
&= \frac{(k-1)n^2}{2(2k+1)} + \frac{1}{8\left(2k^2 - k - 1\right)}(36k^4 - 8k^3n - 52k^3 + 28k^2n - 43k^2 \\
&\quad +38k - 20n + 25).
\end{aligned}$$

But for $n \geq 6k + 3$,

$$\frac{1}{8\left(2k^2 - k - 1\right)}(36k^4 - 8k^3n - 52k^3 + 28k^2n - 43k^2 + 38k - 20n + 25) < 0.$$

Thus,

$$\begin{aligned}
\mathcal{E}(G) &< \frac{(k-1)n^2}{2(2k+1)} \\
&\leq \left\lfloor \frac{n^2}{4} \right\rfloor
\end{aligned}$$

This completes the proof. □

Theorem 2.2. *For $k \geq 4$ and $n \geq 36k$, let $G \in \mathcal{G}(n;\theta_{2k+1})$. Then*

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.$$

Furthermore, the bound is best possible.

Proof. For large n, let G be a graph on n vertices that contains no θ_{2k+1}-graph as a subgraph. If G has no cycle of length $2k+1$, then by Theorem 1.1 we have $\mathcal{E}(G) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$. Thus, in the rest of this theorem we consider the case where G has a cycle C of length $2k+1$. Define $R = G - C$. Observe that $\mathcal{E}(C) = 2k+1$ and from Theorem 2.1 we have $\mathcal{E}(R) \leq \left\lfloor \frac{(n-(2k+1))^2}{4} \right\rfloor$. By following word by word the same argument as in the proof of Theorem 2.1, one can show that for any vertex x of R, $\mathcal{E}(x, C) \leq k$. Moreover, if $B = \{x \in R : \mathcal{E}(x, C) = k\}$, then $|B| \leq 1$. Thus,

$$\begin{aligned}
\mathcal{E}(R, C) &\leq (k-1)\,|R| + 1 \\
&= (k-1)(n-(2k+1)) + 1 \\
&= nk - 2k^2 + k - n + 2. \\
\text{So, } \mathcal{E}(G) &= \mathcal{E}(R) + \mathcal{E}(R, C) + \mathcal{E}(C) \\
&\leq \left\lfloor \frac{(n-(2k+1))^2}{4} \right\rfloor + nk - 2k^2 + k - n + 2 + 2k + 1 \\
&\leq \frac{n^2 - 4k^2 - 6n + 16k + 13}{4} \\
&= \frac{n^2 - 6n + 13}{4} - \frac{4k^2 - 16k}{4}.
\end{aligned}$$

Since $k \geq 4$, we have

$$\begin{aligned}
\mathcal{E}(G) &\leq \frac{n^2 - 6n + 13}{4} \\
&\leq \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 1 \\
&< \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.
\end{aligned}$$

Note that the bound is achievable by the class $\mathcal{G}^*(n)$. This completes the proof of the theorem. $\square$

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