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# Lambda-fold theta graphs: metamorphosis into 6-cycles

Elizabeth J. Billington*, Abdollah Khodkar†,
Dylan Petrusma* and Matthew Sutton*
*School of Mathematics and Physics
The University of Queensland
Queensland 4072, Australia.
e-mail: *ejb@maths.uq.edu.au*
†Department of Mathematics
University of West Georgia
Carrollton, GA 30118, U.S.A.

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**Abstract**

An edge-disjoint decomposition of $\lambda K_n$ into copies of $\Theta(1,3,3)$ is found for all admissible orders $n$ and indices $\lambda$, having a metamorphosis into a $\lambda$-fold 6-cycle system of the same order.

**Keywords:**graph decomposition, metamorphosis, theta graph, 6-cycle.

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## 1. Introduction

The concept of a metamorphosis from a $G$-design into an $H$-design, where $H$ is a subgraph of $G$, is due to Lindner (see, e.g., [9, 11]). To explain this metamorphosis concept, we first need some definitions.

A ($\lambda$-fold) $G$-design of order $n$ is an edge-disjoint decomposition of a complete graph $K_n$ (or $\lambda K_n$) into isomorphic copies of the graph $G$. Here $\lambda K_n$ is used to denote a complete multigraph of order $n$, having $\lambda$ edges between each distinct pair of vertices. Sometimes the parameter $\lambda$ is referred to as the *index* of the $G$-design. If the graph $G$ is a cycle, say $G = C_m$, we sometimes refer to a $G$-design in this case as an *m-cycle system*.

Suppose that $H$ is a subgraph of $G$, and suppose also that the order $n$ and index $\lambda$ are such that it is (theoretically) possible for both a $\lambda$-fold $G$-design of order $n$ and a $\lambda$-fold $H$-design of order $n$ to exist. Letting $V$ denote the vertex set of $K_n$, and $B$ the collection of copies of edge-disjoint copies of $G$ in a $\lambda$-fold $G$-design, we write $(V, B)$ to denote this $\lambda$-fold $G$-design of order $|V| = n$.

Now suppose each copy of $G$ in $B$ is removed, and from each copy of $G$ a subgraph isomorphic to $H$ is retained, and placed in a set $C$. Then the edges from each copy of $G \setminus H$ are rearranged, if possible, into further copies of $H$ and placed in a set $C'$. This then forms a *metamorphosis* from a $\lambda$-fold $G$-design $(V, B)$ into a $\lambda$-fold $H$-design $(V, C \cup C')$.

There are now several papers dealing with metamorphoses of graph designs. See [1, 3, 4, 8, 9, 10] for instance, and [2] for an old survey of some metamorphoses cases when $G$ is $K_4$.

Here we concentrate on the case when $G$ is a theta graph, $\Theta(1,3,3)$. A *theta graph* $\Theta(a,b,c)$, $a \leqslant b \leqslant c$, is a connected graph with $a+b+c-1$ vertices, $a+b+c$ edges, having two vertices of degree 3 and the rest of degree 2, so that the two vertices of degree 3 are joined by three disjoint paths, of lengths $a$, $b$ and $c$. Thus $\Theta(1,3,3)$ can be regarded as a 6-cycle with an extra edge joining two vertices distance 3 apart round the cycle. We shall denote a theta graph $\Theta(1,3,3)$ on vertex set $\{1,2,3,4,5,6\}$ with edges $\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{1,6\},\{1,4\}\}$ by $[1,2,3;4,5,6]$ (or by any of $[1,6,5;4,3,2]$ or $[4,3,2;1,6,5]$ or $[4,5,6;1,2,3]$). The 6-cycle, obtained from this graph by removal of the edge $\{1,4\}$, will be denoted by $(1,2,3,4,5,6)$ or $(1,6,5,4,3,2)$ or any cyclic shift thereof.

Note that the graph $\Theta(1,2,2)$ is the same as the graph $K_4 - e$: a complete graph on four vertices with one edge removed. The problem of finding a metamorphosis from a $K_4 - e$-design into a 4-cycle system was dealt with in [12].

For theta graphs of the form $\Theta(1,k,k)$, removal of the single edge joining two vertices of degree 3 results in a cycle $C_{2k}$. Since [12] dealt with the case $k=2$, it is reasonable to consider graphs $\Theta(1,3,3)$; removal of their *diagonal* edge, that is, the single edge joining the two vertices of degree 3, results in a 6-cycle.

In the rest of this paper, we show that there exists a $\lambda$-fold $\Theta(1,3,3)$-design which has a metamorphosis into a $\lambda$-fold 6-cycle system, for all admissible orders $n$ for which both a $\Theta(1,3,3)$-design of order $n$ exists and a 6-cycle system of the same order $n$ exists. Note that the *spectrum* of a $G$-design of order $n$ is the set of integers $n$ for which a $G$-design exists.

## 2. $\Theta(1,3,3)$ to 6-cycle metamorphosis: necessary conditions

The necessary conditions for the existence of a $\lambda$-fold $\Theta(1,3,3)$-design of order $n$ are straightforward to determine. Since $\Theta(1,3,3)$ contains 6 vertices and 7 edges, the number of vertices $n$ must be at least 6, and the number of edges $\lambda\binom{n}{2}$ must be divisible by 7. (There is no degree requirement, since $\Theta(1,3,3)$ contains vertices of degrees 2 and 3.)

Similarly, necessary requirements for the existence of a $\lambda$-fold 6-cycle system of order $n$ are that $n \geqslant 6$, $\lambda\binom{n}{2} \equiv 0 \pmod 6$, and the degree $\lambda(n-1)$ must be even.

We tabulate these necessary requirements in Table 1.

$C_6$ spectrum

| $\lambda$ (mod 6) | order $n$ ($n \geqslant 6$) |
|---|---|
| 1, 5 | 1 or 9 (mod 12) |
| 2, 4 | 0 or 1 (mod 3) |
| 3 | 1 (mod 4) |
| 6 | any $n \geqslant 6$ |

$\Theta(1,3,3)$ spectrum

| $\lambda$ (mod 7) | order $n$ ($n \geqslant 6$) |
|---|---|
| 1 | 0 or 1 (mod 7), $n > 7$ |
| 2,3,4,5,6 | 0 or 1 (mod 7), $n \geqslant 7$ |
| 7 | any $n \geqslant 6$ |

Expected Spectrum for both $\Theta(1,3,3)$ and $C_6$

| $\lambda$ (mod 42) | order $n$ |
|---|---|
| **1**, 5,11,13,17,19,23,25,29,31,37,41 | 1, 21, 49, 57 (mod 84) |
| **2**, 4,8,10,16,20,22,26,32,34,38,40 | 0, 1, 7, 15 (mod 21) |
| **3**, 9,15,27,33,39 | 1, 21 (mod 28) |
| **6**, 12,18,24,30,36 | 0, 1 (mod 7) |
| **7**, 35 | 1, 9 (mod 12) |
| **14**, 28 | 0, 1 (mod 3) |
| **21** | 1 (mod 4) |
| **42** | any $n \geqslant 6$ |

Table 1. Necessary requirements

It suffices to deal with the bold-faced values of $\lambda$ in Table 1; other higher values of $\lambda$ arise from appropriate multiples of $\lambda$-fold designs.

## 3. Construction and the case $\lambda = 1$

Blinco [6] showed that for index 1, a $\Theta(1,3,3)$-design exists for all orders 0 or 1 (mod 7) with $n > 7$; there is no $\Theta(1,3,3)$-design of order 7. Existence of 6-cycle systems of index $\lambda$ have been known for a long time [13]; the $\lambda$-fold case (for all cycle lengths) also appears in [7].

### 3.1. Some necessary examples

We begin with some crucial building blocks. In the following examples, note that when elements are subscripted, these subscripts are always fixed when starter cycles are taken modulo some integer.

**Example 3.1.** *A $\Theta(1,3,3)$-design of order 21 with a metamorphosis into a 6-cycle system.*

We take the vertex set $\{i_j \mid i \in \mathbb{Z}_3,\ 1 \leqslant j \leqslant 7\}$, and the following ten starter blocks, cycled (mod 3):

$[1_4, 2_4, 2_3; 1_6, 0_6, 2_7]$ $[0_5, 0_4, 2_5; 2_7, 0_3, 2_4]$ $[2_4, 1_1, 1_5; 0_6, 0_3, 1_7]$ $[1_6, 2_2, 1_1; 2_7, 0_2, 0_3]$
$[1_4, 1_2, 2_1; 0_6, 2_2, 0_3]$ $[2_2, 1_2, 0_5; 2_5, 0_3, 0_7]$ $[2_1, 2_2, 1_4; 1_7, 1_1, 1_6]$ $[0_1, 1_1, 1_3; 1_5, 0_3, 2_3]$
$[0_2, 2_3, 1_1; 1_4, 2_1, 1_5]$ $[0_5, 0_6, 0_7; 1_7, 1_2, 1_6]$

The five 6-cycles (not cycled) obtained from the diagonal edges are:

$(0_1, 1_5, 1_2, 2_4, 1_6, 2_7), (1_1, 2_5, 2_2, 0_4, 2_6, 0_7), (2_1, 0_5, 0_2, 1_4, 0_6, 1_7), (0_4, 0_6, 2_4, 2_6, 1_4, 1_6),$
$(0_5, 1_7, 2_5, 0_7, 1_5, 2_7).$

**Example 3.2.** *A* $\Theta(1,3,3)$*-design of order* 49 *with a metamorphosis into a* 6*-cycle system.*

We take the vertex set $\{i_j \mid i \in \mathbb{Z}_7,\ 1 \leqslant j \leqslant 7\}$, and the following 24 starter blocks, cycled (mod 7).

$[0_5, 0_7, 0_2; 2_7, 6_6, 3_7]$ $[0_6, 0_1, 3_6; 2_1, 5_7, 3_1]$ $[0_5, 1_4, 0_7; 2_1, 0_3, 6_2]$
$[0_2, 1_7, 1_6; 2_6, 2_4, 2_3]$ $[0_3, 3_7, 1_4; 3_4, 6_7, 5_7]$ $[0_4, 4_1, 0_5; 3_5, 6_4, 0_7]$
$[0_3, 0_5, 6_4; 6_5, 3_6, 2_5]$ $[0_5, 6_6, 4_1; 6_3, 1_2, 4_7]$ $[0_3, 1_2, 1_4; 1_1, 4_3, 1_3]$
$[0_1, 2_2, 1_6; 3_2, 2_4, 4_5]$ $[0_3, 2_3, 4_4; 3_6, 1_7, 3_5]$ $[1_1, 2_1, 4_1; 5_2, 0_1, 0_2]$
$[0_1, 1_3, 0_4; 1_4, 6_1, 0_5]$ $[0_5, 1_1, 0_6; 3_6, 3_5, 1_5]$ $[0_2, 3_5, 1_2; 6_5, 2_3, 0_6]$
$[0_2, 3_2, 0_4; 3_4, 5_6, 2_4]$ $[0_2, 1_2, 5_5; 6_7, 2_4, 0_5]$ $[0_2, 5_2, 1_3; 3_6, 5_6, 4_3]$
$[0_2, 0_3, 6_7; 1_4, 4_3, 5_7]$ $[0_5, 2_3, 0_4; 5_6, 5_3, 2_6]$ $[0_4, 1_1, 0_7; 1_6, 0_2, 4_6]$
$[0_1, 5_4, 0_2; 4_7, 1_7, 2_5]$ $[0_1, 3_1, 0_3; 0_7, 3_3, 4_4]$ $[0_3, 0_1, 1_7; 6_6, 0_7, 2_7].$

The four starters (mod 7) for 6-cycles obtained from the diagonal edges are:

$(0_1, 3_2, 6_1, 5_3, 4_5, 1_4)$, $(0_1, 5_5, 6_2, 2_4, 1_2, 0_7)$, $(0_1, 5_6, 4_4, 1_3, 2_5, 4_7)$, $(0_2, 3_6, 4_3, 0_6, 4_5, 2_6)$.

**Example 3.3.** *A* $\Theta(1,3,3)$*-design of order* 57 *with a metamorphosis into a* 6*-cycle system.*

We take the vertex set $\{i_j \mid i \in \mathbb{Z}_{19},\ j = 1, 2, 3\}$, and the following twelve starter blocks, cycled (mod 19).

$[9_3, 16_3, 4_2; 18_3, 18_1, 17_3]$, $[6_2, 7_1, 5_2; 7_3, 18_1, 15_3]$, $[6_3, 4_3, 16_2; 12_3, 14_1, 7_3]$,
$[15_1, 6_2, 10_2; 7_3, 8_2, 0_3]$, $[1_1, 9_1, 0_3; 14_2, 3_3, 17_1]$, $[6_1, 8_3, 5_2; 3_2, 3_1, 9_3]$,
$[0_1, 1_1, 3_1; 7_1, 2_1, 3_2]$, $[0_1, 6_1, 15_1; 2_2, 7_1, 11_2]$, $[0_1, 5_2, 12_1; 8_2, 1_1, 14_3]$,
$[0_1, 9_2, 1_2; 1_3, 11_1, 7_3]$, $[0_2, 1_2, 4_2; 9_2, 3_2, 13_3]$, $[0_2, 12_2, 10_3; 6_3, 1_3, 4_3]$.

Two starter 6-cycles (mod 19) are obtained from the diagonal edges above:

$(0_1, 12_1, 9_2, 1_1, 2_3, 11_3)$, $(0_1, 2_2, 12_2, 13_3, 0_3, 13_2)$.

**Example 3.4.** *A* $\Theta(1,3,3)$*-design of order* 85 *with a metamorphosis into a* 6*-cycle system.*

We take the vertex set $\mathbb{Z}_{85}$, and the following six starter blocks, cycled (mod 85):

$[0, 6, 43; 1, 9, 41]$, $[0, 9, 23; 2, 27, 54]$, $[0, 10, 50; 3, 20, 55]$,
$[0, 11, 23; 4, 26, 39]$, $[0, 15, 33; 5, 70, 36]$, $[0, 16, 40; 7, 30, 56]$.

A single starter 6-cycle (mod 85), formed from the diagonal edges, is $(0, 1, 4, 11, 6, 2)$.

**Example 3.5.** *A* $\Theta(1,3,3)$*-decomposition of the complete* 5*-partite graph* $K_{21,21,21,21,21}$*, with a metamorphosis into a* 6*-cycle decomposition.*

We take the vertex set $\bigcup_{1\leqslant j\leqslant 5} \{i_j \mid i \in \mathbb{Z}_{21},\ j\}$, and the following 30 starter blocks, cycled (mod 21).

$$\begin{array}{lll}
[0_2,7_1,17_4;0_3,3_1,13_5] & [1_4,0_2,15_4;8_2,0_4,1_3] & [0_3,13_1,18_2;14_5,3_3,3_5]\\
[0_1,12_2,20_5;7_3,8_1,12_4] & [0_1,3_2,8_4;1_3,19_2,5_5] & [0_2,1_3,10_4;6_5,0_3,1_5]\\
[0_1,3_3,19_2;17_3,4_2,4_5] & [0_2,13_1,19_5;4_4,2_2,20_3] & [0_1,17_2,14_4;0_5,20_1,9_3]\\
[0_1,2_2,10_3;3_4,11_1,8_4] & [0_1,15_5,10_4;19_5,9_3,16_2] & [0_1,0_4,2_1;13_5,13_4,12_3]\\
[0_2,2_5,4_4;6_3,11_2,5_5] & [0_1,4_2,8_3;2_5,9_4,5_3] & [0_2,2_3,7_5;10_3,12_5,0_1]\\
[0_2,7_3,14_5;10_4,1_1,16_4] & [1_1,15_3,12_4;20_3,11_4,8_2] & [0_2,9_4,13_2;12_5,17_4,18_5]\\
[0_1,1_2,0_4;13_2,7_3,1_4] & [0_1,19_2,1_5;2_3,14_2,7_5] & [0_1,0_3,10_2;14_5,10_3,18_5]\\
[0_1,2_4,16_1;3_5,13_4,16_5] & [0_2,12_4,12_2;1_1,16_3,11_5] & [0_1,16_4,10_1;9_5,0_3,17_5]\\
[0_1,9_2,12_1;11_4,3_3,5_4] & [0_2,16_5,8_4;15_1,10_3,6_1] & [0_2,17_3,1_4;19_5,0_3,6_4]\\
[0_4,5_3,14_2;4_1,15_3,6_5] & [0_3,8_1,14_3;3_4,13_2,11_4] & [0_4,2_5,13_4;12_5,3_2,13_5].
\end{array}$$

The following starter 6-cycles (mod 21), are formed from the diagonal edges above:

$(0_1,17_3,11_2,15_4,1_2,13_5), (0_1,6_2,7_1,5_5,7_2,17_4), (0_1,13_2,2_3,16_5,4_4,1_3),$ $(0_1,2_3,4_1,15_4,12_1,0_5), (0_1,7_3,7_2,13_5,20_1,2_5)$.

**Example 3.6.** *A $\Theta(1,3,3)$-decomposition of the complete bipartite graph $K_{6,14}$, with a metamorphosis into a 6-cycle decomposition.*

We take the vertex set $\{a,b,c,d,e,f\} \cup \mathbb{Z}_{14}$, and the following twelve $\Theta(1,3,3)$ blocks:

$$\begin{array}{llll}
[c,10,d;1,f,11], & [d,12,b;4,c,13], & [a,6,c;0,d,7], & [b,6,e;0,f,7],\\
[c,7,e;2,a,8], & [d,6,f;3,b,8], & [f,8,e;4,a,9], & [e,9,c;3,a,10],\\
[b,9,d;2,f,10], & [a,11,b;1,e,12], & [e,11,d;5,a,13], & [f,12,c;5,b,13].
\end{array}$$

The diagonal edges from these blocks can be rearranged into the following 6-cycles: $(c,1,a,0,b,2)$, $(e,5,f,4,d,3)$.

Using the above examples, we can construct some further decompositions with metamorphoses which we shall need for the general construction. We present these as the following lemmas.

First, we remind the reader that a $G$-design of order $n$ having a hole of order $m$ is an edge-disjoint decomposition of the graph $K_n \setminus K_m$ into copies of the graph $G$.

**Lemma 3.7.** *There exists a $\Theta(1,3,3)$-design of order 105, and a $\Theta(1,3,3)$-design of order 105 with a hole of size 21, each having a metamorphosis into a 6-cycle system.*

*Proof.* Since $105 = 5 \times 21$, Examples 3.5 and 3.1 provide a suitable design and metamorphosis. (The five holes of order 21 are filled with copies of the design in Example 3.1.) □

**Lemma 3.8.** *There exists a $\Theta(1,3,3)$-decomposition of $K_{6a,14b}$ having a metamorphosis into a 6-cycle decomposition, for all $a,b \in \mathbb{N}$.*

*Proof.* This follows from Example 3.6 in the obvious way: use $ab$ copies of a decomposition and metamorphosis of $K_{6,14}$. □

The next subsections give some recursive constructions used to obtain appropriate metamorphoses for the expected orders (modulo 84) when the index $\lambda$ is 1.

### 3.2. Recursive constructions

**Lemma 3.9.** *There exists a* $\Theta(1,3,3)$*-design of order* $1 \pmod{84}$ *having a metamorphosis into a* 6*-cycle system.*

*Proof.* Take the vertex set $\{i_j \mid 1 \leqslant i \leqslant x,\ j \in \mathbb{Z}_{84}\} \cup \{\infty\}$. On each set $\{i_j \mid j \in \mathbb{Z}_{84}\} \cup \{\infty\}$, for $1 \leqslant i \leqslant x$, place a decomposition and metamorphosis of order 85 (see Example 3.4). Then on each of the $\binom{x}{2}$ sets $\{i_j \mid j \in \mathbb{Z}_{84}\} \cup \{i'_j \mid j \in \mathbb{Z}_{84}\}$, with $1 \leqslant i < i' \leqslant x$, place a copy of a decomposition with metamorphosis of $K_{84,84}$ (see Lemma 3.8). This gives a suitable design and metamorphosis of order $84x+1$. □

**Lemma 3.10.** *There exists a* $\Theta(1,3,3)$*-design of order* $21 \pmod{84}$ *having a metamorphosis into a* 6*-cycle system.*

*Proof.* Take the vertex set $\{i_j \mid 1 \leqslant i \leqslant x,\ j \in \mathbb{Z}_{84}\} \cup \{X_k \mid 1 \leqslant k \leqslant 21\}$. On the vertex set $\{1_j \mid j \in \mathbb{Z}_{84}\} \cup \{X_k \mid 1 \leqslant k \leqslant 21\}$, we place a copy of Example 3.7 of order 105. Then on the $x-1$ sets $\{i_j \mid j \in \mathbb{Z}_{84}\} \cup \{X_k \mid 1 \leqslant k \leqslant 21\}$, for each $i$ with $2 \leqslant i \leqslant x$, we place a copy of a design with metamorphosis of order 105 having a hole of order 21 (where the hole vertices are naturally $\{X_k \mid 1 \leqslant k \leqslant 21\}$). Finally, on the $\binom{x}{2}$ sets $\{i_j \mid j \in \mathbb{Z}_{84}\} \cup \{i'_j \mid j \in \mathbb{Z}_{84}\}$, with $1 \leqslant i < i' \leqslant x$, we place a copy of a decomposition with metamorphosis of $K_{84,84}$ (see Lemma 3.8). This gives a suitable design and metamorphosis of order $84x+21$. □

**Lemma 3.11.** *There exists a* $\Theta(1,3,3)$*-design of order* $49 \pmod{84}$ *having a metamorphosis into a* 6*-cycle system.*

*Proof.* Let the vertex set, of order $84x+49 = 28(3x+1)+21$, be $\{i_j \mid 1 \leqslant i \leqslant 3x+1,\ 1 \leqslant j \leqslant 28\} \cup H$, where $H = \{X_k \mid 1 \leqslant k \leqslant 21\}$. The component parts for a design and metamorphosis are:

(i) A copy of Example 3.2 on the vertex set $\{(3x+1)_j \mid 1 \leqslant j \leqslant 28\} \cup H$.

(ii) A copy of Example 3.7 on each of the vertex sets $\{i_j, (i+1)_j, (i+2)_j \mid 1 \leqslant j \leqslant 28\} \cup H$, with hole $H$, where $i \in \{3y-2 \mid y = 1, 2, \ldots x\}$.

(iii) Precisely $x$ copies of a decomposition and metamorphosis of $K_{28,84}$ on the sets $\{(3x+1)_j \mid 1 \leqslant j \leqslant 28\} \cup \{i_j, (i+1)_j, (i+2)_j \mid 1 \leqslant j \leqslant 28\}$ for each $i$ with $i \in \{3y-2 \mid y = 1, 2, \ldots x\}$.

(iv) Finally, $\binom{x}{2}$ copies of a decomposition and metamorphosis of $K_{84,84}$ on the sets $\{i_j, (i+1)_j, (i+2)_j \mid 1 \leqslant j \leqslant 28\} \cup \{i'_j, (i'+1)_j, (i'+2)_j \mid 1 \leqslant j \leqslant 28\}$, for $1 \leqslant i < i' \leqslant x$.

This completes the construction in this case. □

**Lemma 3.12.** *There exists a* $\Theta(1,3,3)$*-design of order* 57 (mod 84) *having a metamorphosis into a* 6*-cycle system.*

*Proof.* Let the vertex set of order $84x + 57 = 28(3x+2) + 1$ be $\{i_j \mid 1 \leqslant i \leqslant 3x+2,\ 1 \leqslant j \leqslant 28\} \cup \{\infty\}$. The component parts for a design and metamorphosis are:

(i) A copy of Example 3.3 of order 57 on the vertex set $\{(3x+1)_j,\ (3x+2)_j \mid 1 \leqslant j \leqslant 28\} \cup \{\infty\}$.

(ii) A copy of Example 3.4 of order 85 on each of the $x$ vertex sets $\{i_j, (i+1)_j, (i+2)_j \mid 1 \leqslant j \leqslant 28\} \cup \{\infty\}$, for each $i$ with $i \in \{3y-2 \mid y = 1, 2, \ldots x\}$.

(iii) Precisely $x$ copies of a decomposition and metamorphosis of $K_{56,84}$ on the sets $\{(3x+1)_j,\ (3x+2)_j \mid 1 \leqslant j \leqslant 28\} \cup \{i_j, (i+1)_j, (i+2)_j \mid 1 \leqslant j \leqslant 28\}$ for each $i$ with $i \in \{3y-2 \mid y = 1, 2, \ldots x\}$.

(iv) Repeat case (iv) above, as in Lemma 3.11.

This completes the construction in this case. □

We summarise the results of this section in the following theorem, which follows immediately from the previous four lemmas.

**Theorem 3.13.** *There exists a* $\Theta(1,3,3)$*-design of order* $n$ *with a metamorphosis into a* 6*-cycle system of order* $n$ *if and only if* $n \geqslant 21$ *and* $n \equiv 1, 21, 49$ *or* 57 (mod 84).

## 4. Higher indices $\lambda$

### 4.1. The case $\lambda = 2$

The order is $0, 1, 7$ or 15 (mod 21). Examples of orders 7 and 15 appear in the Appendix.

**Example 4.1.** *A decomposition and metamorphosis of* $2K_{3,7}$*:*

With the vertex set $\{a, b, c\} \cup \{1, 2, 3, 4, 5, 6, 7\}$, take six $\Theta(1,3,3)$ graphs:

$(a, 2, b; 1, c, 4)$, $(b, 4, a; 1, c, 6)$, $(a, 2, c; 3, b, 7)$,
$(c, 7, a; 3, b, 4)$, $(b, 2, c; 5, a, 6)$, $(c, 7, b; 5, a, 6)$.

The diagonal edges, when removed, form a further 6-cycle for the metamorphosis: $(a, 1, b, 5, c, 3)$.

The following corollaries are now immediate.

**Corollary 4.2.** *There is a decomposition into* $\Theta(1,3,3)$ *graphs, with a metamorphosis into a* 6*-cycle system, of* $2K_{3a,7b}$*, for all* $a, b \in \mathbb{N}$.

**Corollary 4.3.** *There exists a decomposition of* $2K_{22}$ *into* $\Theta(1,3,3)$ *with a metamorphosis into a* 2*-fold* 6*-cycle system.*

*Proof.* Let the vertex set be $A \cup B$ where $A = \{a_i \mid 1 \leqslant i \leqslant 15\}$ and $B = \{b_i \mid 1 \leqslant i \leqslant 7\}$. On $A$, place a decomposition and metamorphosis of $2K_{15}$ (see the Appendix); on $B$, place a decomposition and metamorphosis of $2K_7$ (also in the Appendix). Then use five copies of $2K_{3,7}$ with $A$ partitioned into five parts, and $B$ as the part of size 7. □

Now we take the vertex set $X \cup \{i_j \mid 1 \leqslant i \leqslant 3x + \epsilon,\ 1 \leqslant j \leqslant 7\}$, where $\delta = |X|$, with $X = \{\infty\}$ if $\delta = 1$, $X = \emptyset$ if $\delta = 0$, and where $(\epsilon, \delta) = (0,0), (0,1), (1,0), (2,1)$, according as the order is $0, 1, 7$ or 15 (mod 21), respectively.

**The construction, order 0 (mod 21)**

(i) Use a copy of $2K_{21}$ on the sets $\{(3i-2)_j, (3i-1)_j, 3i_j \mid 1 \leqslant j \leqslant 7\}$ for $i = 1, 2, \ldots, x$.

(ii) Use a copy of $2K_{21,21}$ on the sets $\{(3i-2)_j, (3i-1)_j, 3i_j \mid 1 \leqslant j \leqslant 7\} \cup \{(3i'-2)_j, (3i'-1)_j, 3i'_j \mid 1 \leqslant j \leqslant 7\}$ for $\binom{x}{2}$ values $1 \leqslant i < i' \leqslant 3x$.

**The construction, order 1 (mod 21)**

(i) Use a copy of $2K_{22}$ on the sets $\{(3i-2)_j, (3i-1)_j, 3i_j \mid 1 \leqslant j \leqslant 7\} \cup \{\infty\}$ for $i = 1, 2, \ldots, x$.

(ii) Same as (ii) above.

**The construction, order 7 (mod 21)**

(i) Use $2K_7$ once on $\{(3x+1)_j \mid 1 \leqslant j \leqslant 7\}$. Then use $2K_{21}$ on the sets $\{(3i-2)_j, (3i-1)_j, 3i_j \mid 1 \leqslant j \leqslant 7\}$, for $i = 1, 2, \ldots, x$.

(ii) Same as (ii) above.

(iii) Use $2K_{21,7}$ precisely $x$ times, on $\{(3i-2)_j, (3i-1)_j, 3i_j \mid 1 \leqslant j \leqslant 7\} \cup \{(3x+1)_j \mid 1 \leqslant j \leqslant 7\}$, for $i = 1, 2, \ldots, x$.

**The construction, order 15 (mod 21)**

(i) Use $2K_{15}$ on $\{\infty\} \cup \{i_j \mid i = 3x+1, 3x+2,\ 1 \leqslant j \leqslant 7\}$. Then use $2K_{22}$ on $\{\infty\} \cup \{(3i-2)_j, (3i-1)_j, 3i_j \mid 1 \leqslant j \leqslant 7\}$, for each $i$ with $i = 1, 2, \ldots, x$.

(ii) Same as (ii) above.

(iii) Use $2K_{21,14}$ precisely $x$ times, on
$\{(3i-2)_j, (3i-1)_j, 3i_j \mid 1 \leqslant j \leqslant 7\} \cup \{(3x+1)_j, (3x+2)_j \mid 1 \leqslant j \leqslant 7\}$,
for $i = 1, 2, \ldots, x$.

This completes the constructions for the case $\lambda = 2$.

For the remaining values of $\lambda$, since the constructions are similar, we simply list the "ingredients" we used.

### 4.2. The case $\lambda = 3$

The order is 1 or 21 (mod 28). From a decomposition and metamorphosis of $K_{21}$, taking three copies gives us one of $3K_{21}$. We also have decompositions and metamorphoses of $3K_{29}$ and $3K_{4,14}$ (see the Appendix for both these).

For <u>order $28x+1$</u> we use $x$ copies of $3K_{29}$, and $\binom{x}{2}$ lots of $3K_{28,28}$ (which comes from $7 \times 2$ appropriate copies of $3K_{4,14}$).

For <u>order $28x+21$</u> we use one copy of $3K_{21}$ (which arises from three copies of Example 3.1), $x$ copies of $3K_{29}$, $\binom{x}{2}$ lots of $3K_{28,28}$, and $x$ lots of $3K_{20,28}$ (which arises from taking $5 \times 2$ lots of $3K_{4,14}$).

### 4.3. The case $\lambda = 6$

The order is 0 or 1 (mod 7).

See the Appendix for orders 7 and 8. Now from order 21 when $\lambda = 1$ (see Example 3.1), we obtain a decomposition and metamorphosis of $6K_{21}$. Also from order 22 when $\lambda = 2$ we obtain a decomposition and metamorphosis of $6K_{22}$.

Now for order 14, with vertex set $\{\infty\} \cup \mathbb{Z}_7 \cup \{a_1, b_1, c_1\} \cup \{a_2, b_2, c_2\}$, we take 6-fold designs with metamorphoses on the sets: $\{\infty\} \cup \mathbb{Z}_7$ (of order 8); $\{\infty\} \cup \{a_1, b_1, c_1\} \cup \{a_2, b_2, c_2\}$ (of order 7); and on $\{a_i, b_i, c_i\} \cup \mathbb{Z}_7$ we use $6K_{3,7}$, for each $i = 1, 2$.

For order 15 we may treble the case of order 15 when $\lambda = 2$; see the Appendix.

We also have $2K_{3,7}$ (Example 4.1) from which we obtain ones of $6K_{3,7}$, $6K_{21,7}$ and $6K_{21,21}$.

Now for order $7x$ and $7x+1$ when $x \geqslant 3$, we tabulate the "ingredients" we can use:

| | order $7x$ | order $7x+1$ |
|---|---|---|
| $x \equiv 0 \pmod 3$ | $6K_{21}$, $6K_{21,21}$ | $6K_{22}$, $6K_{21,21}$ |
| $x \equiv 1 \pmod 3$ | $6K_7$, $6K_{21}$, $6K_{21,7}$, $6K_{21,21}$ | $6K_8$, $6K_{22}$, $6K_{21,7}$, $6K_{21,21}$ |
| $x \equiv 2 \pmod 3$ | $6K_{14}$, $6K_{21}$, $6K_{21,14}$, $6K_{21,21}$ | $6K_{15}$, $6K_{22}$, $6K_{21,14}$, $6K_{21,21}$ |

### 4.4. The case $\lambda = 7$

The order is 1 or 9 (mod 12).

For <u>order $12x+1$</u>, we use $7K_{13}$ and $7K_{6,6}$. (Both of these are in the Appendix.)

For <u>order $12x+9$</u>, we use $7K_9$ (in the Appendix), $7K_{13}$, $7K_{6,6}$ and $7K_{4,6}$ (see the Appendix).

### 4.5. The case $\lambda = 14$

The order is 0 or 1 (mod 3). The case $14K_6$ appears in the Appendix. Now from $2K_7$ (see Appendix) we obtain a decomposition and metamorphosis of $14K_7$. And $14K_9$ comes from two copies of $7K_9$ in the Appendix, while $14K_{10}$ appears in the Appendix.

So when the order is $6x$, we can use a decomposition and metamorphosis of $14K_6$ and $14K_{6,6}$ (this latter comes from two copies of $7K_{6,6}$).

When the order is $6x + 1$, we use $14K_7$ and $14K_{6,6}$.

When the order is $6x + 3$ we use $14K_9$ (once), $14K_7$ ($x - 1$ times), $14K_{4,6}$ ($2(x - 1)$ times) and $14K_{6,6}$ ($\binom{x-1}{2}$ times).

Finally, when the order is $6x+4$, we use $14K_{10}$ (once), $14K_6$ ($x-1$ times), $14K_{4,6}$ ($x-1$ times) and $14K_{6,6}$ ($\binom{x}{2}$ times).

### 4.6. The case $\lambda = 21$

The order is 1 (mod 4).

By using $\lambda = 7$, we only need consider orders 5 (mod 12). We use $21K_{17}$ (see below), $21K_{13}$ (from the $\lambda = 7$ case), and $21K_{4,4}$ (see Appendix).

For $21K_{17}$, we take vertex set $\{\infty\} \cup \{a_i, b_i, c_i, d_i \mid i = 1, 2, 3, 4\}$. On the sets $\{\infty\} \cup \{a_i, b_i \mid i = 1, 2, 3, 4\}$ and $\{\infty\} \cup \{c_i, d_i \mid i = 1, 2, 3, 4\}$, we place designs and metamorphoses of $21K_9$ (using three copies of $7K_9$). Then on $\{x_i \mid i = 1, 2, 3, 4\} \cup \{y_i \mid i = 1, 2, 3, 4\}$, for $\{x, y\} = \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}$, we place decompositions and metamorphoses of $21K_{4,4}$.

### 4.7. The case $\lambda = 42$

Here any order greater than or equal to 6 is possible. The orders 0 or 1 (mod 3) follow from the case $\lambda = 14$ by taking three copies. So we consider just orders 2 (mod 3).

For order $6x + 8$ we use $6K_8$, $6K_7$, and $7K_{4,6}$; these all appear in the Appendix.

For order $6x + 5$ we use $42K_{11}$, $14K_6$, $6K_7$, $7K_{4,6}$ and $7K_{6,6}$; these all appear in the Appendix.

## 5. Concluding comments

A $\Theta(1, 3, 3)$ design of order $v$ and index $\lambda$, with a metamorphosis to a $\lambda$-fold 6-cycle system of order $v$, exists for all values given in Table 1 (see Section 2). For values of $\lambda$ not directly considered above, taking appropriate multiple copies of designs with smaller values of $\lambda$ will suffice.

More generally, partial work on existence of $\Theta(1, k, k)$-designs appears in [6] and [5], for some cases when $k$ is odd. Further work on existence of these designs is underway by the fourth author, to partially complete the spectrum. So a natural progression is now to find a metamorphosis from a $\Theta(1, k, k)$-design to a $2k$-cycle system; needless to say, this is a hard problem!

## Appendix

$\boxed{\lambda = 2;\text{ order } 7}$

Vertex set $\{\infty\} \cup \mathbb{Z}_6$. One starter block mod 6: $[0, 5, \infty\,; 1, 4, 2]$; a further 6-cycle from the diagonal edges is $(0, 1, 2, 3, 4, 5)$.

$\boxed{\lambda = 2;\text{ order } 15}$

Vertex set $\mathbb{Z}_{15}$; two starters: $[0, 4, 9; 3, 11, 14]$, $[0, 7, 3; 2, 4, 9]$ (mod 15). Then five 6-cycles can be formed from the diagonal edges:

$(0, 3, 5, 8, 10, 13), (1, 4, 6, 7, 11, 14), (2, 5, 7, 8, 12, 0), (3, 6, 8, 9, 13, 1), (4, 7, 9, 10, 14, 2)$.

$\boxed{\lambda = 3;\text{ order } 29}$

Vertex set $\mathbb{Z}_{29}$; six starters mod 29:

$[0, 1, 4; 27, 26, 22]$, $[0, 2, 8; 3, 1, 24]$, $[0, 9, 22; 25, 16, 3]$,
$[0, 8, 23; 5, 26, 11]$, $[0, 10, 18; 6, 25, 11]$, $[0, 10, 26; 9, 16, 17]$.

One 6-cycle starter (mod 29) is formed from the diagonal edges: $(0, 2, 5, 9, 14, 20)$ (mod 29).

$\boxed{\lambda = 3;\ 3K_{4,14}}$

With the vertex set $\{a, b, c, d\} \cup \{1, 2, \ldots, 14\}$, take the following 24 blocks:

| | | | |
|---|---|---|---|
| $[a, 2, d; 1, c, 5]$ | $[b, 2, c; 1, a, 3]$ | $[b, 1, d; 2, a, 4]$ | $[c, 1, a; 2, b, 3]$ |
| $[a, 5, b; 3, d, 6]$ | $[c, 2, d; 3, a, 4]$ | $[a, 6, b; 4, d, 7]$ | $[c, 3, d; 4, b, 6]$ |
| $[c, 4, d; 5, b, 7]$ | $[d, 1, b; 5, c, 7]$ | $[d, 7, b; 6, c, 8]$ | $[a, 5, d; 6, c, 7]$ |
| $[d, 10, b; 14, a, 13]$ | $[c, 12, d; 14, b, 13]$ | $[c, 11, d; 13, a, 14]$ | $[b, 12, c; 13, d, 14]$ |
| $[d, 9, a; 12, c, 10]$ | $[b, 11, d; 12, a, 13]$ | $[d, 8, a; 11, c, 9]$ | $[b, 9, c; 11, a, 12]$ |
| $[b, 8, c; 10, a, 11]$ | $[a, 8, b; 10, c, 14]$ | $[a, 7, b; 9, c, 8]$ | $[d, 8, b; 9, a, 10]$. |

There are four 6-cycles from the diagonal edges:
$(a, 1, b, 2, c, 3), (a, 4, c, 5, d, 6), (c, 13, b, 12, d, 14), (a, 9, d, 11, b, 10)$.

$\boxed{\lambda = 6;\text{ order } 7}$

Take the vertex set $\{\infty\} \cup \mathbb{Z}_6$. We have 18 $\Theta(1, 3, 3)$ graphs, from three starters mod 6: $(2, \infty, 4; 0, 5, 3), (5, \infty, 3; 0, 1, 4), (3, \infty, 2; 0, 1, 5)$. There are 18 diagonal edges, which can be formed into three new 6-cycles for the metamorphosis:
$(0, 1, 5, 2, 3, 4), (0, 2, 1, 4, 5, 3), (0, 3, 1, 4, 2, 5)$.

$\lambda = 6$; order 8

With vertex set $\mathbb{Z}_8$ we take three starters mod 8:
$[0, 3, 4; 1, 5, 2], [0, 4, 5; 2, 3, 6], [0, 4, 6; 2, 3, 1]$.
Then the diagonal edges use differences 1 once and 2 twice; these 24 edges rearrange into four 6-cycles as follows:
$(0, 1, 7, 5, 4, 2), (1, 2, 0, 6, 5, 3), (2, 3, 1, 7, 6, 4), (3, 4, 6, 0, 7, 5)$.

$\lambda = 7$; $7K_{4,6}$

With vertex set $\mathbb{Z}_4 \cup \{a, b, c, d, e, f\}$, we take six starters mod 4:
$[0, a, 1; b, 2, c], [0, b, 1; c, 2, a], [0, c, 1; a, 2, b], [0, d, 1; e, 2, f], [0, e, 1; f, 2, d], [0, f, 1; d, 2, e]$.

Then there are four 6-cycles from the diagonal edges, to complete the metamorphosis:

$(0, a, 1, b, 2, c), (3, a, 2, d, 1, c), (3, b, 0, e, 1, f), (0, d, 3, e, 2, f)$.

$\lambda = 7$; $7K_{6,6}$

With vertex set $\mathbb{Z}_6 \cup \{a, b, c, d, e, f\}$, we take six starters mod 6:
$[0, a, 1; b, 2, c], [0, b, 1; c, 2, a], [0, c, 1; a, 2, b], [0, d, 1; e, 2, f], [0, e, 1; f, 2, d], [0, f, 1; d, 2, e]$.

Then the diagonal edges, 36 in number, precisely cover the graph $K_{6,6}$, which easily has a decomposition into 6-cycles. (For example, see Sotteau [14].)

$\lambda = 7$; order 9

Take vertex set $\mathbb{Z}_9$; we have four starters mod 9:
$[0, 2, 5; 1, 6, 8], [0, 1, 4; 2, 8, 5], [0, 4, 2; 3, 5, 1], [0, 6, 5; 4, 2, 6]$.

Then the diagonal edges form six 6-cycles: $(0, 1, 3, 6, 7, 4), (1, 2, 4, 8, 3, 5)$, $(2, 3, 4, 6, 0, 8), (1, 4, 5, 0, 7, 6), (0, 2, 6, 5, 7, 3), (1, 7, 2, 5, 8, 6)$.

$\lambda = 7$; order 13

Take vertex set $\mathbb{Z}_{13}$, and six starters mod 13:

$$[0, 2, 6; 1, 12, 8], \quad [0, 1, 11; 4, 3, 6], \quad [0, 1, 3; 11, 10, 7],$$
$$[0, 12, 2; 8, 9, 6], \quad [0, 11, 2; 7, 9, 5], \quad [0, 2, 11; 3, 1, 5].$$

The diagonal edges form the following starter 6-cycle (mod 13): $(0, 1, 3, 6, 2, 7)$.

$\lambda = 14$; order 6

Vertex set $\{\infty\} \cup \mathbb{Z}_6$. We have 6 starters mod 5:

$$[\infty, 3, 1; 0, 2, 4], \quad [\infty, 3, 2; 1, 0, 4], \quad [0, \infty, 1; 2, 3, 4],$$
$$[2, \infty, 3; 4, 1, 0], \quad [4, \infty, 0; 3, 1, 2], \quad [1, \infty, 0; 3, 2, 4].$$

Then the diagonal edges, when removed, form one starter 6-cycle mod 5:
$(1, \infty, 0, 2, 4, 3)$.

$\boxed{\lambda = 14;\ \text{order } 10}$

Vertex set $\{i_j \mid i \in \mathbb{Z}_5,\ j = 1, 2\}$. Take the following 18 starter blocks:

$$\begin{array}{llll}
[0_1, 1_1, 2_1; 3_1, 4_1, 0_2] & [0_1, 1_1, 2_1; 3_1, 4_1, 1_2] & [0_1, 1_1, 2_1; 3_1, 4_1, 2_2] & [0_1, 1_1, 2_1; 0_2, 4_1, 3_2] \\
[0_1, 2_1, 4_1; 0_2, 1_2, 4_2] & [0_1, 2_1, 4_1; 0_2, 2_2, 3_1] & [0_1, 2_1, 4_1; 0_2, 2_2, 1_2] & [0_1, 2_1, 4_1; 0_2, 2_2, 3_2] \\
[0_1, 4_2, 1_1; 3_2, 4_1, 2_2] & [0_1, 0_2, 4_1; 1_2, 3_2, 2_2] & [0_1, 0_2, 4_1; 1_2, 3_2, 4_2] & [0_1, 0_2, 4_1; 2_2, 4_2, 3_2] \\
[0_1, 0_2, 1_2; 4_2, 1_1, 3_2] & [0_1, 0_2, 1_2; 4_2, 2_1, 2_2] & [0_1, 0_2, 1_2; 4_2, 2_2, 3_2] & [0_1, 3_2, 1_1; 4_2, 0_2, 2_2] \\
[0_1, 4_2, 3_1; 2_2, 1_2, 2_1] & [0_2, 0_1, 2_1; 1_2, 3_2, 2_2] & &
\end{array}$$

Three 6-cycle starters (mod 5) with the same differences as the differences produced by the diagonals of the above 18 starter blocks:

$(0_1, 2_1, 4_1, 1_1, 0_2, 1_2)$, $(0_1, 0_2, 4_1, 2_2, 2_1, 4_2)$, $(0_1, 4_2, 4_1, 1_2, 1_1, 0_2)$.

$\boxed{\lambda = 21;\ 21K_{4,4}}$

Vertex set $\mathbb{Z}_4 \cup \{a, b, c, d\}$, and 12 starter blocks mod 4:

$$\begin{array}{lllll}
[0, a, 1; b, 2, c], & [0, c, 1; a, 2, b], & [0, b, 1; c, 2, a], & [0, b, 1; c, 2, d], & [0, d, 1; b, 2, c], \\
[0, c, 1; d, 2, b] & [0, c, 1; d, 2, a], & [0, a, 1; c, 2, d], & [0, d, 1; a, 2, c], & [0, d, 1; a, 2, b], \\
[0, b, 1; d, 2, a], & [0, a, 1; b, 2, d]. & & &
\end{array}$$

The diagonal edges here precisely cover $3K_{4,4}$, which has a simple decomposition into 6-cycles; they may be taken as follows (mod 4): $(0, a, 1, b, 2, c)$, $(0, b, 2, d, 1, c)$.

$\boxed{\lambda = 42;\ \text{order } 11}$

Vertex set is $\mathbb{Z}_{11}$. We take the following 30 starters, mod 11:

$$\begin{array}{llll}
[0, 1, 2; 3, 4, 5] & [0, 1, 2; 3, 4, 6] & [0, 1, 2; 3, 4, 7] & [0, 1, 2; 3, 4, 8] \\
[0, 1, 2; 3, 4, 9] & [0, 1, 2; 3, 4, 10] & [0, 1, 2; 3, 5, 4] & [0, 1, 2; 3, 5, 6] \\
[0, 1, 2; 3, 5, 7] & [0, 1, 2; 3, 5, 9] & [0, 1, 2; 4, 7, 3] & [0, 2, 4; 6, 8, 5] \\
[0, 2, 4; 6, 9, 3] & [0, 2, 4; 6, 9, 5] & [0, 2, 4; 6, 9, 7] & [0, 2, 4; 6, 10, 3] \\
[0, 2, 4; 6, 10, 7] & [0, 2, 4; 6, 10, 8] & [0, 2, 4; 7, 1, 3] & [0, 2, 4; 7, 1, 5] \\
[0, 2, 4; 7, 1, 8] & [0, 2, 4; 7, 3, 6] & [0, 3, 7; 4, 9, 5] & [0, 3, 7; 4, 9, 6] \\
[0, 3, 7; 4, 10, 6] & [0, 3, 8; 5, 1, 4] & [0, 3, 8; 5, 1, 7] & [0, 3, 9; 4, 10, 6] \\
[0, 4, 10; 6, 1, 5] & [0, 4, 10; 6, 2, 7] & &
\end{array}$$

The five 6-cycle starters, with the same differences as the differences produced by the diagonals of the above 30 theta starters, mod 11:

$(0, 3, 6, 9, 1, 5)$ $(0, 3, 6, 2, 9, 4)$ $(0, 4, 8, 2, 9, 5)$ $(0, 4, 9, 6, 3, 8)$ $(0, 6, 1, 7, 2, 5)$.

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