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Restricted domination in arc-colored digraphs

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Abstract

Let $H = (V(H), A(H))$ be a digraph possibly with loops and $D = (V(D), A(D))$ a digraph whose arcs are colored with the vertices of H (this is what we call an H-colored digraph); i.e. there exists a function $c\colon A(D) \to V(H)$; for an arc of D, $f = (u, v) \in A(D)$, we call $c(f) = c(u, v)$ the color of f. A directed walk (directed path) $P = (u_0, u_1, \ldots, u_n)$ in D will be called an H-walk (H-path) whenever $(c(u_0, u_1), c(u_1, u_2), \ldots, c(u_{n-2}, u_{n-1}), c(u_{n-1}, u_n))$ is a directed walk (directed path) in H. We introduce the concept of H-kernel N, as a generalization of the two properties that define a kernel (Recall that a kernel N of a digraph D is a set of vertices $N \subseteq V(D)$ which is independent and for each $x \in V(D) - N$, there exists an xN-arc in D). A set $N \subseteq V(D)$ is called *H-independent* whenever for every two different vertices $x, y \in N$ there is no H-path between them, and N is called *H-absorbent* whenever for each $x \in V(D) - N$ there exists a vertex $y \in N$ and an xy-H-path in D. The set $N \subseteq V(D)$ will be called H-kernel if and only if it is H-independent and H-absorbent.

This new concept generalizes the concepts of kernel, kernel by monochromatic paths and kernel by alternating paths.

In this paper we show sufficient conditions for an infinite digraph to have an H-kernel.

Keywords: kernel, domination, arc-coloration.

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1. Introduction

The concept of H-path was first introduced by Linek and Sands [7] and was later used by Arpin and Linek [1] to work on three classes of digraphs: $\mathfrak{B}_3$, the class of all H such that any H-colored multidigraph D has a set of vertices S that is both H-independent by walks ($S \subseteq V(D)$ is H-independent by walks whenever for every two different vertices $x, y \in S$ there is no H-walk between them) and H-absorbent by walks. (A set $S \subseteq V(D)$ is H-absorbent by walks whenever for each $x \in V(D) - S$, there exists $s \in S$ and an xy-H-walk in D). $\mathfrak{B}_2$, the class of all H such that any H-colored multidigraph D has an independent set of vertices that is H-absorbent by walks; and the class $\mathfrak{B}_1$ of all H such that any H-colored tournament has a single vertex H-absorbent by walks.

In the same spirit of the definitions given by Arpin and Linek, we could define a new class: $\widehat{\mathfrak{B}}_3$, of all H such that any H-colored digraph D has an H-kernel. In this paper we do not try to characterize this class, not only because it may be a difficult task but rather because we found another attractive perspective: that of actually characterizing the H-colored digraphs D with an H-kernel (a task that might be as difficult as the one mentioned before). So our main results focus on properties about the H-coloration of D that guarantee the existence of an H-kernel in D.

In order to prove the existence of H-kernels in infinite H-colored digraphs, throughout this paper we gather results about infinite H-colored digraphs and use a new concept that we call H semikernel.

It is important to mention that there are two obstacles one can anticipate within our problem. On one side, the fact that the existence of an H-walk between two vertices does not guarantee the existence of an H-path between those vertices. And on the other, that the concatenation of two H-paths is not always an H-path (this is precisely the reason Arpin and Linek [1] prefer to use walks instead of paths).

1.1. Terminology and notation

Throughout this paper all the paths, cycles, and walks considered are directed paths, directed walks and directed cycles.

We use the standard terminology on digraphs as given in [3]. However we provide most of the necessary definitions and notation for the convenience of the reader. We also refer the reader to [2]. For a digraph D, the vertex set is denoted by $V(D)$ and the arc set by $A(D)$. If $S \subseteq V(D)$ is a nonempty set then the *subdigraph* of D *induced by the vertex set* S, $D[S]$, is that digraph having vertex set S, whose arc set consists of all those arcs of D joining vertices of S. Likewise, if $F \subseteq A(D)$ is a nonempty set then $D[F]$, the *subdigraph* of D *induced by the arc set* F, is the digraph with F as the arc set, whose vertices are the end points of the arcs in F. An arc $(z_1, z_2) \in A(D)$ is called an *asymmetrical arc* (symmetrical) if $(z_2, z_1) \notin A(D)$ $(z_2, z_1) \in A(D))$. The *asymmetrical part of* D denoted by $Asym(D)$ is the spanning subdigraph of D whose arcs are the asymmetrical arcs of D. The *symmetrical part of* D can be likewise defined, considering the *symmetrical arcs*, and it is denoted by $Sym(D)$. The arc $(z_1, z_2) \in A(D)$ is called an S_1S_2*-arc* whenever $z_1 \in S_1 \subseteq V(D)$ and $z_2 \in S_2 \subseteq V(D)$.

Let $z \in V(D)$. The set $N^+(z) = \{x \in V(D)$ such that $(z, x) \in A(D)\}$ is called the *out-neighborhood of* z, while the set $A^+(z) = \{(z, x) \in A(D)$ such that $x \in V(D)\}$ is the *arc out-neighborhood of* z.

$I \subseteq V(D)$ is an *independent set* in D whenever $A(D[I]) = \emptyset$. If W is a directed path or cycle in D then $\ell(W)$ will denote its length. If $\{z_1, z_2\} \subseteq V(D)$ then a z_1z_2*- walk* will denote a walk from z_1 to z_2 in D and if we restrict z_1 and z_2 to $V(W)$ then the z_1z_2*- walk* contained in W will be denoted by (z_1, W, z_2). If $I \subseteq V(D)$ and $z \in V(D)$ then a *zI-walk* is a zx-walk for some $x \in I$. If $P = (z_0, z_1, z_2, \ldots)$ is an infinite sequence of different vertices such that $(z_i, z_{i+1}) \in A(D)$ for each $i \in \{1, 2, 3, \ldots\}$, then it will be called an infinite exterior path. By C_n we will denote the cycle of length n. Let $C = (0, 1, \ldots, m, 0)$ be a cycle of D. A *chord* of C is an arc $f = (i, j) \in A(D) - A(C)$.

1.2. Kernels

A kernel of a digraph D is an independent set of vertices $K \subseteq V(D)$ such that for each $z \in (V(D) - N)$ there exists a zN-arc.

A semikernel S of a digraph D is an independent set of vertices $S \subseteq V(D)$ such that for each $z \in V(D) - S$ for which thare exists an sz-arc, there also exists a zS-arc.

Consider D an *arc-colored digraph*. For an arc (z_0, z_1) of D we will denote by $c(z_0, z_1)$ its color. $F \subseteq A(D)$ is a *monochromatic set* if all of its arcs are colored alike and $F \subseteq A(D)$ is a *quasimonochromatic set* if all its arcs but at most one are colored alike. Then $K \subseteq D$ is a *monochromatic subdigraph* of an arc-colored digraph if $A(K)$ is a monochromatic set. On the other hand $K \subseteq D$ is called an *alternating subdigraph* whenever $\{(u, v), (v, w)\} \subseteq A(D)$ implies that $(c, v) \neq c(v, w)$.

We will write

$$x \underset{H}{\rightarrow} y$$

when there exists an *H-path from x to y in D*. Similarly we write $S \underset{H}{\rightarrow} y$ when there exists an *H-path in D from a vertex $s \in S$, $S \subseteq V(D)$ to a vertex y*.

A cycle C in D will be called an *H-cycle* if every path of length 2 in C is an H-path.

Notice that when H consists only of loops then an H-path in an H-colored digraph D is a monochromatic path. In the opposite case, if H has no loops and it has any other arc then an H-path in D is an alternating path. We will say that a path of length one (an arc) is an H-path.

1.3. H-kernels

It is important to mention that the concept of H-paths and therefore that of H-kernels has indeed its origins in the concept of monotonic paths introduced by Linek and Sands [7]: They assign a color to the arcs of a tournament with the vertices of a poset and say that a walk (path) is monotone if the colors displayed on it form a nondecreasing sequence in the partial order. As it was already mentioned in the Introduction, this was later reconsidered by Arpin and Linek [1] in order to work on the three clases of digraphs $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $\mathfrak{B}_3$.

The concept of monotone paths came out of a result due to Sands, Sauer, and Woodrow, who proved in [11] that if H is the reflexive, disconnected digraph with two vertices (the graph of the antichain of two elements) then the tournament coloring number of H, denoted by $tc(H)$, is one[1]. In other words, what the authors proved is that if the arcs of a tournament T are colored with two colors then there exists a single vertex x in T such that for every vertex $y \neq x$ in T there is a monochromatic path from y to x (i.e. there is an H-kernel of just one vertex). It is not even known if $tc(D)$ exists when H is the graph of an antichain of three or more elements. In [10] Reid discusses results concerning the tournament coloring number and exoplores it for reflexive digraphs with three vertices.

There is an important concept related to kernels, the semikernel. The fact that this concept is weaker than that of the kernel is precisely its advantage because it allows us to find sufficient conditions for the existence of kernels. Here, in the case of H-kernels the H-semikernel will also play an important role. In an H-colored digraph D, $S \subseteq V(D)$ will be called an *H-semikernel* of D if it is an H-independent set and for every $x \in V(D) - S$, such that there exists T an H-path in D from S to x, there exists T' an H-path in D from x to S. Notice that if N is an H-kernel of D then it is an H-semikernel of D. Observe as well that the empty set is an an H-semikernel of every digraph.

1.4. Restricted semikernels

The following Lemma shows how to grow an H-semikernel.

Lemma 1.1. *Let H be a digraph and D be an H-colored digraph such that for every $z \in V(D)$ it holds that $A^+(z)$ is a monochromatic set. Let S be a nonempty H-semikernel of D. Consider*

$$B = \{v \in V(D) - S \mid \neg\exists v \underset{H}{\rightarrow} S\}$$

and take $F = D[B]$. If S_F is an H-semikernel of F, then $S \cup S_F$ is an H-semikernel of D.

Proof. Using the definition of an H-semikernel let us prove the following facts:

I. *$S \cup S_F$ is an H-independent set of D.*

We proceed by contradiction. Let us suppose that $S \cup S_F$ is not an H-independent set of D. Then there exists $T = (s_0, s_1, s_2, \ldots, s_n) \subseteq D$, an H-path in D from s_0 to s_n, with $\{s_0, s_n\} \subseteq S \cup S_F$.

First of all we will see that $\{s_0, s_n\} \subseteq S_F$. S is an H-independent set in D and so $\{s_0, s_n\} \nsubseteq S$. On the other hand, since $S_F \subseteq B$, from the definition of B we get that there is no H-path in D from S_F to S. Even more, there is no H-path in D from S to S_F, because if we suppose the contrary, since S is an H-semikernel, we would have that there exists an H-path in D from S_F to S, in contradiction with the definition of B. Then $\{s_0, s_n\} \subseteq S_F$.

[1] For those readers not familiarized with this number, $tc(H)$ is the smallest positive integer (provided it exists) such that for any H-coloring of any tournament T, there exists a set S of at most $tc(H)$ vertices of T with the property of being an H-absorbent set.

Now, consider $A = V(D) - (B \cup S)$. We will prove that there exists an internal vertex of T belonging to A. That is:

$$\{j \mid 1 \leq j \leq n-1 \text{ such that } s_j \in A\} \neq \emptyset$$

Suppose the contrary. Since $V(D) = A \cup B \cup S$, then every internal vertex s_j of T would belong to $S \cup B$. If there exists some j, $1 \leq j \leq n-1$, such that $s_j \in S$, then (s_0, T, s_j) is an H-path in D from S_F to S, in contradiction with the definition of B. So we would have that for every j such that $1 \leq j \leq n-1$, it holds that $s_j \in B$. Then $T \subseteq F$, which means that T is an H-path in F from S_F to S_F, a contradiction (S_F is an H-independent set in F). This proves that there exists an internal vertex of T in A.

Next, consider

$$i = \min_{1 \leq j \leq n-1} \{j \mid s_j \in A\}$$

(notice that i is well defined because of the previous paragraph). From the definition of A we have that there exists $s_i \underset{H}{\to} S$. Let $P = (s_i = p_0, p_1, \ldots, p_m) \subseteq D$ be such path. Notice that $(c(s_{i-1}, s_i), c(s_i, s_{i+1})) \in A(H)$ since T is an H-path and that $\{(s_i, s_{i+1}), (s_i = p_0, p_1)\} \subset F^+(s_i)$. Since by hypothesis $F^+(s_i)$ is a monochromatic set, then $c(s_i, s_{i+1}) = c(s_i = p_0, p_1)$ and so $(s_0, T, s_i) \cup (s_i, P, p_m)$ is an H-path in D from S_F to S (it is certainly a path because of the choice of i and because $P \subseteq A$). This contradicts the definition of B. And so it is now proved that $S \cup S_F$ is an H-independent set of D.

II. *Let $x \in V(D) - (S \cup S_F)$. If there exists $(S \cup S_F) \underset{H}{\to} x$, then there exists $x \underset{H}{\to} (S \cup S_F)$.*

If $x \notin B$ then the affirmation holds ($x \underset{H}{\to} S$ exists from the definition of B). In what follows we will prove the statement also holds if $x \in B$. Let $Q = (x_0, x_1, \ldots, x_k = x) \subseteq D$ be the $(S \cup S_F) \underset{H}{\to} x$ supposed by hypothesis.

1. $x_0 \in S_F$:
   If it is not the case, then $x_0 \in S$. This means Q is an $S \underset{H}{\to} x$ and so there exists $x \underset{H}{\to} S$ (because S is an H-semikernel of D). This contradicts that $x \in B$ and so $x_0 \in S_F$.

2. Every internal vertex of Q is not in $S \cup S_F$ (this follows from statement I).

3. Every internal vertex of Q is not in A:
   Suppose the contrary and consider the first vertex of Q in A, that is

   $$t = \min_{1 \leq i \leq k-1} \{i \mid x_i \in A\},$$

   then because of the definition of B and A, there exists $x_t \underset{H}{\to} S$, namely $R = (x_t = r_0, r_1, \ldots, r_m)$. On the other hand, since Q is an H-path we have that $(c(x_{t-1}, x_t),$

$c(x_t, x_{t+1})) \in A(H)$. We also have that $\{(x_t, x_{t+1}), (x_t, r_1)\} \subset F^+(x_t)$ and by hypothesis $F^+(x_t)$ is a monochromatic set. Then $c(x_t, x_{t+1}) = c(x_t = r_0, r_1)$ and so $(x_0, Q, x_t) \cup (x_t, R, r_m)$ is an $S_F \underset{H}{\rightarrow} S$ (it is a path because of the choice of t). This contradicts statement I and so it is proved that every internal vertex of Q is not in A.

This last claim allows us to state that $Q \subseteq F$ (recall that $A = V(D) - (B \cup S)$). Moreover, since $x_0 \in S_F$ (where S_F is an H-semikernel of F), then there exist $x \underset{H}{\rightarrow} S_F$ and so statement II holds. Statements I and II prove that $S \cup S_H$ is an H-semikernel of D. □

Our next lemma gives us a sufficient condition for a digraph to have an H-kernel. It assures (by using the previous result) that it is sufficient to ask for every induced subdigraph of D to have a nonempty H-semikernel. Notice that this result also holds for possible infinite digraphs. However we need to prove the following proposition first.

Proposition 1.2. *Let D be an H-colored digraph and $\mathfrak{S}$, the set of H–semikernels of D ordered by inclusion. Then the hypothesis of Zorn's Lemma holds.*

Proof. Let $\mathfrak{C}$ be a chain in $\mathfrak{S}$ and let us consider

$$\mathfrak{U} = \bigcup \{S \mid S \in \mathfrak{C}\}.$$

We must prove that $\mathfrak{U}$ is an H-semikernel of D:

1. $\mathfrak{U}$ is an H-independent set by of D:

   Suppose, by the contrary, that there exist u and v in $\mathfrak{U}$ such that there is an uv-H-path in D. $u \in S_1$ for some $S_1 \in \mathfrak{C}$ and $v \in S_2$ for some $S_2 \in \mathfrak{C}$ ($S_1 \neq S_2$ as they are H-independent sets). Notice that $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$ (since $S_1, S_2 \in \mathfrak{C}$). Let us suppose without loss of generality that $S_1 \subseteq S_2$ and so $u, v \in S_2$. This contradicts the fact that S_2 is an H-independent set (since it is an H-semikernel).

2. $\mathfrak{U}$ satisfies the second property of an H-semikernel: $x \in V(D) - \mathfrak{U}$ and suppose that there exists $\mathfrak{U} \underset{H}{\rightarrow} x$, call it T. From the definition of $\mathfrak{U}$ there exists $u \in S$ for some $S \in \mathfrak{C}$ such that T is an $u \underset{H}{\rightarrow} x$. Since S is an H-semikernel, there exists T', an $x \underset{H}{\rightarrow} s$ for some $s \in S$. Notice that T' is an $x \underset{H}{\rightarrow} \mathfrak{U}$ (since $s \in S \subseteq \mathfrak{U}$).

□

Now, our Lemma.

Lemma 1.3. *Let D be an (possibly infinite) H-colored digraph such that for every $z \in V(D)$ it holds that $A^+(z)$ is a monochromatic set. If every induced subdigraph of D has a nonempty H-semikernel, then D has an H-kernel.*

Proof. Take $(\mathfrak{S}, \subseteq)$, the set of H-semikernels of D ordered with the inclusion. It follows from Proposition 1.2 and Zorn's Lemma that $(\mathfrak{S}, \subseteq)$ has a maximal element $S^\star$. We will prove that $S^\star$ is an H-kernel of D:
By contradiction, suppose that $S^\star$ is not an H-kernel of D. From the definition of an H-kernel we have that there exists $x \in V(D) - S^\star$ such that there is no $x \underset{H}{\to} S^\star$ (this is because the other condition, that of H-independence, holds because $S^\star$ is an H-semikernel). So we can consider the nonempty set

$$B = \{v \in (V(D) - S^\star) \mid \neg\exists v \underset{H}{\to} S^\star\}$$

and let $F = D[B]$. Now, take S_F, a nonempty H-semikernel of F (it exists by hypothesis). Because of Lemma 1.1 we have that $S^\star \cup S_F$ is an H-semikernel of D. This contradicts the maximality of $S^\star$. Then our Lemma holds. □

The previous Lemma gives as a corollary a result first proved by V. Neumann-Lara [8]:

Corollary 1.4. *Let D be a possibly infinite digraph. If every induced subdigraph of D has a nonempty semikernel, then D has a kernel.*

Proof. Take H the digraph with two vertices and no arcs; clearly the only H-paths of D are the arcs of D, give any partition of $V(D)$ in two sets V_1 and V_2 and color all the arcs in $A^+(z)$ with color i if and only if $z \in V_i$ □

Lemma 1.3 gives us even more: The following sufficient condition for an H-colored digraph to have an H-kernel.

Theorem 1.5. *Let D be an H-colored digraph such that:*

a) For every $z \in V(D)$, $A^+(z)$ is a monochromatic set,

b) Every cycle in D is an H-cycle, and

c) There are no infinite exterior paths in D.

Then D has an H-kernel.

Proof. Take an arbitrary subset $U \subseteq V(D)$. According to Lemma 1.3 it would be sufficient to prove that $G = D[U]$ has a nonempty semikernel. Suppose, by the contrary, that G does not have a nonempty H-semikernel. The following statements will allow us to get a contradiction.

1. *Consider $\{u, v\} \subseteq V(D)$. Then every H-walk in D from u to v contains as a subsequence an H-path in D from u to v.*
   Let W be an H-walk in D from u to v. We will prove this by induction over

$\ell(W)$, the length of W. If $\ell(W) = 1$ then W is already an H-path. Assume that the statement holds for every H-walk from u to v of length $\ell < n$. Let $W = (u = z_0, z_1, z_2, \ldots, z_n = v) \subseteq D$ be an H-walk from u to v with length n. If $z_i \neq z_j$ for every $i \neq j$ then W is the desired H-path. Suppose then that there exist i and j, $i \neq j$, such that $z_i = z_j$. Without loss of generality let us suppose that $i < j$. Now, W is a walk, so $(c(z_{i-1}, z_i), c(z_i, z_{i+1})) \in A(H)$. Even more, since $\{(z_i, z_{i+1}), (z_j = z_i, z_{j+1})\} \subseteq F^+(z_i = z_j)$, which is a monochromatic set by hypothesis, then $(c(z_{i-1}, z_i), c(z_j = z_i, z_{j+1})) \in A(H)$. Notice that $W' = (u = z_0, W, z_i) \cup (z_i = z_j, W, z_n = v) \subset W \subseteq D$ is an H-walk from u to v with length $\ell(W') < \ell(W)$. It follows from the inductive hypothesis that W' contains as a subsequence an H-path from u to v. Finally notice that $T \subset W' \subset W$.

2. *Every closed walk in D is an H-walk.*
   Let W be a closed walk in D. We will again proceed by induction over $\ell(W)$. If $\ell(W) = 2$ then W is a cycle and it is an H-cycle by hypothesis. Now, suppose that the statements holds for every walk with length $\ell < n$ and consider $W = (u = z_0, z_1, z_2, \ldots, z_n = u) \subseteq D$ a closed walk in D of length n. If $z_i \neq z_j$ for every $i \neq j$, then W is a cycle and again it is an H-cycle by hypothesis. Let us assume then that there exist i and j, $i \neq j$, such that $z_i = z_j$. Without loss of generality suppose $i < j$. Consider $W_1 = (z_i, W, z_j) \subset W$ and $W_2 = (z_0, W, z_i) \cup (z_i = z_j, W, z_n) \subset W$. Both, W_1 and W_2, are closed walks of length strictly less that $\ell(W)$, so it follows from the induction hypothesis that W_1 and W_2 are H-walks. Then $W_1 \cup W_2 = W$ is also an H-walk (since $A^+(z_i) = F^+(z_j)$ is a monochromatic set).

3. *For every $u \in U$ it holds that $A^+(u) \neq \emptyset$.*
   By the contrary, suppose there exists $u \in U$ such that $A^+(u) = \emptyset$. Clearly $\{u\}$ is an H-semikernel of G, a contradiction with what we supposed about G (i.e. that G does not have a nonempty H-semikernel).

4. *There exists a sequence of vertices*

   $$S = (u_i)_{i \in \mathbb{N}}$$

   defined as follows. For every i there exists $u_i \underset{H}{\to} u_{i+1}$, namely

   $$T_i = (u_i = x_0^i, x_1^i, \ldots, x_{n_i}^i = u_{i+1}) \subseteq D$$

   and there is no $u_{i+1} \underset{H}{\to} u_i$.
   This is a consequence of both, the previous statement and the fact that $\{u_i\}$ is not an H-semikernel.

5. *For every i such that $i \geq 0$, and for every $j \notin \{i-1, i+1\}$, it holds that $T_i \cap T_j = \emptyset$.*
   Proceeding by contradiction, suppose that there exist i and j as in the statement

and such that there exists $w \in T_i \cap T_j$. Without loss of generality assume that $i < j$. Then the closed walk

$$W = (w, T_i, u_{i+1}) \cup (\bigcup_{k=i+1}^{j-1} T_k) \cup (u_j, T_j, w)$$

is an H-walk because of statement 2. So, in particular, $W' = (u_j, W, u_{j-1})$ is an H-walk from u_j to u_{j-1}. From statement 1 we get that W' has an H-path from u_j to u_{j-1}, in contradiction with statement 4.

6. *S is not an infinite sequence of different vertices.*
Suppose it is. Now, if for every $i \neq j$ it holds that T_i and T_j are internally disjoint, then $\bigcup T_i$ contains an infinite exterior path in D, a contradiction. Otherwise, there exist i and j, $i \neq j$, such that T_i and T_j intersect each other in something more than in their end points. If follows from the previous statement that $j \in \{i-1, i+1\}$. Then consider, for each t, the vertex $x_t \in V(T_{t+1})$ as the last vertex in T_{t+1} which is also in T_t. Notice that the walk

$$\bigcup [(u_i = x_0^i, T_i, x_i) \cup (x_i, T_{i+1}, u_{i+2})]$$

contains again an infinite exterior path in D, a contradiction.

From the last statement we conclude there exist two different natural numbers m and r such that $m < r$ and $u_m{=}u_r$. Take the closed walk

$$W = \bigcup_{t=m}^{r} T_t.$$

By using (2) we know that W is an H-walk. So

$$W' = (u_{m+1}, W, u_m)$$

is an H-walk in D. From statement 1 we then conclude that W' contains an H-path in D from u_{m+1} to u_m, a contradiction (recall that $u_m \in S$). □

2. Applications

Taking into account that our results are in the same spirit as those established by Arpin and Linek [1], we are compelled to think not only about the wide range of applications of their results, but also about their possible echo on other fields, such as Computation Theory and Symbolic Dynamics.

By introducing a suitable generalization of the concept of H-coloration of a digraph D, Arpin and Linek [1] considered H to be an automaton and replaced D by the language accepted by D in the following way: Let Σ be an alphabet and let $L \subseteq \Sigma^\star$ be a language ($\Sigma^\star$ denotes the set of sequences with elements in Σ); if the arcs of a digraph D are colored with the alphabet Σ then an L-walk in D is a walk whose colors spell out a word (a finite sequence) in L. Then the notion of a B_i-language arises (for $i \in \{1, 2, 3\}$), which generalizes the notion of a B_i-digraph since each digraph has a particular language associated with it.

In the case of alternating coloration, several applications appear on topics of Graph Theory and Algorithms [6, 9, 12], Genetics [5] and Social Sciences [4].

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References

[1] P. Arpin and V. Linek, V. Reachability problems in edge-colored digraphs, *Discrete Math.*, **307** (17-18), (2007), 2276–2289.

[2] J. Bang-Jense, and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer-Verlag, London, 2000.

[3] C. Berge, *Graphs and Hypergraphs*, Nort-Holland, Amsterdam, 1973.

[4] W. S. Chow, Y. Manoussakis, O. Megalakaki, M. Spyratos and Z. Tuza, Paths through fixed vertices in edge-colored graphs, *Math. Inf. Sci. hum.*, **32** (127), (1994), 49–58.

[5] D. Dorninger, Hamiltonian circuits determining the order of chromosomes, *Discrete Appl. Math.*, **50** (1994), 159–168.

[6] R. Haggkvist, On F-Hamiltonian graphs, *Graph Theory Relat. Top.*, **35** (1979), 219–231.

[7] V. Linek and B. Sands, A note on paths in edge-coloured tournaments, *Ars Combin.*, **44**, (1996),

[8] V. Neumann-Lara, Seminúcleos y núcleos, *An. Inst. Mat. UNAM*, **11** (1971), 55–62.

[9] J. Petersen, Die Theorie der regularen graphs, *Acta Math.*, **15** (1891), 193–220.

[10] K. B. Reid, Monotone reachability in arc-colored tournaments, *Congr. Numer.*, **146** (2000), 131–141.

[11] B. Sands, N. Sauer and R. E. Woodrow, On monochromatic paths in edge-colored digraphs, *J. Combin. Theory Ser. B*, **33** (1982), 271–275.

[12] D. Woodall, *Improper colouring of graphs*, R. Nelson and R.J. Wilson, editors: Graph Colourings, Pitman Research Notes in Math. Series, **218** (1990), 45–63.