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Vertices contained in all or in no minimum k-dominating sets of a tree*

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Abstract

Let k be a positive integer and $G = (V, E)$ be a simple graph. A subset $S \subseteq V$ is dominating in G, if for each vertex $v \in V \setminus S$, $N(v) \cap S \neq \emptyset$. In 1985, Fink and Jacobson gave a generalization of the concept of dominating sets in graphs. A subset S of V is k-dominating in G, if every vertex of $V \setminus S$ is adjacent to at least k vertices in S. In this paper, we characterize vertices that are in all or in no minimum k-dominating sets in a tree for $k \geq 2$.

Keywords: domination, k-domination.

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1. Introduction and preliminary results

Let $G = (V, E)$ be a simple graph. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a set S, we denote by $\langle S \rangle$ the *subgraph* induced by the vertices of S. A subset $S \subseteq V$ is a *dominating* if for each vertex $v \in V \setminus S$, $N(v) \cap S \neq \emptyset$. For more details on domination in graphs, see [9, 10]. In [5, 6], Fink and Jacobson generalized the concept of dominating sets in graphs. For a positive integer k, a subset $S \subseteq V$ is *k-dominating*, if every vertex of $V \setminus S$ has at least k neighbors in S. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. We call a k-dominating set of cardinality $\gamma_k(G)$ a *$\gamma_k(G)$-set.* A recent survey on k-independence and k-domination can be found in [3].

The *degree* of a vertex v, denoted by $\deg_G(v)$, is the number of vertices adjacent to v. A *leaf* of T is a vertex of degree one, while a *support vertex* of T is a vertex adjacent to a leaf. We denote the set of leaves and the set of support vertices of T by $L(T)$ and $S(T)$, respectively. Specifically, for a vertex v in a rooted tree T, we denote the set of children

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and descendants, of v, by $C(v)$ and $D(v)$, respectively, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. The *diameter* of G is $diam(G) = \max\{d(x,y) \mid x, y \in V(G)\}$ where $d(x,y)$ is the length of the shortest path between x and y. Let T be a rooted tree. We denote by $L(v)$ the set of leaves of T descendant from v, that is, $L(v) = D(v) \cap L(T)$. For $k \geq 2$, a vertex of degree at least $k+1$ is called a $(k-1)$ *-branch vertex.* Also, we denote by $B^k(T)$ the set of all $(k-1)$-branch vertices of T.

For a property P of a vertex subset of a graph G, let $\mu_P(G)$ denote the minimum (or maximum) cardinality of a set with the property P. Many researchers were interested in characterizing the vertices of G that are in all or in no set with the property P and cardinality $\mu_P(G)$, see [8, 11, 4, 1].

In this paper, we give a characterization of vertices belonging to all or to no minimum k-dominating sets of a tree T. For this purpose, we introduce some more notations. We define the sets $\mathcal{A}_k(T), \mathcal{N}_k(T)$ and $L^j(v)$ by

$$\begin{aligned} \mathcal{A}_k(T) &= \{v \in V(T) \mid v \text{ is in every } \gamma_k(T)\text{-set}\}, \\ \mathcal{N}_k(T) &= \{v \in V(T) \mid v \text{ is in no } \gamma_k(T)\text{-set}\}, \text{ and} \\ L^j(v) &= \{x \in C(v) \mid \deg_T(x) = j\}. \end{aligned}$$

A vertex v of a tree T is said to good (resp. bad), if it belongs (resp. does not belong) to at least $\gamma_k(T)$-set (resp. to no $\gamma_k(T)$-set). We denote by $b_k(T) = |\mathcal{N}_k(T)|$ the number of bad vertices, and by $g_k(T) = |V(T)| - |\mathcal{N}_k(T)|$ the number of good vertices, in T.

The following observation is straightforward.

Observation 1.1. *For a graph G, a k-dominating set contains all vertices of degree less than k.*

For the purpose of characterizing the sets $\mathcal{A}_k(T)$ and $\mathcal{N}_k(T)$ for any nontrivial tree T and any integer $k \geq 2$, we define the following trees. A nontrivial tree T is called an $\mathcal{H}_{k-1}$-tree if T has a vertex w such that $d_T(w) \geq k-1$, and $d_T(x) \leq k-1$ for every vertex $x \in V(T) \setminus \{w\}$. The vertex w will be called the special vertex of T. An $\mathcal{H}_{k-1}$-tree with special vertex w is called *exact* if $d_T(w) = k-1$. Note that a star with center vertex of degree at least $k-1$ is an example of $\mathcal{H}_{k-1}$-tree.

Lemma 1.2. *Let $k \geq 2$ be an integer and let u be a vertex of a tree T' of degree at most $k-1$. Let T be a tree obtained from T' and an exact $\mathcal{H}_{k-1}$-tree H with a special vertex w by adding the edge uw. Then*

1. $\gamma_k(T) = \gamma_k(T') + |V(H)| - 1$;
2. *For every vertex $v \neq u$ of T', $v \in \mathcal{A}_k(T')$ if and only if $v \in \mathcal{A}_k(T)$;*
3. *For every vertex $v \neq u$ of T', $v \in \mathcal{N}_k(T')$ if and only if $v \in \mathcal{N}_k(T)$.*

Proof. Suppose that the tree T is obtained from T' and an exact $\mathcal{H}_{k-1}$-tree H with a special vertex w by adding the edge uw. From Observation 1.1, u (resp. $V(H) \setminus \{w\}$) is in $\mathcal{A}_k(T')$ (resp. is in $\mathcal{A}_k(T)$).

1. Let D be a $\gamma_k(T)$-set. Then by Observation 1.1, D contains $V(H) \setminus \{w\}$, and without loss of generality, we assume that $w \notin D$ (else replace w by u), hence $u \in D$. Thus $D \cap V(T')$ is a k-dominating set of T', and so $\gamma_k(T') \leq \gamma_k(T) - |V(H)| + 1$. Now, let D' be a $\gamma_k(T')$-set. Since $\deg_{T'}(u) \leq k-1$, $u \in D'$ and $D' \cup (V(H) \setminus \{w\})$ is a k-dominating set of T. Therefore $\gamma_k(T) \leq \gamma_k(T') + |V(H)| - 1$ and the equality follows.

2. Suppose that $v \in \mathcal{A}_k(T')$ and let D be a $\gamma_k(T)$-set. Clearly by item 1, $D' = D \cap T'$ is a $\gamma_k(T')$-set of order $\gamma_k(T) - |V(H)| + 1$, and since $v \notin V(H) \setminus \{w\}$, $v \in D' \subset D$. Therefore $v \in \mathcal{A}_k(T)$.

   Conversely, suppose that $v \in \mathcal{A}_k(T)$, and let D' be a $\gamma_k(T')$-set. By item 1, $D = D' \cup (V(H) \setminus \{w\})$ is a $\gamma_k(T)$-set of order $\gamma_k(T') + |V(H)| - 1$. Also since $v \in D \setminus V(H), v \in D'$ and thus $v \in \mathcal{A}_k(T')$.

3. Suppose that $v \in \mathcal{N}_k(T')$ and let D be a $\gamma_k(T)$-set. By item 1, $D' = D \cap T'$ is a $\gamma_k(T')$-set of order $\gamma_k(T) - |V(H)| + 1$, and since $v \notin V(H) \setminus \{w\}$, $v \notin D' \subset D$. Therefore $v \in \mathcal{N}_k(T)$.

   Conversely, suppose that $v \in \mathcal{N}_k(T)$, and let D' be a $\gamma_k(T')$-set. By item 1, $D = D' \cup (V(H) \setminus \{w\})$ is a $\gamma_k(T)$-set of order $\gamma_k(T') + |V(H)| - 1$. Also since $v \notin D \setminus V(H), v \notin D'$ and thus $v \in \mathcal{N}_k(T')$.

□

2. The pruning of a tree

In order to characterize the sets $\mathcal{A}_k(T)$ and $\mathcal{N}_k(T)$ for any nontrivial tree T, we will use a technique called *tree pruning* introduced by Mynhardt [11] and used later by Cockayne et al. and Blidia et al., see [4, 1, 2].

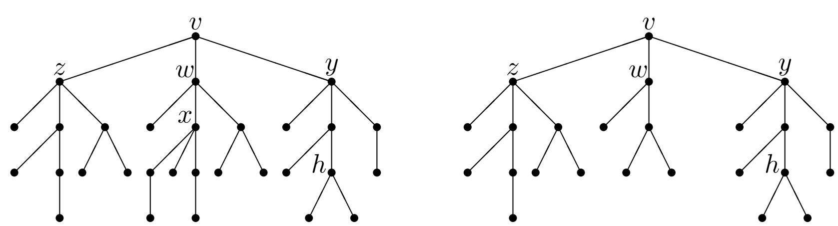
Let v be a vertex of a nontrivial tree T_v with degree at least $k \geq 2$. Using the process described below, on every $(k-1)$-branch vertex different from v, the tree T_v is transformed to another tree $\overline{T}_v$, called the pruning of T_v, in which every vertex different to v has degree at most k and all vertices of degree k are in $C(v)$. We show that the properties of the vertex v to be in $\mathcal{A}_k(T)$ or $\mathcal{N}_k(T)$ will be preserved in $\overline{T}_v$.

We now describe the process of tree pruning for the parameter $\gamma_k(T)$. Let T_v be a nontrivial tree rooted at a vertex v. If $d_{T_v}(u) \leq k$ for every $u \in D(v)$, then by Lemma 1.2, we can remove consecutively all exact $\mathcal{H}_{k-1}$-trees attached at the descendant vertices from v, with degree k and at distance at least 2 from v, and under this condition at maximum distance from v. Hence after this transformation we obtain the tree $\overline{T}_v$. Otherwise, let $u \neq v$ be a $(k-1)$-branch vertex at maximum distance from v and let $w = p(u)$ be the parent of u in T_v. Since $d_{T_v}(x) \leq k$ for every $x \in D(u)$, we apply on T_u the similar process described above on T_v, by using Lemma 1.2. So, and without loss of generality, in the subtree T_u, all vertices of degree k are in $C(u)$. Since $|L^k(u)| + \sum_{j=1}^{k-1} \left|L^j(u)\right| \geq k$, we apply the following process:

- If $\left|L^k(u)\right| \geq 1$, then delete $D(u)$.
- If $\left|L^k(u)\right| = 0$, then we distinguish between two situations :
  a) If $w \notin B^k(T)$, then delete $D(u)$ and attach $(k-1)$ vertices at u.
  b) If $w \in B^k(T)$, then delete $D[u]$.

We repeat this process until we obtain a tree $\overline{T}_v$ such that $d_{\overline{T}_v}(u) \leq k$ for every $u \in V(\overline{T}_v) \setminus \{v\}$ and all vertices of degree k are in $C(v)$.

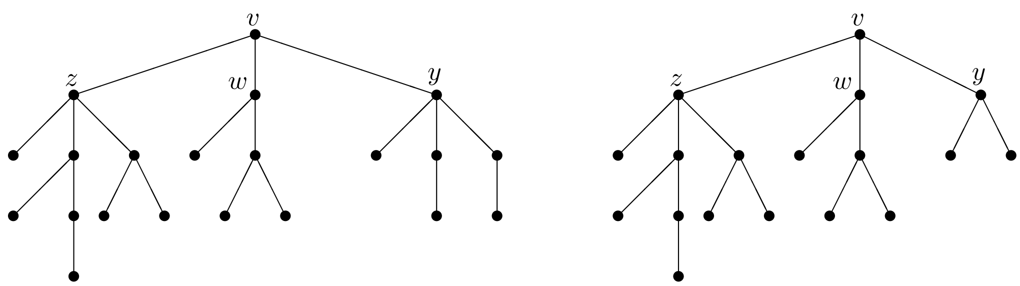
To illustrate this process for a given integer, say $k = 3$, we consider the tree T_v in Figure 1 (a) where x, y, z and w are the 2-branch vertices of T. First, x is the 2-branch vertex at maximum distance 2 from v, and since every vertex of $D(x) \setminus C(x)$ has degree less than 3, we don't apply Lemma 1.2. Also, since $\left|L^3(x)\right| = 0$ and the parent w of x is in $B^3(T)$, we delete $D[x]$ (Figure 1 (b)).



(a) *The tree* T_v (b)

Figure 1: The first step of the pruning of T_v

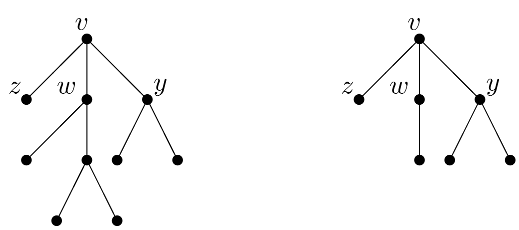
Now it remains two 2-branch vertices, y and z at maximum distance 1 from v. Let us consider y. Since there is a vertex h descendant from y of degree 3 not in $C(y)$, we apply Lemma 1.2 with $H = T_h$, so we delete $D[h]$ (Figure 2 (c)). Hence we obtain $\left|L^3(y)\right| = 0$ and the vertex v, the parent of y, is not in $B^3(T)$. So we delete $D(y)$ and we attach two vertices at y (Figure 2 (d)).



(c) (d)

Figure 2: The second and the third step of the pruning of T_v

Finally we consider the 2-branch vertex z. Since every vertex of $D(z)\setminus C(z)$ has degree less than 3 and $\left|L^3(z)\right| = 2 \geq 1$, we delete $D(z)$ (Figure 3 (e)). Also, since in the final step of the pruning (Figure 3.(e)) there is a vertex in $D(v) \setminus C(v)$ of degree 3, we apply Lemma 1.2, and we obtain the tree $\overline{T}_v$ of Figure 3 (f), in which all vertices of degree three (if any) are in $C(v)$.



(e) (f) *The tree* $\overline{T}_v$

Figure 3: The final step of the pruning of T_v

Let T be a tree rooted at a vertex v. In the following lemma we will assume for a $(k-1)$-branch vertex u, at maximum distance from v, that all its descendant vertices of degree k are in $C(u)$. Otherwise, there exists at least one vertex of degree k in $D(u)\setminus C(u)$. Among all these vertices, let x be a vertex at maximum distance from u, so x fulfills the condition on the vertex w in Lemma 1.2. Apply Lemma 1.2 on T, by considering $H = T_x$. Thus the new tree is obtained from the old T by removing the vertices of $V(H)$.

Lemma 2.1. *Let $k \geq 2$ be an integer and let T be a tree rooted at a vertex v. Let u be a $(k-1)$-branch vertex at maximum distance from v and $l_k = \left|L^k(u)\right|$.*

1. *If $l_k \geq 1$, then let T' be the tree obtained from T by deleting $D(u)$.*
2. *If $l_k = 0$, then:*

- If $w \notin B^k(T)$, then let T' be the tree obtained from T by deleting $D(u)$ and attaching $k-1$ vertices to u.

- If $w \in B^k(T)$, then let T' be the tree obtained from T by deleting $D[u]$.

Then for each case we have:

a. $v \in \mathcal{A}_k(T')$ if and only if $v \in \mathcal{A}_k(T)$.

b. $v \in \mathcal{N}_k(T')$ if and only if $v \in \mathcal{N}_k(T)$.

Proof. Let $C(u) = \{x_1, x_2, ..., x_{l_k}, y_1, y_2, ..., y_{l_m}\}$ where $d_T(x_i) = k$ for every i $(1 \leq i \leq l_k)$, $l_m = \sum_{j=1}^{k-1} \left|L^j(u)\right|$ and $d_T(y_i) = j$ for every i $(1 \leq i \leq l_m)$ and $1 \leq j \leq k-1$. Since u is a $(k-1)$-branch vertex, $l_k + l_m \geq k$. Let w denote the parent of u in the rooted tree T.

1. $l_k \geq 1$. Let $T' = T \setminus D(u)$.

We shall show that $\gamma_k(T') = \gamma_k(T) - |D(u)| + l_k$. Every $\gamma_k(T')$-set can be extended to a k-dominating set of T by adding the set $D(u) \setminus \{x_1, x_2, ..., x_{l_k}\}$, so we have $\gamma_k(T) \leq \gamma_k(T') + |D(u)| - l_k$. Now, let D be a $\gamma_k(T)$-set and $D' = D \cap T'$. If $u \notin D$, then $\{x_1, x_2, ..., x_{l_k}\} \subset D$ and so $\{u\} \cup (D \setminus \{x_1, x_2, ..., x_{l_k}\})$ is a k-dominating set of order less than $|D|$ or equal to $|D|$ (when $l_k = 1$). Hence without loss of generality, we can assume that $u \in D$. As seen previously, $D \setminus (D(u) \setminus \{x_1, x_2, ..., x_{l_k}\})$ is a k-dominating set of T', implying that $\gamma_k(T') \leq \gamma_k(T) - (|D(u)| - l_k)$, and the equality follows.

Now we prove item a. Suppose that $v \in \mathcal{A}_k(T')$ and let D be a $\gamma_k(T)$-set. As seen previously, $D' = D \cap T' = D \setminus (D(u) \setminus \{x_1, x_2, ..., x_{l_k}\})$ is a $\gamma_k(T')$-set of order $\gamma_k(T) - |D(u)| + l_k$. Now since $v \notin D(u)$ and $v \in D' \subset D$, it follows that $v \in \mathcal{A}_k(T)$.

Conversely, suppose that $v \in \mathcal{A}_k(T)$, and let D' be a $\gamma_k(T')$-set. Likewise, we have $D = D' \cup (D(u) \setminus \{x_1, x_2, ..., x_{l_k}\})$ of order $\gamma_k(T') + |D(u)| - l_k$ is a $\gamma_k(T)$-set. Now since $v \in D \setminus D[u], v \in D'$ and thus $v \in \mathcal{A}_k(T')$.

For item b, suppose that $v \in \mathcal{N}_k(T')$ and let D be a $\gamma_k(T)$-set. Clearly, $D' = D \cap T' = D \setminus (D(u) \setminus \{x_1, x_2, ..., x_{l_k}\})$ is a $\gamma_k(T')$-set of order $\gamma_k(T) - |D(u)| + l_k$. Since $v \notin D' \subset D$, it follows that $v \in \mathcal{N}_k(T)$.

Conversely, suppose that $v \in \mathcal{N}_k(T)$, and let D' be a $\gamma_k(T')$-set. The set $D = D' \cup (D(u) \setminus \{x_1, x_2, ..., x_{l_k}\})$ is a $\gamma_k(T)$-set of order $\gamma_k(T') + |D(u)| - l_k$. Now since $v \notin D \setminus D[u], v \notin D'$ and thus $v \in \mathcal{N}_k(T')$.

2. $l_k = 0$. We distinguish between two situations:

- $w \notin B^k(T)$: Let $T' = T \setminus (D(u) \setminus \{y_1, y_2, ..., y_{k-1}\})$. Clearly $D(u) \subset D$. We shall show that $\gamma_k(T') = \gamma_k(T) - |D(u)| + k - 1$. First every $\gamma_k(T')$-set can be extended to a k-dominating set of T by adding the set $D(u) \setminus \{y_1, y_2, ..., y_{k-1}\}$, so $\gamma_k(T) \leq \gamma_k(T') + |D(u)| - (k-1)$. Now let D be a $\gamma_k(T)$-set and $D' = D \cap T' = D \setminus (D(u) \setminus \{y_1, y_2, ..., y_{k-1}\})$. If $w \in D$, then u is k dominated in T', and so D' is a k-dominating set of T'. If $w \notin D$, then $u \in D$ since $w \notin B^k(T)$, and so D' is a k-dominating set of T'. In both cases we have $\gamma_k(T') \leq \gamma_k(T) - |D(u)| + (k-1)$, and the equality follows.

Now we prove item a. Suppose that $v \in \mathcal{A}_k(T')$ and let D be a $\gamma_k(T)$-set. As seen previously, $D' = D \setminus (D(u) \setminus \{y_1, y_2, ..., y_{k-1}\})$ of order $\gamma_k(T) - |D(u)| + (k-1)$, is a $\gamma_k(T')$-set. Now since $v \notin D[u]$ and $v \in D' \subset D$, $v \in \mathcal{A}_k(T)$.

Conversely, suppose that $v \in \mathcal{A}_k(T)$ and let D' be a $\gamma_k(T')$-set. Likewise, we have $D = D' \cup (D(u) \setminus \{y_1, y_2, ..., y_{k-1}\})$ of order $\gamma_k(T') + |D(u)| - (k-1)$, is a $\gamma_k(T)$-set. Now since $v \in D \setminus D(u)$, $v \in D'$ and thus $v \in \mathcal{A}_k(T')$.

For item b, suppose that $v \in \mathcal{N}_k(T')$ and let D be a $\gamma_k(T)$-set. Clearly, $D' = D \setminus (D(u) \setminus \{y_1, y_2, ..., y_{k-1}\})$ is a $\gamma_k(T')$-set of order $\gamma_k(T) - |D(u)| + (k-1)$ and $v \notin D'$. Since $v \notin D(u)$ and $D' \subset D$, $v \notin D$ and thus $v \in \mathcal{N}_k(T)$.

Conversely, suppose that $v \in \mathcal{N}_k(T)$ and let D' be a $\gamma_k(T')$-set. The set $D = D' \cup (D(u) \setminus \{y_1, y_2, ..., y_{k-1}\})$ of order $\gamma_k(T') + |D(u)| - (k-1)$, is a $\gamma_k(T)$-set. Since $v \notin D \setminus D(u)$, $v \notin D'$ and thus $v \in \mathcal{N}_k(T')$.

- $w \in B^k(T)$: Let $T' = T \setminus D[u]$. We first show that $\gamma_k(T') = \gamma_k(T) - |D(u)|$. Inequality $\gamma_k(T) \leq \gamma_k(T') + |D(u)|$ follows from the fact that every $\gamma_k(T')$-set can be extended to a k-dominating set of T by adding $D(u)$. Also, let D be a $\gamma_k(T)$-set and $D' = D \cap T'$. If $w \in D$, then D' is a k-dominating set of T'. Now if $w \notin D$, then we consider two situations: If $u \notin D$, then D' is a k-dominating set of T', so $\gamma_k(T') \leq \gamma_k(T) - |D(u)|$. Also, if $u \in D$, then $D_1 = \{w\} \cup (D \setminus \{u\})$ is a $\gamma_k(T)$-set, and thus $D' = D_1 \cap T'$ is a k-dominating set of T'. Hence $\gamma_k(T') \leq \gamma_k(T) - |D(u)|$. In both situations, we have $\gamma_k(T') = \gamma_k(T) - |D(u)|$.

Next we prove item a. Suppose that $v \in \mathcal{A}_k(T')$ and let D be a $\gamma_k(T)$-set. Without loss of generality, let $u \notin D$. Then $D' = D \cap T' = D \setminus D(u)$ is a $\gamma_k(T')$-set of order $\gamma_k(T) - |D(u)|$. Since $v \notin D(u)$ and $v \in D' \subset D$, $v \in \mathcal{A}_k(T)$.

Conversely, suppose that $v \in \mathcal{A}_k(T)$ and let D' be a $\gamma_k(T')$-set. The set $D = D' \cup D(u)$ is a $\gamma_k(T)$-set of order $\gamma_k(T') + |D(u)|$. Since $v \in D \setminus D(u), v \in D'$ and thus $v \in \mathcal{A}_k(T')$.

For item b, suppose that $v \in \mathcal{N}_k(T')$ and let D be a $\gamma_k(T)$-set. Clearly $D' = D \cap T' = D \setminus D(u)$ is a $\gamma_k(T')$-set of order $\gamma_k(T) - |D(u)|$. Since $v \notin D(u)$ and $v \notin D' \subset D, v \notin D$ and so $v \in \mathcal{N}_k(T)$.

Conversely, suppose that $v \in \mathcal{N}_k(T)$ and let D' be a $\gamma_k(T')$-set. The set $D = D' \cup D(u)$ is a $\gamma_k(T)$-set of order $\gamma_k(T') + |D(u)|$. Since $v \notin D \setminus D(u), v \notin D'$ and so $v \in \mathcal{N}_k(T')$. □

3. Characterization

The following theorem gives a necessary and sufficient condition for the root v of a nontrivial tree T to be in $\mathcal{A}_k(T)$ (resp. in $\mathcal{N}_k(T)$). Note that the tree T in Theorem 3.1, is the tree obtained after applying the process of pruning.

Theorem 3.1. *Let $k \geq 2$ be an integer and let T be a nontrivial tree rooted at a vertex v of degree at least k, such that $\deg_T(u) < k$ for every vertex $u \in V(T) \setminus C(v)$ $(u \neq v)$. Then*

1. *$v \in \mathcal{A}_k(T)$ if and only if $|L^k(v)| \geq 2$;*
2. *$v \in \mathcal{N}_k(T)$ if and only if $L^k(v) = \emptyset$.*

Proof. Since v has degree at least k, $|L^k(v)| + \sum_{j=1}^{k-1} \left|L^j(v)\right| \geq k$. Let D be a $\gamma_k(T)$-set. First we show the sufficient condition.

1. $|L^k(v)| \geq 2$. Let $x, y \in L^k(v)$, clearly by Observation 1.1, $\cup_{j=1}^{k-1} L^j(v) \subset D$. If $\{x, y\} \subset D$, then $v \notin D$, and so $\{v\} \cup (D \setminus \{x, y\})$ is a k-dominating set of size less than $|D|$, a contradiction. Therefore $v \in \mathcal{A}_k(T)$.
2. $\left|L^k(v)\right| = 0$. By Observation 1.1, $\cup_{j=1}^{k-1} L^j(v) \subset D$, and since $\sum_{j=1}^{k-1} \left|L^j(v)\right| \geq k$, v is k dominated by $\cup_{j=1}^{k-1} L^j(v)$. Therefore $v \in \mathcal{N}_k(T)$.

Conversely, we proceed by contrapositive. The negation of ($|L^k(v)| \geq 2$) and ($\left|L^k(v)\right| = 0$) produces $|L^k(v)| = 1$. Let $L^k(v) = \{x\}$. Hence by Observation 1.1, $\cup_{j=1}^{k-1} L^j(v) \subset D$ where $\sum_{j=1}^{k-1} \left|L^j(v)\right| \geq k - 1$. Now if $v \notin D$, then $x \in D$, and if $v \in D$, then $x \notin D$. In both cases D is a $\gamma_k(T)$-set, and thus $v \notin \mathcal{A}_k(T) \cup \mathcal{N}_k(T)$. □

For the above example, the tree obtained in Figure 3.(f) verifies $|L^3(v)| = 1$. Hence by Theorem 3.1, $v \notin \mathcal{A}_3(T) \cup \mathcal{N}_3(T)$.

According to Lemma 2.1 and Theorem 3.1, we have our main result:

Theorem 3.2. *Let v be a vertex of the tree T. Then*

- *$v \in \mathcal{A}_k(T)$ if and only if $v \in \mathcal{A}_k(\overline{T}_v)$.*
- *$v \in \mathcal{N}_k(T)$ if and only if $v \in \mathcal{N}_k(\overline{T}_v)$.*

4. Conclusion

By applying Theorem 3.1, we can test all the vertices of a tree T and give the occurrence of these vertices to be in all or in no minimum k-dominating sets of T (for $k \geq 2$). Therefore we can determine the sets $\mathcal{A}_k(T)$ and $\mathcal{N}_k(T)$. Also, it is easy to verify that a pruning tree can be found in polynomial time by the process defined above. So, the nature of the tree according to the values of $b_k(T)$ and $g_k(T)$ (see [7]) can be recognized in polynomial time. Indeed, if $b_k(T) = 0$, then T is called γ_k-excellent. If $g_k(T) > b_k(T) \geq 1$, then T is called γ_k-recommendable. When $b_k(T) > g_k(T) \geq 1$, the tree T is called γ_k-undesirable. Also, if $g_k(T) = b_k(T)$, then T is called γ_k-exact. Finally, a nontrivial tree T does not have a unique γ_k-set if and only if $\mathcal{A}_k \cup \mathcal{N}_k = |V|$. As example, the tree of Figure 1.(a) is not γ_3-excellent (because x is not contained in any γ_3-set) but it is γ_3-recommendable (because $g_3 > b_3 \geq 1$) and since $v \notin \mathcal{A}_3(T) \cup \mathcal{N}_3(T)$, the tree T does not have a unique γ_3-set.

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