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On super (a, d)-edge-antimagic total labeling of generalized extended w-trees

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Abstract

Let $G = (V(G), E(G))$ be a graph with v vertices and e edges. An (a, d)-edge-antimagic total labeling is a bijection $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, v + e\}$, such that the set of edge-weights $\{w(xy) : w(xy) = \lambda(x) + \lambda(y) + \lambda(xy), xy \in E(G)\}$ forms an arithmetic progression with the initial term a and common difference d. Additionally, if $\lambda(V(G)) = \{1, 2, \ldots, v\}$ then λ is a super (a, d)-edge-antimagic total labeling. In this paper, we define a generalized extended w-tree denoted by $GEwt(n_1, n_2, ..., n_k; m_1, m_2, ..., m_k; r; k)$ and prove that it admits a super (a, d)-EAT labeling.

Keywords: Super (a, d)-EAT labeling, w-tree, extended w-tree and generalized extended w-tree.

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1. Introduction

All graphs in this paper are finite, simple and undirected. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set, respectively. A (v, e)-graph G is a graph such that $|V(G)| = v$ and $|E(G)| = e$. A general reference for graph-theoretic ideas can be seen in [19]. A *labeling* (or *valuation*) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper, the domain will be the set of all vertices and edges and such a labeling is called *a total labeling.* Some labelings use the vertex-set only or the edge-set only and we shall call them *vertex-labelings* or *edge-labelings*, respectively. In this paper, we focus on a super (a, d)-edge-antimagic total labeling. More details on an antimagic labeling can be seen in [1, 5, 6].

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Definition 1.1. *An (s,d)-edge-antimagic vertex ((s,d)-EAV) labeling of a graph G is a bijective function $\lambda : V(G) \to \{1,2,\dots,v\}$ such that the set of edge-sums of all edges in G, $\{w(xy) = \lambda(x) + \lambda(y) : xy \in E(G)\}$, forms an arithmetic progression $\{s, s+d, s+2d, \dots, s+(e-1)d\}$, where $s > 0$ and $d \geq 0$ are two fixed integers.*

Definition 1.2. *An (a,d)-edge-antimagic total ((a,d)-EAT) labeling of a graph G is a bijective function $\lambda : V(G) \cup E(G) \to \{1,2,\dots,v+e\}$ such that the set of edge-weights of all edges in G, $\{w(xy) = \lambda(x) + \lambda(xy) + \lambda(y) : xy \in E(G)\}$, forms an arithmetic progression $\{a, a+d, a+2d, \dots, a+(e-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists then G is said to be an (a,d)-EAT graph.*

Definition 1.3. *An (a,d)-EAT labeling λ is called a super (a,d)-edge-antimagic total (super (a,d)-EAT) labeling of G if $\lambda(V(G)) = \{1,2,\dots,v\}$. Thus, a super (a,d)-EAT graph is a graph that admits a super (a,d)-EAT labeling.*

The concept of an edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [14, 15]. Simanjuntak et al. [17] defined an (a,d)-EAT labeling as a natural extension of an edge-magic total labeling defined by Kotzig and Rosa. The notion of a super $(a,0)$-EAT labeling was introduced by Enomoto et al. [4] and they proposed the conjecture that every tree admits a super $(a,0)$-EAT labeling. In the effort of attacking this conjecture, many authors have considered a super (a,d)-EAT labeling for some particular classes of trees, for example path like-trees [3], banana trees [7], w-trees [8], extended w-trees [9, 10], subdivided stars [11, 12], caterpillars [14, 18] and subdivided caterpillars [13]. Lee and Shah [16] have verified this conjecture for trees on at most 17 vertices with the help of computer. However, this conjecture is still open.

Let us consider the following proposition which we will use in the main results.

Proposition 1.4. [2] *If a (v,e)-graph G has an (s,d)-EAV labeling then*

(i) G has a super $(s+v+1, d+1)$-EAT labeling,

(ii) G has a super $(s+v+e, d-1)$-EAT labeling.

2. Definitions of w-tree, extended w-tree and generalized extended w-tree

In this section, the definitions of a w-tree and an extended w-tree are presented. The known results related to a super $(a,0)$-EAT labeling of w-trees and extended w-trees are stated. In the end, the concept of a generalized extended w-tree is also defined and explained with a suitable example.

Definition 2.1. [8] *Let n be a positive integer. Consider a path P on 5 vertices as $V(P) = \{b, c_1, w_1, c_2, d\}$. A w-graph $W(n)$, is a graph derived from the path P by hanging n leaves $x_i^1, x_i^2, \dots, x_i^n$ from each vertex c_i. Now we consider k copies of the w-graph $W(n)$ with end vertices $d_1, d_2, ..., d_k$ on the paths $P_1, P_2, ..., P_k$ respectively. A w-tree $WT(n,k)$ is obtained by joining all vertices $d_1, d_2, \dots, d_k$ to a further vertex a.*

Theorem 2.2. [8] *For $k \geq 3$, $G \cong WT(n,k)$ admits a super $(a,0)$-EAT labeling if $n \geq k-1$, where k is even and $n \geq k$, where k is odd.*

Theorem 2.3. [8] *For $h \geq 2$, $G \cong \bigcup_{m=1}^{h} WT(n,k_m)$ admits a super $(a,0)$-EAT labeling if $n \geq 2\sum_{m=1}^{h} k_m - 2$ and $3 \leq k_1 \leq k_2 \leq, ..., \leq k_n$.*

Definition 2.4. [9] *Let n and r be positive integers. Consider a path P on $2r+1$ vertices as $V(P) = \{b, c_1, w_1, c_2, w_2, \ldots, w_{r-2}, c_{r-1}, w_{r-1}, c_r, d\}$. An extended w-graph $Ew(n,r)$, is a graph derived from the path P by hanging n leaves $x_i^1, x_i^2, \ldots, x_i^n$ from each vertex c_i. Now we consider k copies of the extended w-graph $Ew(n,r)$ with end vertices $d_1, d_2, ..., d_k$ on the paths $P_1, P_2, ..., P_k$ respectively. An extended w-tree $Ewt(n,r,k)$ is obtained by joining all vertices $d_1, d_2, ..., d_k$ to a further vertex a.*

Theorem 2.5. [9] *For r, $k \geq 3$, $G \cong Ewt(n,k,r)$ admits a super $(a,0)$-EAT labeling if $n \geq r\lceil \frac{k-1}{2} \rceil$.*

Theorem 2.6. [9] *For $h \geq 2$, $G \cong \bigcup_{m=1}^{h} Ewt(n,k_m,r)$ admits a super $(a,0)$-EAT labeling if $\quad n \geq r\sum_{m=1}^{h} k_m - r$, $r \geq 3$ and $3 \leq k_1 \leq k_2 ... \leq k_h$.*

Definition 2.7. *Let n_i, m_i, r and k be positive integers. Consider a family of disjoint paths $P_1, P_2, ..., P_k$ such that $V(P_i) = \{c_1^i, c_2^i, ..., c_r^i; 1 \leq i \leq k\}$. Suppose that the path P_i for each i has n_i hanging leaves $x_{ij}^1, x_{ij}^2, ..., x_{ij}^{n_i}$ (respectively, m_i leaves $y_i^1, y_i^2, ..., y_i^{m_i}$) from each vertex c_j^i if $1 \leq j \leq r-1$ (respectively, if $j = r$). A generalized extended w-tree is obtained by joining all vertices $y_1^{m_1}, y_2^{m_2}, ..., y_k^{m_k}$ as a last hanging leave from $c_r^1, c_r^2, ..., c_r^k$ on each path $P_1, P_2, ..., P_k$ respectively to a further vertex a.*

$$
\begin{aligned}
\text{Let } V(G) = \{a\} & \\
&\cup \{c_s^i : 1 \leq i \leq k,\ \ 1 \leq s \leq r\} \\
&\cup \{x_{is}^{l_i} : 1 \leq i \leq k,\ \ 1 \leq s \leq r-1,\ \ 1 \leq l_i \leq n_i\} \\
&\cup \{y_i^{p_i} : 1 \leq i \leq k,\ \ 1 \leq p_i \leq m_i\} \text{ and} \\
E(G) = \{ay_i^{m_i} : 1 \leq i \leq k\} & \\
&\cup \{c_s^i c_{s+1}^i : 1 \leq i \leq k,\ \ 1 \leq s \leq\ \ r-1\} \\
&\cup \{x_{is}^{l_i} c_s^i : 1 \leq i \leq k,\ \ 1 \leq s \leq r-1,\ \ 1 \leq l_i \leq\ \ n_i\} \\
&\cup \{y_i^{p_i} c_r^i : 1 \leq p_i \leq m_i,\ \ 1 \leq i \leq k\}
\end{aligned}
$$

be the set of vertices and edges of a generalized extended w-tree $G \cong GEwt(n_1, n_2, ..., n_k; m_1, m_2, ..., m_k; r; k)$. Thus $v = |V(G)| = \sum_{i=1}^{k}(rn_i - n_i + m_i + r) + 1$ and $e = |E(G)| = \sum_{i=1}^{k}(rn_i - n_i + m_i + r)$.

An example of a generalized extended w-tree is shown in Figure 1 for $n_1 = 2, n_2 = 1, n_3 = 3, m_1 = 7, m_2 = 6, m_3 = 8, r = 4$ and $k = 3$.

3. Super (a,d)-EAT labeling of generalized extended w-trees

In this section, we prove the existence of a super (a,d)-EAT labeling of the generalized extended w-tree for different values of d.

Theorem 3.1. *For $n_i \geq 1$, $k \geq 3$, and $m_i \geq \frac{r}{2}\sum_{t=1}^{k}(n_t + 1) + 1$, $G \cong GEwt(n_1, n_2, ..., n_k; m_1, m_2, ..., m_k; r; k)$ admits a super $(a,0)$-EAT labeling with $a = 2v + s - 1$ and a super $(a,2)$-EAT labeling with $a = v + s + 1$ if r is even, $n_1 \leq n_2 \leq, ... \leq n_{k-1} \leq n_k$ and $m_1 \leq m_2 \leq, ... \leq m_{k-1} \leq m_k$, where $v = |V(G)|$ and $s = \frac{r}{2}\sum_{i=1}^{k}(n_i + 1) + 3$.*

Proof. We define $\lambda : V(G) \rightarrow \{1, 2, ..., v\}$ as follows:

$$\lambda(a) = \frac{r}{2}(n_1 + 1) + 1.$$

For $s = 1, 3, ..., r-1$ and $1 \leq l_i \leq n_i$;

$$\lambda(c_s^i) = \begin{cases} \frac{1}{2}[\sum_{t=1}^{k} r(n_t + 1) + (s-1)(n_1 + 1)] + 2, & for\ \ i = 1, \\ \frac{r}{2}\sum_{t=1}^{k}(n_t + 1) + \frac{r-2}{2}\sum_{t=1}^{i}(n_t + 1) + \sum_{t=1}^{i} m_t + i + 1 - \\ \frac{1}{2}(n_i + 1)(s-1), & for\ \ 2 \leq i \leq k. \end{cases}$$

and

$$\lambda(x_{is}^{l_i}) = \begin{cases} \frac{1}{2}(n_1 + 1)(s-1) + l_1, & for\ \ i = 1, \\ \frac{r}{2}\sum_{t=1}^{i}(n_t + 1) - \frac{1}{2}(n_i + 1)(s-1) + 2 - l_i, & for\ \ 2 \leq i \leq k. \end{cases}$$

For $s = 2, 4, ..., r$;

$$\lambda(c_s^i) = \begin{cases} \frac{1}{2}(n_1 + 1)s, & for\ \ i = 1, \\ \frac{r}{2}\sum_{t=1}^{i}(n_t + 1) - \frac{1}{2}(n_i + 1)(s-2) - n_i + 1, & for\ \ 2 \leq i \leq k. \end{cases}$$

For $s = 2, 4, ..., r-2$ and $1 \le l_i \le n_i$;

$$\lambda(x_{is}^{l_i}) = \begin{cases} \frac{r}{2}\sum\limits_{t=1}^{k}(n_t+1) + \frac{1}{2}(n_1+1)(s-2) + 2 + l_1, & for\ \ i = 1, \\ \\ \frac{r}{2}\sum\limits_{t=1}^{k}(n_t+1) + \frac{r-2}{2}\sum\limits_{t=1}^{i}(n_t+1) + \sum\limits_{t=1}^{i} m_t - \\ \frac{1}{2}(n_i+1)(s-2) + i + 1 - l_i, & for\ \ 2 \le i \le k. \end{cases}$$

For $1 \le i \le 2$;

$$\lambda(y_i^{p_i}) = \begin{cases} \frac{r}{2}\sum\limits_{t=1}^{k}(n_t+1) + \frac{r-2}{2}(n_1+1) + m_1 + 1 + i, & for\ \ p_i = m_i, \\ \\ \frac{r}{2}\sum\limits_{t=1}^{k}(n_t+1) + \frac{r-2}{2}(n_1+1) + (m_2+1)(i-1) + m_1 + \\ 2 - p_i, & for\ \ 1 \le p_i \le m_i - 1. \end{cases}$$

For $3 \le i \le k$ and $\alpha^i = \frac{r}{2}\sum\limits_{t=1}^{i-1}(n_t+1) - \frac{r}{2}(n_1+1)$;

$$\lambda(y_i^{m_i}) = \frac{r}{2}\sum_{t=1}^{k}(n_t+1) + (r-1)\sum_{t=1}^{i-1}(n_t+1) + \sum_{t=1}^{i-1} m_t - \frac{r}{2}(n_1+1) + i + 1.$$

and

$$\lambda(y_i^{p_i}) = \begin{cases} \frac{r}{2}\sum\limits_{t=1}^{k}(n_t+1) + \frac{r-2}{2}\sum\limits_{t=1}^{i-1}(n_t+1) + \sum\limits_{t=1}^{i-1} m_t + i + p_i, \\ \qquad\qquad for\ \ 1 \le p_i \le \alpha^i, \\ \\ \frac{r}{2}\sum\limits_{t=1}^{k}(n_t+1) + \frac{r-2}{2}\sum\limits_{t=1}^{i-1}(n_t+1) + \sum\limits_{t=1}^{i-1} m_t \\ +i + 1 + p_i, \quad for\ \ \alpha^i + 1 \le p_i \le m_i - 1. \end{cases}$$

The set of all edge-sums generated by the above formulas forms a consecutive integer sequence $[\frac{r}{2}\sum\limits_{i=1}^{k}(n_i+1)+2]+1$, $[\frac{r}{2}\sum\limits_{i=1}^{k}(n_i+1)+2]+2$, $\cdots$, $[\frac{r}{2}\sum\limits_{i=1}^{k}(n_i+1)+2]+e$. Therefore, by Proposition 1.4, λ can be extended to a super $(a, 0)$-EAT labeling with $a = v + e + s = \frac{1}{2}\sum\limits_{i=1}^{k}(5rn_i + 4m_i - 4n_i + 5r) + 4$ and to a super $(a, 2)$-EAT labeling with $a = v + 1 + s = \frac{1}{2}\sum\limits_{i=1}^{k}(3rn_i + 2m_i - 2n_i + 3r) + 5$. □

Theorem 3.2. *For $n_i \geq 1$, $k \geq 3$, and $m_i \geq \frac{r}{2}\sum_{t=1}^{k}(n_t+1)+1$, $G \cong GEwt(n_1,n_2,...,n_k;m_1,m_2,...,m_k;r;k)$ admits a super $(a,1)$-EAT labeling with $a = s+\frac{3}{2}v$ if r is even, v is even, $n_1 \leq n_2 \leq,... \leq n_{k-1} \leq n_k$ and $m_1 \leq m_2 \leq,... \leq m_{k-1} \leq m_k$, where $v = |V(G)|$ and $s = \frac{r}{2}\sum_{i=1}^{k}(n_i+1)+3$.*

Proof. We define $V(G)$, $E(G)$ and $\lambda : V(G) \to \{1,2,...,v\}$ as in Theorem 3.1. Thus, the set of edge-sums $A = \{a_j : 1 \leq j \leq e\}$, where $a_j = \left[\frac{r}{2}\sum_{i=1}^{k}(n_i+1)+2\right]+j$ constitutes an arithmetic sequence with common difference 1. Consequently, the set of edge-labels is $B = \{b_j : 1 \leq j \leq e\}$, where $b_j = v+j$. The set of edge-weights is defined as $C = \{a_{2i-1}+b_{e-i+1} : 1 \leq i \leq \frac{e+1}{2}\} \cup \{a_{2j}+b_{\frac{e-1}{2}-j+1} : 1 \leq j \leq \frac{e+1}{2}-1\}$. It is easy to see that C constitute an arithmetic sequence with $d=1$ and $a = s+\frac{3}{2}v$. Since all vertices receive the smallest labels, so, λ is a super $(a,1)$-EAT labeling. □

The following theorem also can prove on the same lines:

Theorem 3.3. *For $n_i \geq 1$, $k \geq 3$, $m_i \geq \frac{r-1}{2}\sum_{i=1}^{k}(n_i+1)+k+1$, $n_1 \leq n_2 \leq,... \leq n_{k-1} \leq n_k$ and $m_1 \leq m_2 \leq,... \leq m_{k-1} \leq m_k$, $v = |V(G)|$ and $s = \frac{r-1}{2}\sum_{i=1}^{k}(n_i+1)+k+3$, $G \cong GEwt(n_1,n_2,...,n_k;m_1,m_2,...,m_k;r;k)$ admits*

- *a super $(a,0)$-EAT labeling with $a = 2v+s-1$ and a super $(a,2)$-EAT labeling with $a = v+s+1$ if r is odd,*
- *a super $(a,1)$-EAT labeling with $a = s+\frac{3}{2}v$ if r is odd and v is even.*

4. Super (a,d)-EAT labeling for disjoint union of generalized extended w-trees

This section deals with the results related to a super (a,d)-EAT labeling for disjoint union of isomorphic and non isomorphic copies of generalized extended w-trees.

$$\begin{aligned}\text{Let} V(G) = &\{a^p : 1 \leq p \leq h\}\\ &\cup \{c_{is}^{p} : 1 \leq i \leq k,\ 1 \leq s \leq r, 1 \leq p \leq h\}\\ &\cup \{x_{is}^{pl_i^p} : 1 \leq i \leq k,\ 1 \leq s \leq r-1, 1 \leq p \leq h,\ 1 \leq l_i^p \leq n_i^p\}\\ &\cup \{y_i^{pq_i^p} : 1 \leq i \leq k, 1 \leq p \leq h,\ 1 \leq q_i^p \leq m_i^p\} \text{ and}\end{aligned}$$

$$
\begin{aligned}
E(G) = &\{a^p y_i^{pm_i^p} : 1 \le i \le k, 1 \le p \le h\} \\
&\cup \{c_{is}^p c_{is+1}^p : 1 \le i \le k, 1 \le p \le h,\ \ 1 \le s \le r-1\} \\
&\cup \{x_{is}^{pl_i^p} c_{is}^p : 1 \le i \le k,\ \ 1 \le s \le r-1, 1 \le p \le h, 1 \le l_i^p \le n_i^p\} \\
&\cup \{y_i^{pq_i^p} c_{ir}^p : 1 \le q_i^p \le m_i^p,\ \ 1 \le i \le k, 1 \le p \le h\}
\end{aligned}
$$

be the set of vertices and edges of $G \cong \bigcup_{p=1}^{h} GEwt(n_1^p, n_2^p, ..., n_k^p\ ;\ m_1^p, m_2^p, ..., m_k^p\ ;\ r; k)$. Thus

$$v = |V(G)| = \sum_{p=1}^{h}[\sum_{i=1}^{k}(rn_i^p - n_i^p + m_i^p + r)] + h$$

and

$$e = |E(G)| = \sum_{p=1}^{h}[\sum_{i=1}^{k}(rn_i^p - n_i^p + m_i^p + r)].$$

Theorem 4.1. *For $h \ge 2$, $k \ge 3$, $n_i^p \ge 1$ and $m_i^p \ge \frac{r}{2}\sum_{u=1}^{h}\left[\sum_{t=1}^{k}(n_t^u + 1)\right] + h$, $G \cong \bigcup_{p=1}^{h} At(n_1^p, n_2^p, ..., n_k^p\ ;\ m_1^p, m_2^p, ..., m_k^p\ ;\ r; k)$ admits a super $(a, 0)$-EAT labeling with $a = 2v+s-h$ and a super $(a, 2)$-EAT labeling with $a = v+s+1$ if r is even, $n_1^p \le n_2^p \le, ..., \le n_k^p$ and $m_1^p \le m_2^p \le, ..., \le m_k^p$, where $v = |V(G)|$ and $s = \frac{r}{2}\sum_{p=1}^{h}\left[\sum_{i=1}^{k}(n_i^p + 1)\right] + 2h + 1$.*

Proof. Through out the labeling we will consider

$\alpha = \frac{r}{2}\sum_{u=1}^{h}[\sum_{t=1}^{k}(n_t^u + 1)]$, $\beta^0 = \gamma^0 = 0$,

$\beta^p = \frac{r}{2}\sum_{u=1}^{p-1}[\sum_{t=1}^{k}(n_t^u + 1)]$ and

$\gamma^p = \frac{1}{2}\sum_{u=1}^{p-1}\left[\sum_{t=1}^{k}[(r-2)(n_t^u + 1) + 2 + 2m_t^u]\right]$, where $2 \le p \le h$.

Now, we define the labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows;

When $1 \le i \le k$, $1 \le p \le h$ and $1 \le l_i^p \le n_i^p$;

$$\lambda(a^p) = h - p + 1,$$

$$\lambda(c_{is}^p) = \begin{cases} \beta^p + \frac{r}{2}\sum_{t=1}^{i}(n_t^p+1) - \frac{1}{2}(n_i^p+1)(s-2) - n_i^p + h, \\ \qquad\qquad\qquad\qquad\qquad for\ \ s = 2,4,...,r, \\ \alpha + \gamma^p + \frac{1}{2}\sum_{t=1}^{i}[(r-2)(n_t^p+1) + 2 + 2m_t^p] - \\ \frac{1}{2}(n_i^p+1)(s-1) + h, \qquad for\ \ s = 1,3,...,r-1, \end{cases}$$

and

$$\lambda(x_{is}^{pl_i^p}) = \begin{cases} \beta^p + \frac{r}{2}\sum_{t=1}^{i}(n_t^p+1) - \frac{1}{2}(n_i^p+1)(s-1) + h - l_i^p + 1, \\ \qquad\qquad\qquad\qquad\qquad for\ \ s = 1,3,...,r-1, \\ \alpha + \gamma^p + \frac{1}{2}\sum_{t=1}^{i}[(r-2)(n_t^p+1) + 2 + 2m_t^p] - \\ \frac{1}{2}(n_i^p+1)(s-2) + h - l_i^P, \qquad for\ \ s = 2,4,...,r-2. \end{cases}$$

When $\quad \delta_i^p = \frac{r}{2}\sum_{t=1}^{i-1}(n_t^p+1) \quad$ and $\quad p = 1;$

$$\lambda(y_i^{pq_i^p}) = \begin{cases} \alpha + h + 1 + q_1^p \ \ for\ \ 1 \le q_1^p \le m_1^p - 1\ \ and\ \ i = 1, \\ \alpha + \frac{1}{2}\sum_{t=1}^{i-1}[(r-2)(n_t^p+1) + 2 + 2m_t^p] + h + q_i^p, \\ \qquad\qquad for\ \ 1 \le q_i^p \le \delta_i^p\ \ and\ \ 2 \le i \le k, \\ \alpha + \frac{1}{2}\sum_{t=1}^{i-1}[(r-2)(n_t^p+1) + 2 + 2m_t^p] + h + q_i^p + 1, \\ \qquad\qquad for\ \ \delta_i^p + 1 \le q_i^p \le m_i^p - 1\ \ and\ \ 2 \le i \le k. \end{cases}$$

When $\quad \delta_i^p = \beta^p + \frac{r}{2}\sum_{t=1}^{i-1}(n_t^p+1) + p - 1,\ 1 \le i \le k$ and $2 \le p \le h;$

$$\lambda(y_i^{pq_i^p}) = \begin{cases} \alpha + \gamma^p + \frac{1}{2}\sum_{t=1}^{i-1}[(r-2)(n_t^p+1) + 2 + 2m_t^p] + h + q_i^P, \\ \qquad\qquad\qquad\qquad\qquad for\ \ 1 \le q_i^p \le \delta_i^p, \\ \alpha + \gamma^p + \frac{1}{2}\sum_{t=1}^{i-1}[(r-2)(n_t^p+1) + 2 + 2m_t^p] + \\ h + q_i^P + 1, \qquad\qquad for\ \ \delta_i^p + 1 \le q_i^p \le m_i^p - 1. \end{cases}$$

When $q_i^p = m_i^p,\ 1 \le i \le k;$

$$\lambda(y_i^{pq_i^p}) = \begin{cases} \alpha + \frac{1}{2}\sum\limits_{t=1}^{i-1}[(r-2)(n_t^p+1)+2+2m_t^p]+ \\ \frac{r}{2}\sum\limits_{t=1}^{i-1}(n_t^p+1)+h+1, \quad for \quad p=1, \\ \\ \alpha+\beta^p+\gamma^p+\frac{1}{2}\sum\limits_{t=1}^{i-1}[(r-2)(n_t^p+1)+2+2m_t^p]+ \\ \frac{r}{2}\sum\limits_{t=1}^{i-1}(n_t^p+1)+h+p, \quad for \quad 2 \le p \le h, \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $\frac{r}{2}\sum\limits_{p=1}^{h}\left[\sum\limits_{i=1}^{k}(n_i^p+1)\right]+2h+1, \frac{r}{2}\sum\limits_{p=1}^{h}\left[\sum\limits_{i=1}^{k}(n_i^p+1)\right]+2h+2, \cdots, \frac{r}{2}\sum\limits_{p=1}^{h}\left[\sum\limits_{i=1}^{k}(n_i^p+1)\right]+2h+e$. Therefore by Proposition 1.4, λ can be extended to a super $(a,0)$-EAT labeling with $a=v+e+s=2v+s-h=\frac{1}{2}\sum\limits_{p=1}^{h}\left[\sum\limits_{i=1}^{k}(5rn_i^p+4m_i^p-4n_i^p+5r)\right]+3h+1$ and to a super $(a,2)$-EAT labeling with $a=v+s+1=\frac{1}{2}\sum\limits_{p=1}^{h}\left[\sum\limits_{i=1}^{k}(3rn_i^p+2m_i^p-2n_i^p+3r)\right]+3h+2$. □

Theorem 4.2. *For $h \ge 2$, $k \ge 3$, $n_i^p \ge 1$ and $m_i^p \ge \frac{r}{2}\sum\limits_{u=1}^{h}\left[\sum\limits_{t=1}^{k}(n_t^u+1)\right]+h$, $G \cong \bigcup\limits_{p=1}^{h} GEwt(n_1^p, n_2^p, ..., n_k^p\ ;\ m_1^p, m_2^p, ..., m_k^p\ ;\ r;k)$ admits a super $(a,1)$-EAT labeling with $a=s+v+1+\frac{e-1}{2}$ if r is even, e is odd, $n_1^p \le n_2^p \le, ..., \le n_k^p$ and $m_1^p \le m_2^p \le, ..., \le m_k^p$, where $v=|V(G)|$ and $s=\frac{r}{2}\sum\limits_{p=1}^{h}\left[\sum\limits_{i=1}^{k}(n_i^p+1)\right]+2h+1$.*

Proof. Define $V(G)$, $E(G)$ and $\lambda : V(G) \to \{1,2,...,v\}$ as in Theorem 4.1. Thus, the set of edge-sums $A=\{a_j : 1 \le j \le e\}$, where $a_j=\frac{r}{2}\sum\limits_{p=1}^{h}\left[\sum\limits_{i=1}^{k}(n_i^p+1)\right]+2h+j$ constitutes an arithmetic sequence with common difference 1. Consequently, the set of edge-labels is $B=\{b_j\ :\ 1 \le j \le e\}$, where $b_j=v+j$. The set of edge-weights is defined as $C=\left\{a_{2i-1}+b_{e-i+1}, 1 \le i \le \frac{e+1}{2}\right\} \cup \left\{a_{2j}+b_{\frac{e-1}{2}-j+1}\ :\ 1 \le j \le \frac{e+1}{2}-1\right\}$. It is easy to see that C constitute an arithmetic sequence with $d=1$ and $a=s+v+1+\frac{e-1}{2}$. Since all vertices receive the smallest labels, so, λ is a super $(a,1)$-EAT labeling. □

The following theorem also can prove on the same lines:

Theorem 4.3. *For* $h \geq 2$, $k \geq 3$, $n_i^p \geq 1$, $m_i^p \geq \frac{1}{2} \sum_{u=1}^{h} \left[\sum_{t=1}^{k} (rn_t^u - n_t^u + r + 1) \right] + h$, $n_1^p \leq n_2^p \leq, ..., \leq n_k^p$, $m_1^p \leq m_2^p \leq, ..., \leq m_k^p$, $v = |V(G)|$ *and* $s = \frac{1}{2} \sum_{p=1}^{h} \left[\sum_{i=1}^{k} (rn_i^p - n_i^p + r + 1) \right] + 2h + 1$, $G \cong \bigcup_{p=1}^{h} GEwt(n_1^p, n_2^p, ..., n_k^p \ ; \ m_1^p, m_2^p, ..., m_k^p \ ; \ r; k)$ *admits*

- *a super* $(a, 0)$*-EAT labeling with* $a = 2v + s - h$ *and a super* $(a, 2)$*-EAT labeling with* $a = v + s + 1$ *if* r *is odd,*
- *a super* $(a, 1)$*-EAT labeling with* $a = s + v + 1 + \frac{e-1}{2}$ *if* r *and* e *both are odd.*

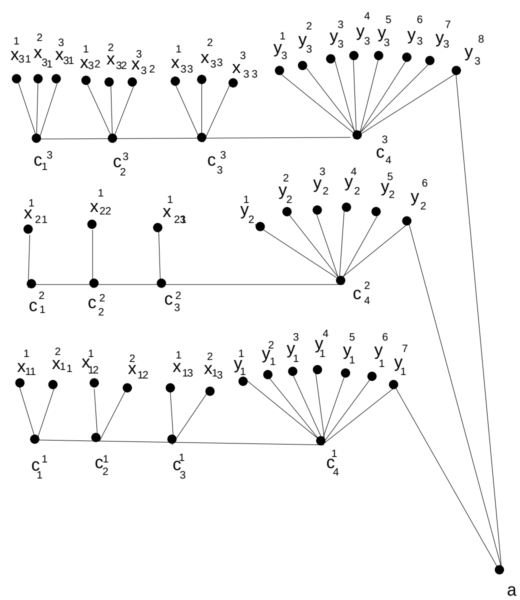

Figure 1: $GEwt(2, 1, 3; 7, 6, 8; 4; 3)$

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References

[1] M. Bača and M. Miller, *Super Edge-Antimagic Graphs*, Brown Walker Press, Boca Raton, Florida USA, (2008).

[2] M. Bača, Y. Lin, M. Miller and R. Simanjuntak, New constructions of magic and antimagic graph labelings, *Util Math.*, **60** (2001), 229–239.

[3] M. Bača, Y. Lin, and F. A. Muntaner-Batle, Super edge-antimagic labeling of path like-trees, *Util Math.*, **73** (2007), 117–128.

[4] H. Enomoto, A. S. Llado, T. Nakamigawa, and G. Ringle, Super edge-magic graphs, *SUT J. Math.* **34** (1980), 105–109.

[5] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, The place of super edge-magic labeling among other classes of labeling, *Discrete Math.,* **231** (2001), 153–168.

[6] J. A. Gallian, A dynamic survey of graph labeling, *The Electronic J. of Combin.* **16** (2013), DS#6.

[7] M. Hussain, E. T. Baskoro and Slamin, On super edge-magic total labeling of banana trees, *Util. Math.*, **79** (2009), 243–251.

[8] M. Javaid, M. Hussain, K. Ali and K. H. Dar, Super edge-magic total labeling on w-trees, *Util. Math.*, **86** (2011), 183–191.

[9] M. Javaid, A. A. Bhatti, M. Hussain and K. Ali, Super edge-magic total labeling on forest of extended w-trees, *Util. Math.*, **91** (2013), 155–162.

[10] M. Javaid M., A. A. Bhatti and M. Hussain, On (a, d)-edge-antimagic total labeling of extended w-trees, *Util. Math..* **87** (2012), 293–303.

[11] M. Javaid, M. Hussain, K. Ali and H. Shaker, On super edge-magic total labeling on subdivision of trees, *Util. Math.*, **89** (2012), 169–177.

[12] M. Javaid and A. A. Bhatti, On super (a, d)-edge-antimagic total labeling of subdivided stars, *Ars Combin.*, **105** (2012), 503–512.

[13] M. Javaid M., A. A. Bhatti and M. Hussain, On super (a, d)-edge-antimagic total labeling of subdivided caterpillar, *Util. Math.*, (To appear).

[14] A. Kotzig and A. Rosa, Magic valuation of complete graphs, *Centre de Recherches Mathematiques, Universite de Montreal*, (1972), CRM-175.

[15] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970), 451–461.

[16] S. M. Lee, and Q. X. Shah, *All trees with at most 17 vertices are super edge-magic*, 16th MCCCC Conference, Carbondale, Southern Illinois University, (2002).

[17] R. Simanjuntak, F. Bertault and M. Millar, *Two new (a, d)-antimagic graph labelings*, Proc. of Eleventh Australian Workshop on combinatorial algorithm, (2000), 179–189.

[18] K. A. Sugeng, M. Miller and M. Baca, (a, d)-edge-antimagic total labeling of caterpillars, *Lecture Notes Comput. Sci.*, **3330** (2005), 169–180.

[19] D. B. West, *An Introduction to Graph Theory*, Prentice-Hall, (1996).