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Note on strict-double-bound graphs and numbers

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Abstract

For a poset $P = (X, \leq_P)$, the *strict-double-bound graph* (*sDB-graph* sDB(P)) is the graph on X for which u and v are adjacent if and only if $u \neq v$ and there exist elements $x, y \in X$ distinct from u and v such that $x \leq u \leq y$ and $x \leq v \leq y$. The *strict-double-bound number* $\zeta(G)$ of a graph G is defined as $\min\{n; G \cup \overline{K_n}$ is a strict-double-bound graph $\}$.

We prove bounds for strict-double-bound numbers of some graphs. That is, $\zeta(K_{1,n}) = \lceil 2\sqrt{n} \rceil$, and for a non-trivial tree T, $\zeta(T) \leq \sum_{v \in \mathrm{IN}(T)} \lceil 2\sqrt{\deg_T(v)} \rceil - 2(|\mathrm{IN}(T)| - 1)$, where $\mathrm{IN}(T)$ is the vertex set of non-leaves of T.

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1. Introduction

In this paper we consider finite undirected simple graphs. $\overline{G}$ is the complement of a graph G. A *clique* in a graph G is the vertex set of a maximal complete subgraph of G. A family $\mathcal{Q} = \{Q_1, Q_2, ..., Q_l\}$ is an *edge cover* of G if each Q_i is a vertex subset of G, and for $uv \in E(G)$, there exists $Q_i \in \mathcal{Q}$ such that $u, v \in Q_i$. In particular a family $\mathcal{Q} = \{Q_1, Q_2, ..., Q_l\}$ is an *edge clique cover* of G if each Q_i is a clique of G, and $\mathcal{Q}$ is an edge cover of G. For a graph G and $S \subseteq V(G)$, $\langle S \rangle_G$ is the induced subgraph on S. For a graph G and $v \in V(G)$, $N_G(v) = \{u; uv \in E(G)\}$.

For a poset $P = (X, \leq_P)$ and an element $x \in X$ of P, we put $U_P(x) = \{y \in X \ ; \ x \leq_P y\}$ and $L_P(x) = \{y \in X \ ; \ y \leq_P x\}$, and denote by $\mathrm{Max}(P)$ the set of all maximal elements of P and $\mathrm{Min}(P)$ the set of all minimal elements of P.

McMorris and Zaslavsky [4] introduced a concept of double bound graphs. Diny [2] characterized double bound graphs.

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We consider strict-double-bound graphs and strict-double-bound numbers. For a poset $P = (X, \leq)$, the *strict-double-bound graph* (*sDB-graph*) of $P = (X, \leq)$ is the graph $\mathrm{sDB}(P)$ on X for which u and v are adjacent if and only if $u \neq v$ and there exist elements $x, y \in X$ distinct from u and v such that $x \leq u \leq y$ and $x \leq v \leq y$. We say that a graph G is a *strict-double-bound graph* if there exists a poset whose strict-double-bound graph is isomorphic to G.

Maximal elements and minimal elements of posets are isolated vertices of sDB-graphs. So a connected graph is not an sDB-graph. Era, Tsuchiya [3] and Scott [5] dealt with strict-double-bound graphs. Scott [5] gave the following result.

Proposition 1.1. [5] *Any graph that is the disjoint union of a non-trivial component and a large enough number of isolated vertices is a strict-double-bound graph.*

We introduce the strict-double-bound number of a graph G. The *strict-double-bound number* $\zeta(G)$ of a graph G is defined as $\min\{n\ ;\ G \cup \overline{K_n}$ is a strict-double-bound graph $\}$. In this paper, we consider strict-double-bound numbers of some families of graphs.

2. Strict-double-bound numbers

Scott [5] obtained the following result, using a concept of transitive double competition numbers.

Theorem 2.1. [5] *For a non-trivial connected graph G, and a minimal edge clique cover $\mathcal{Q}$,* $\left\lceil 2\sqrt{|\mathcal{Q}|} \right\rceil \leq \zeta(G) \leq |\mathcal{Q}| + 1$.

Proposition 2.2 shows that there exists a graph G with $\zeta(G) = \left\lceil 2\sqrt{|\mathcal{Q}|} \right\rceil$.

Proposition 2.2. *Let G be a non-trivial graph. If G has a minimal edge clique cover $\mathcal{Q} = \{Q_1, Q_2, ..., Q_l\}$ such that there exists a non-maximal complete subgraph $H \neq \emptyset$ satisfying*

1. *$Q_i \cap Q_j = V(H)$ for each pair Q_i, $Q_j \in \mathcal{Q}$, and*
2. *$Q_i - V(H) \neq \emptyset$ for all $Q_i \in \mathcal{Q}$,*

then $\zeta(G) = \left\lceil 2\sqrt{|\mathcal{Q}|} \right\rceil$.

Proof. We consider two cases on $\left\lceil 2\sqrt{|\mathcal{Q}|} \right\rceil = \left\lceil 2\sqrt{l} \right\rceil$.

Case 1. $\left\lceil 2\sqrt{l} \right\rceil$ is even.

Let $\left\lceil 2\sqrt{l} \right\rceil = 2n$. Then $n^2 - n + 1 \leq l \leq n^2$. We construct an edge cover $\mathcal{Q}'$ from $\mathcal{Q}$ as follows:

$\mathcal{Q}' = \{Q_{1,1}, Q_{1,2}, ..., Q_{1,n},\ Q_{2,1}, Q_{2,2}, ..., Q_{2,n},\ ...,\ Q_{n,1}, Q_{n,2}, ..., Q_{n,n}\}$ such that

(1) $Q_{i,j} = Q_{(i-1)n+j}$ for $i < n$, or $i = n$ and $n^2 - n + j \leq l$ and

(2) $Q_{n,k} = V(H)$ for $n^2 - n + k > l$.

We construct a poset P from $\mathcal{Q}'$ as follows:

(1) $V(P) = V(G) \cup \{x_1, x_2, ..., x_n\} \cup \{y_1, y_2, ..., y_n\}$,

(2) for i, j, $y_j \leq_P x_i$, and

(3) for $v \in Q_{i,j}$, $y_j \leq_P v \leq_P x_i$.

Then $\mathrm{sDB}(P) = G \cup \overline{K_{2n}}$. Since $\left\lceil 2\sqrt{l} \right\rceil = 2n$, $\zeta(G) \leq \left\lceil 2\sqrt{l} \right\rceil$. By Theorem 2.1, $\left\lceil 2\sqrt{l} \right\rceil \leq \zeta(G)$. So $\zeta(G) = \left\lceil 2\sqrt{l} \right\rceil$.

Case 2. $\left\lceil 2\sqrt{l} \right\rceil$ is odd.

Let $\left\lceil 2\sqrt{l} \right\rceil = 2n-1$. Then $n^2 - 2n + 2 \leq l \leq n^2 - n$. We construct an edge cover $\mathcal{Q}'$ from $\mathcal{Q}$ as follows: $\mathcal{Q}' = \{Q_{1,1}, Q_{1,2}, ..., Q_{1,n},\ Q_{2,1}, Q_{2,2}, ..., Q_{2,n},\ ...,\ Q_{n-1,1}, Q_{n-1,2}, ..., Q_{n-1,n}\}$ such that

(1) $Q_{i,j} = Q_{(i-1)n+j}$ for $i < n-1$, or $i = n-1$ and $n^2 - 2n + j \leq l$, and

(2) $Q_{n-1,k} = V(H)$ for $n^2 - 2n + k > l$.

Similar to Case 1, we construct a poset P from $\mathcal{Q}'$. Then $\mathrm{sDB}(P) = G \cup \overline{K_{2n-1}}$. Since $\left\lceil 2\sqrt{l} \right\rceil = 2n-1$, $\zeta(G) \leq \left\lceil 2\sqrt{l} \right\rceil$. By Theorem 2.1, $\left\lceil 2\sqrt{l} \right\rceil \leq \zeta(G)$. So $\zeta(G) = \left\lceil 2\sqrt{l} \right\rceil$. □

We have the following result by Proposition 2.2. Let v be the center vertex of $K_{1,n}$ and $E(K_{1,n}) = \{vv_i\ ;\ i = 1, 2, ..., n\}$. Then $\mathcal{Q} = \{\{v, v_i\}\ ;\ vv_i \in E(K_{1,n}), i = 1, 2, ..., n\}$ is an edge clique cover of $K_{1,n}$ satisfying the condition of Proposition 2.2.

Corollary 2.3. *For a star graph $K_{1,n}$, $\zeta(K_{1,n}) = \left\lceil 2\sqrt{n} \right\rceil$.*

Bergstrand and Jones [1] dealt with novas in the study of upper bound graphs. A *nova* NOVA(n) is a graph obtained from a star $K_{1,n}$ ($n \geq 1$) by replacing each edge with a complete graph with at least two vertices. We have the following result by Proposition 2.2.

Corollary 2.4. *For a nova* NOVA(n), $\zeta(\mathrm{NOVA}(n)) = \left\lceil 2\sqrt{n} \right\rceil$.

3. Strict-double-bound number of graphs with bridges

Next we consider upper bounds of strict-double-bound numbers of graphs with bridges.

Proposition 3.1. *Let G be a graph with a bridge $e = xy$, where the degrees of x and y are greater than or equal to 2. Let $\mathcal{H} = \{H_1, H_2, ..., H_l\}$ be the set of components of $G - \{x, y\}$. For a partition $(\mathcal{H}_1, \mathcal{H}_2)$ of $\mathcal{H}$, let G_k be an induced subgraph $\langle(\bigcup_{H_i \in \mathcal{H}_k} V(H_i)) \cup \{x, y\}\rangle_G$ for $k = 1, 2$. Then $\zeta(G) \leq \zeta(G_1) + \zeta(G_2) - 2$.*

Proof. For $k = 1, 2$, let P_k be a poset whose $\mathrm{sDB}(P_k) \cong G_k \cup \overline{K_{l_k}}$, where $l_k = \zeta(G_k)$. Since $xy \in E(G_k)$, there exists a pair of vertices $m_k \in U_{P_k}(x) \cap U_{P_k}(y) \cap \mathrm{Max}(P_k)$ and $n_k \in L_{P_k}(x) \cap L_{P_k}(y) \cap \mathrm{Min}(P_k)$. Then we construct a poset P such that

(1) $V(P) = (V(P_1) - \{m_1, n_1\}) \cup (V(P_2) - \{m_2, n_2\}) \cup \{m, n\}$,

(2-1) $u \leq_P m$ if $u \in L_{P_k}(m_k)$,

(2-2) $n \leq_P v$ if $v \in U_{P_k}(n_k)$, and

(2-3) $u \leq_P v$ if $u, v \in V(P_k)$ and $u \leq_{P_k} v$.

For $u, v \in V(P_k)$, $U_{P_k}(u) \cap U_{P_k}(v) \neq \emptyset$ and $L_{P_k}(u) \cap L_{P_k}(v) \neq \emptyset$ if and only if $U_P(u) \cap U_P(v) \neq \emptyset$ and $L_P(u) \cap L_P(v) \neq \emptyset$. Thus $uv \in E(\mathrm{sDB}(P_k))$ if and only if $uv \in E(\mathrm{sDB}(P))$. By the definition of P,

$$\begin{aligned} L_P(m) &= (L_{P_1}(m_1) - \{n_1\}) \cup (L_{P_2}(m_2) - \{n_2\}) \cup \{n\}, \quad \text{and} \\ U_P(n) &= (U_{P_1}(n_1) - \{m_1\}) \cup (U_{P_2}(n_2) - \{m_2\}) \cup \{m\}. \end{aligned}$$

And

$$\begin{aligned} L_P(m) \cap U_P(n) &= ((L_{P_1}(m_1) - \{n_1\}) \cup (L_{P_2}(m_2) - \{n_2\})) \\ &\quad \cap((U_{P_1}(n_1) - \{m_1\}) \cup (U_{P_2}(n_2) - \{m_2\})) \\ &= \{x, y\}. \end{aligned}$$

Since $\{x, y\}$ is a clique of G_k (k=1,2), for $s \in \mathrm{Max}(P_k)$ and $t \in \mathrm{Min}(P_k)$, if $L_{P_k}(s) \cap U_{P_k}(t) \supseteq \{x, y\}$, then $L_{P_k}(s) \cap U_{P_k}(t) = \{x, y\}$.
For $s_1 \in \mathrm{Max}(P_1) - \{m_1\}$ and $t_2 \in \mathrm{Min}(P_2) - \{n_2\}$, $L_P(s_1) \cap U_P(t_2) \subseteq \{x, y\}$.
For $s_2 \in \mathrm{Max}(P_2) - \{m_2\}$ and $t_1 \in \mathrm{Min}(P_1) - \{n_1\}$, $L_P(s_2) \cap U_P(t_1) \subseteq \{x, y\}$.
Thus $\{x, y\}$ is also a clique of $\mathrm{sDB}(P)$.

Since for $u \in V(P_1) - \{x, y\}$ and $v \in V(P_2) - \{x, y\}$, $U_P(u) \cap U_P(v) = \emptyset$ or $L_P(u) \cap L_P(v) = \emptyset$, $uv \notin E(\mathrm{sDB}(P))$. Hence $\mathrm{sDB}(P)$ is constructed by G with some isolated vertices. In sDB(P) the number of isolated vertices is

$$\begin{aligned} |\operatorname{Max}(P)\cup\operatorname{Min}(P)| &= |((\operatorname{Max}(P_1)\cup\operatorname{Max}(P_2)-\{m_1,m_2\})\cup\{m\}) \\ &\quad \cup((\operatorname{Min}(P_1)\cup\operatorname{Min}(P_2)-\{n_1,n_2\})\cup\{n\})| \\ &= |\operatorname{Max}(P_1)\cup\operatorname{Min}(P_1)-\{m_1,n_1\}| \\ &\quad +|\operatorname{Max}(P_2)\cup\operatorname{Min}(P_2)-\{m_2,n_2\}|+|\{m,n\}| \\ &= (\zeta(G_1)-2)+(\zeta(G_2)-2)+2 \\ &= \zeta(G_1)+\zeta(G_2)-2. \end{aligned}$$

Since there exists a poset P_G that sDB$(P_G)\cong G\cup\overline{K_l}$, where $l=\zeta(G_1)+\zeta(G_2)-2$, we obtain that $\zeta(G)\le\zeta(G_1)+\zeta(G_2)-2$. □

Since every edge of a tree is a bridge, we obtain the following result by Proposition 3.1. For a tree T, $\mathrm{IN}(T)=\{v\in V(T)\ ;\ \deg_T(v)\ge 2\}$ is the vertex set of non-leaves of T.

Proposition 3.2. *For a non-trivial tree* T*,*

$$\zeta(T)\le\sum_{v\in\mathrm{IN}(T)}\left\lceil 2\sqrt{\deg_T(v)}\right\rceil-2(|\mathrm{IN}(T)|-1).$$

Proof. We use induction on $l=|\mathrm{IN}(T)|$. In the case $l=0$, $T\cong K_2$ and $\zeta(T)=\zeta(K_2)=2$. In the case $l=1$, T is isomorphic to $K_{1,\deg_T(v)}$, where $v\in\mathrm{IN}(T)$. By Corollary 2.3, $\zeta(T)=\left\lceil 2\sqrt{\deg_T(v)}\right\rceil$.

We assume that $l=|\mathrm{IN}(T)|\ge 2$. Let v_1v_2 be an edge in T such that $v_1,v_2\in\mathrm{IN}(T)$. For the set of components $\mathcal{H}=\{H_1,H_2,...,H_l\}$ of $T-\{v_1,v_2\}$, let $(\mathcal{H}_1,\mathcal{H}_2)$ be a partition of $\mathcal{H}$ such that $H_i\in\mathcal{H}_k$ if $V(H_i)\cap N_T(v_k)\ne\emptyset$ for $k=1,2$. Then for $k=1,2$, the induced subgraph $T_k=\langle(\bigcup_{H_i\in\mathcal{H}_k}V(H_i))\cup\{v_1,v_2\}\rangle_G$ is a tree. Then $\mathrm{IN}(T)=\mathrm{IN}(T_1)\cup\mathrm{IN}(T_2)$, and $|\mathrm{IN}(T_k)|<|\mathrm{IN}(T)|$. By the induction hypothesis $\zeta(T_k)\le\sum_{v\in\mathrm{IN}(T_k)}\left\lceil 2\sqrt{\deg_{T_k}(v)}\right\rceil$ $-\ 2(|\mathrm{IN}(T_k)|-1)$. By Proposition 3.1,

$$\begin{aligned} \zeta(T) &\le \zeta(T_1)+\zeta(T_2)-2 \\ &\le \left(\sum_{v\in\mathrm{IN}(T_1)}\left\lceil 2\sqrt{\deg_{T_1}(v)}\right\rceil-2(|\mathrm{IN}(T_1)|-1)\right) \\ &\qquad +\left(\sum_{v\in\mathrm{IN}(T_2)}\left\lceil 2\sqrt{\deg_{T_2}(v)}\right\rceil-2(|\mathrm{IN}(T_2)|-1)\right)-2 \end{aligned}$$

$$= \sum_{v\in \mathrm{IN}(T_1)} \left\lceil 2\sqrt{\deg_{T_1}(v)}\right\rceil + \sum_{v\in \mathrm{IN}(T_2)} \left\lceil 2\sqrt{\deg_{T_2}(v)}\right\rceil - 2(|\,\mathrm{IN}(T_1)| + |\,\mathrm{IN}(T_2)| - 1)$$

$$= \sum_{v\in \mathrm{IN}(T)} \left\lceil 2\sqrt{\deg_{T}(v)}\right\rceil - 2(|\,\mathrm{IN}(T)| - 1).$$

□

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