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4-Factors in Graphs Which Do Not Contain a Small Star

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Abstract

We show that every 7-connected $K_{1,5}$-free graph, as well as every 5-connected $K_{1,4}$-free graph with minimum degree is at least 6, has a 4-factor.

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1. Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges.

Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, $\deg_G(x)$ denotes the degree of x in G. We let $\delta(G)$ denote the minimum of $\deg_G(x)$ as x ranges over $V(G)$. For an integer $r \geq 1$, a subgraph F of G such that $V(F) = V(G)$ and $\deg_F(x) = r$ for all $x \in V(F)$ is called an r-factor of G. The complete bipartite graph $K_{1,t}$ with partite sets of cardinalities 1 and t is called a t-star. We say that G is $K_{1,t}$-free or t-star-free if G does not contain $K_{1,t}$ as an induced subgraph.

The following result concerning 4-factors is proved in [3].

Theorem A. *Let $t \geq 3$ be an integer, and let G be a 2-connected $K_{1,t}$-free graph such that $\delta(G) \geq \lceil (3t+1)/2 \rceil$. Then G has a 4-factor.*

For $t \geq 6$, the following result is proved in [4].

Theorem B. *Let $t \geq 6$ be an integer, and let G be a $\lceil (3t-3)/2 \rceil$-connected $K_{1,t}$-free graph. Then G has a 4-factor.*

In [4], it is also shown that Theorems A and B are best possible in the sense that for each $t \geq 6$, there exist infinitely many $\lceil (3t-5)/2 \rceil$-connected graphs G with $\delta(G) \geq \lceil (3t-1)/2 \rceil$ such that G has no 4-factor. In this paper, we take up the case where $t \leq 5$, and prove theorems which correspond to Theorem B. Our main results are as follows.

Theorem 1.1. *Let G be a 7-connected $K_{1,5}$-free graph. Then G has a 4-factor.*

Theorem 1.2. *Let G be a 5-connected $K_{1,4}$-free graph, and suppose that $\delta(G) \geq 6$. Then G has a 4-factor.*

In view of Theorem A, Theorem 1.1 is best possible in the sense that there exist infinitely many 6-connected $K_{1,5}$-free graphs G with $\delta(G) \geq 7$ such that G has no 4-factor (see Example 1 below). Similarly, Theorem 1.2 is best possible in the sense that there exist infinitely many 4-connected $K_{1,4}$-free graphs G with $\delta(G) \geq 6$ such that G has no 4-factor (Example 2), and there also exist infinitely many 5-connected $K_{1,4}$-free graphs G such that G has no 4-factor (Example 3). Further, for $K_{1,3}$-free graphs, results like Theorems 1.1 and 1.2 do not hold because there exist infinitely many 4-connected $K_{1,3}$-free graphs G such that G do not have a 4-factor (Example 4). For related results about r-factors, we refer the reader to Ota and Tokuda [5], Aldred te al.[1] and Yashima[7].

Example 1.3. *There exist infinitely many 6-connected $K_{1,5}$-free graphs G with $\delta(G) \geq 7$ such that G has no 4-factor. Let $m \geq 5$ be an arbitrary odd integer. Let $I_1, I_2, \ldots, I_m$, $J_1, J_2, \ldots, J_m$, $H_1, H_2, \ldots, H_{4m}$, L_1, L_2 be disjoint graphs such that I_k is isomorphic to the complete graph of order 3 for each $1 \leq k \leq m$, J_k is isomorphic to the complete graph of order 2 for each $1 \leq k \leq m$, H_i is isomorphic to the complete graph of order 3 for each $1 \leq i \leq 4m$, L_1 is isomorphic to the complete graph of order $2m-1$, and L_2 is isomorphic to the complete graph of order $2m+1$.*

Write $V(H_i) = \{v_{i,1}, v_{i,2}, v_{i,3}\}$ for each $1 \leq i \leq 4m$, and $V(L_1) = \{w_1, w_2, \ldots, w_{2m-1}\}$ and $V(L_2) = \{w_{2m}, w_{2m+1}, \ldots, w_{4m}\}$. For each $1 \leq k \leq m$, set

$$T_k = \bigcup_{1 \leq j \leq 4} V(H_{4(k-1)+j}),$$
$$T'_k = \bigcup_{1 \leq j \leq 4} V(H_{(j-1)m+k}).$$

Now define a graph G by

$$V(G) = \left(\bigcup_{1 \leq k \leq m} \Big(V(I_k) \cup V(J_k) \Big) \right) \cup \Big(\bigcup_{1 \leq i \leq 4m} V(H_i) \Big) \cup V(L_1) \cup V(L_2),$$

$$E(G) = \Bigg(\bigcup_{1 \le k \le m} \Big(E(I_k) \cup E(J_k) \Big)$$

$$\cup \{xy \,|\, x \in V(I_k), y \in T_k\} \cup \{xy \,|\, x \in V(J_k), y \in T'_k\} \Bigg)$$

$$\bigcup \Big(\bigcup_{1 \le i \le 4m} E(H_i)) \Big) \cup \{v_{i,1}w_i \,|\, 1 \le i \le 4m\}.$$

Then G is 6-connected and $K_{1,5}$-free, and satisfies $\delta(G) = 7$. However, we easily see that G does not have a 4-factor (for example, if we apply Lemma 2.1 with $S = \bigcup_{1 \le k \le m}(V(I_k) \cup V(J_k))$ and $T = \bigcup_{1 \le i \le 4m} V(H_i)$, then we get $\theta(S, T) = -2$).

Example 1.4. *There exist infinitely many 4-connected $K_{1,4}$-free graphs G with $\delta(G) \ge 6$ such that G has no 4-factor. Let $m \ge 4$ be an arbitrary integer relatively prime to 3. Let $I_1, I_2, \ldots, I_m, J_1, J_2, \ldots, J_m, H_1, H_2, \ldots, H_{3m}$ be disjoint graphs such that I_k and J_k are isomorphic to the complete graph of order 2 for each $1 \le k \le m$, and H_i is isomorphic to the complete graph of order 3 for each $1 \le i \le 3m$. For each $1 \le k \le m$, set*

$$T_k = \bigcup_{1 \le j \le 3} V(H_{3(k-1)+j}),$$

$$T'_k = \bigcup_{1 \le j \le 3} V(H_{(j-1)m+k}).$$

Now define a graph G by

$$V(G) = \Bigg(\bigcup_{1 \le k \le m} \Big(V(I_k) \cup V(J_k) \Big) \Bigg) \cup \Big(\bigcup_{1 \le i \le 3m} V(H_i) \Big),$$

$$E(G) = \Bigg(\bigcup_{1 \le k \le m} \Big(E(I_k) \cup E(J_k) \Big)$$

$$\cup \{xy \,|\, x \in V(I_k), y \in T_k\} \cup \{xy \,|\, x \in V(J_k), y \in T'_k\} \Bigg)$$

$$\bigcup \Big(\bigcup_{1 \le i \le 3m} E(H_i)) \Big).$$

Then G is 4-connected and $K_{1,4}$-free, and satisfies $\delta(G) = 6$. However, we easily see that G does not have a 4-factor (for example, if we apply Lemma 2.1 with $S = \bigcup_{1 \le k \le m}(V(I_k) \cup V(J_k))$ and $T = \bigcup_{1 \le i \le 3m} V(H_i)$, then we get $\theta(S, T) = -2m$).

Example 1.5. *There exist infinitely many 5-connected $K_{1,4}$-free graphs G such that G has no 4-factor. Let $m \ge 3$ be an arbitrary odd integer. Let $I_1, I_2, \ldots, I_m, H_1, H_2,$*

…, H_{3m}, L_1, L_2 be disjoint graphs such that I_k is isomorphic to the complete graph of order 3 for each $1 \le k \le m$, H_i is isomorphic to the complete graph of order 3 for each $1 \le i \le 3m$, and L_1 and L_2 are isomorphic to the complete graph of order $3m$. Write $V(H_i) = \{v_{i,1}, v_{i,2}, v_{i,3}\}$ for each $1 \le i \le 3m$, and $V(L_j) = \{w_{j,1}, w_{j,2}, \ldots, w_{j,3m}\}$ for each $j = 1, 2$. Now define a graph G by

$$\begin{aligned}
V(G) =& \Big(\bigcup_{1\le k\le m} V(I_k) \Big) \cup \Big(\bigcup_{1\le i\le 3m} V(H_i) \Big) \cup V(L_1) \cup V(L_2), \\
E(G) =& \Big(\bigcup_{1\le k\le m} E(I_k) \\
& \cup \{xy \mid x \in V(I_k), y \in (V(H_{3(k-1)+1}) \cup V(H_{3(k-1)+2}) \cup V(H_{3(k-1)+3}))\} \Big) \\
& \cup \Big(\bigcup_{1\le i\le 3m} E(H_i) \Big) \cup \{v_{i,1}w_{1,i} \mid 1 \le i \le 3m\} \cup \{v_{i,2}w_{2,i} \mid 1 \le i \le 3m\}.
\end{aligned}$$

Then G is 5-connected and $K_{1,4}$-free. However, we easily see that G does not have a 4-factor (for example, if we apply Lemma 2.1 with $S = \bigcup_{1\le k\le m} V(I_k)$ and $T = \bigcup_{1\le i\le 3m} V(H_i)$, then we get $\theta(S,T) = -2$).

Example 1.6. *There exist infinitely many 4-connected $K_{1,3}$-free graphs G such that G has no 4-factor. Let $m \ge 3$ be an arbitrary integer. Let $I_1, I_2, \ldots, I_m$, H_1, H_2, …, H_{2m}, L_1, L_2 be disjoint graphs such that I_k is isomorphic to the complete graph of order 2 for each $1 \le k \le m$, H_i is isomorphic to the complete graph of order 3 for each $1 \le i \le 2m$, L_1 is isomorphic to the complete graph of order $2m+1$, and L_2 is isomorphic to the complete graph of order $2m-1$. Write $V(H_i) = \{v_{i,1}, v_{i,2}, v_{i,3}\}$ for each $1 \le i \le 2m$, $V(L_1) = \{w_{1,1}, w_{1,2}, \ldots, w_{1,2m+1}\}$, and $V(L_2) = \{w_{2,1}, w_{2,2}, \ldots, w_{2,2m-1}\}$. Now define a graph G by*

$$\begin{aligned}
V(G) =& \Big(\bigcup_{1\le k\le m} V(I_k) \Big) \cup \Big(\bigcup_{1\le i\le 2m} V(H_i) \Big) \cup V(L_1) \cup V(L_2), \\
E(G) =& \Big(\bigcup_{1\le k\le m} E(I_k) \\
& \cup \{xy \mid x \in V(I_k), y \in (V(H_{2(k-1)+1}) \cup V(H_{2(k-1)+2})\} \Big) \\
& \cup \Big(\bigcup_{1\le i\le 2m} E(H_i) \Big) \\
& \cup \{v_{i,1}w_{1,i} \mid 1 \le i \le 2m\} \cup \{v_{i,2}w_{2,i} \mid 1 \le i \le 2m-1\} \\
& \cup \{v_{2m,2}w_{1,2m+1}\}.
\end{aligned}$$

Then G is 4-connected and $K_{1,3}$-free. However, we easily see that G does not have a 4-factor (for example, if we apply Lemma 2.1 with $S = \bigcup_{1\le k\le m} V(I_k)$ and $T = \bigcup_{1\le i\le 2m} V(H_i)$, then we get $\theta(S,T) = -2$).

Our notation is standard, and is mostly taken from Diestel [2]. Possible exceptions are as follows. Let G be a graph. For $x \in V(G)$, $N(x) = N_G(x)$ denotes the set of vertices adjacent to x in G; thus $\deg_G(x) = |N_G(x)|$. For $A \subseteq V(G)$, we let $N(A)$ denote the union of $N(x)$ as x ranges over A. For A, $B \subseteq V(G)$ with $A \cap B = \emptyset$, $E(A,B)$ denotes the set of those edges of G which join a vertex in A and a vertex in B. For $A \subseteq V(G)$, the graph obtained from G by deleting all vertices in A together with the edges incident with them is denoted by $G - A$. For $A \subseteq V(G)$, we let $G[A]$ denote the subgraph induced by A in G. We often identify a vertex x of G with the set $\{x\}$; for example, when B is a subset of $V(G)$ with $x \notin B$, we write $E(x,B)$ for $E(\{x\},B)$.

2. Preliminary results

In this section, we state preliminary lemmas, which we use in the proof of Theorems 1.1 and 1.2.

Let G be a graph. For S, $T \subseteq V(G)$ with $S \cap T = \emptyset$, define $\theta(S,T)$ by

$$\theta(S,T) = 4|S| + \Sigma_{y \in T}(\deg_{G-S}(y) - 4) - h(S,T),$$

where $h(S,T)$ denotes the number of those components C of $G - S - T$ such that $|E(T,V(C))|$ is odd. The following lemma is a special case of the f-Factor Theorem of Tutte [6].

Lemma 2.1. *The graph G has a 4-factor if and only if $\theta(S,T) \geq 0$ for all S, $T \subseteq V(G)$ with $S \cap T = \emptyset$.*

The following two lemmas are proved in [4].

Lemma 2.2. *Let S, $T \subseteq V(G)$ be subsets of $V(G)$ with $S \cap T = \emptyset$ for which $\theta(S,T)$ becomes smallest. Then the following hold.*

(i) Let C be a component of $G - S - T$ such that $|E(T,V(C))| = 1$. Then $|V(C)| \geq 2$.

(ii) Suppose that S and T are chosen $|T|$ is as small as possible, subject to the condition that $\theta(S,T)$ is smallest. Then $\deg_{G[T]}(y) \leq 2$ for every $y \in T$.

Lemma 2.3. *Suppose that $\delta(G) \geq 6$ and $|V(G)| \leq \delta(G) + 3$. Then G has a 4-factor.* □

3. Notation

We prove Theorems 1.1 and 1.2 simultaneously. Thus let G be a $K_{1,t}$-free graph with $t = 5$ or 4. Let $\kappa = 7$ or 5 according as $t = 5$ or 4, and assume that G is a κ-connected. In the case where $t = 4$, assume further that $\delta(G) \geq 6$. In this section, we fix notation for the proof of the theorems.

We have $\delta(G) \geq 6$ by assumption. Thus in view of Lemma 2.3, we may assume $|V(G)| \geq \delta(G) + 4 \geq \kappa + 4$.

Let S, T be subsets of $V(G)$ with $S \cap T = \emptyset$ for which $\theta(S,T)$ becomes smallest. We choose S, $T \subseteq V(G)$ so that $|T|$ is as small as possible, subject to the condition that $\theta(S,T)$ is smallest. We show that $\theta(S,T) \geq 0$. By Lemma 2.1, this will imply that Theorems 1.1 and 1.2 hold.

If $T = \emptyset$, then $h(S,T) = 0$, and hence $\theta(S,T) = 4|S| \geq 0$. Thus we may assume that $T \neq \emptyset$.

We call a component C of $G-S-T$ an odd component or an even component according as $|E(T, V(C))|$ is odd or even.

Let $C_1, \dots, C_k$ be the components of $G - S - T$. We may assume that there exists a with $0 \leq a \leq k$ such that $|E(T, V(C_i))| = 1$ for each $1 \leq i \leq a$ and $|E(T, V(C_i))| \neq 1$ for each $a+1 \leq i \leq k$. Then the components $C_1, \dots, C_a$ are odd components. We may further assume that there exists b with $0 \leq b \leq k-a$ such that C_i is an odd component for each $a+1 \leq i \leq a+b$ and C_i is an even component for each $a+b+1 \leq i \leq k$. Then $h(S,T) = a+b$. Set $U = \cup_{i=1}^{a} V(C_i)$ and $U' = V(G) - S - T - U$. For each $y \in T$, set $\alpha(y) = |N(y) \cap U|$ and $\beta(y) = |N(y) \cap U'|$. We also set $\beta(A) = \sum_{y \in A} \beta(y)$ for a subset A of T. The following claim is proved in [4] (in [4], it is proved under the assumption of Theorem B, but the argument works with no change in the present situation).

Claim 3.1.

(i) $a = \sum_{y \in T} \alpha(y)$.

(ii) $b \leq (\sum_{y \in T} \beta(y))/3$.

Let $1 \leq i \leq a$. Since G is κ-connected and $\kappa \geq 2$, G is 2-connected. Since $|V(C_i)| \geq 2$ by Lemma 2.2(i), this implies that there exists an edge joining S and $V(C_i) - N(T)$. Let $x_i u_i$ be such an edge ($x_i \in S$, $u_i \in V(C_i) - N(T)$). Now for $x \in S$, let $L(x) = \{u_i \,|\, 1 \leq i \leq a, x_i = x\}$. Then

$$\sum_{x \in S} |L(x)| = a. \tag{3.1}$$

We now look at components of $G[T]$. Recall that $T \neq \emptyset$. Let H_1, $H_2, \dots, H_m$ be the components of $G[T]$. By Lemma 2.2 (ii), H_i is a path or a cycle for each $1 \leq i \leq m$. In the remainder of this section, we assign a real number θ_i to each H_i, and show that $\theta(S,T) \geq \sum_{1 \leq i \leq m} \theta_i$.

Claim 3.2. *Let $1 \leq i \leq m$, and suppose that H_i is a cycle of order 3. Let z, $z' \in V(H_i)$ with $z \neq z'$. Then $|N(\{z,z'\}) \cap (S \cup U \cup U')| \geq \kappa - 1$.*

Proof. Suppose that $|N(\{z,z'\}) \cap (S \cup U \cup U')| \leq \kappa - 2$. Then $|N(\{z,z'\}) \cap (V(G) - V(H_i))| = |(N(\{z,z'\}) \cap (S \cup U \cup U')) \cup (V(H_i) - \{z,z'\})| \leq \kappa - 1$. Since G is κ-connected, this implies $|V(G)| = |V(H_i) \cup (N(\{z, z'\}) \cap (S \cup U \cup U'))| \leq \kappa + 1$, which contradicts the assumption that $|V(G)| \geq \kappa + 4$. □

Claim 3.3. *Let $1 \leq i \leq m$, and suppose that H_i is a cycle of order 3. Then $|N(V(H_i)) \cap (S \cup U \cup U')| \geq \kappa$.*

Proof. If $|N(V(H_i)) \cap (S \cup U \cup U')| \leq \kappa - 1$, then since G is κ-connected, it follows that $|V(G)| = |V(H_i) \cup (N(V(H_i)) \cap (S \cup U \cup U'))| \leq \kappa + 2$, a contradiction. □

Before defining the numbers θ_i ($1 \leq i \leq m$), we choose, for each $1 \leq i \leq m$, disjoint subsets $T_{i,1}$, $T_{i,2} \dots, T_{i,j_i}$ of $V(H_i)$ such that $E(T_{i,j}, T_{i,j'}) = \emptyset$ for any j, j' with $j \neq j'$ as follows.

(D1) Assume that H_i is a path of order l, where l is even, or H_i is a cycle of order l, where $l \geq 4$. In this case, we let j_i be the independence number of H_i. Note that we can take two independent sets Y, $Y' \subseteq V(H_i)$ having cardinality j_i so that $Y \cap Y' = \emptyset$. We may assume $\beta(Y) \leq \beta(Y')$. Write $Y = \{z_1, \dots, z_{j_i}\}$. Under this notation, we set $T_{i,j} = \{z_j\}$ for each $1 \leq j \leq j_i$. We have $\beta(V(H_i)) \geq \beta(Y) + \beta(Y') \geq 2\beta(Y) = 2\sum_{1\leq j\leq j_i} \beta(T_{i,j})$.

(D2) Assume that H_i is a path of order l, where l is odd and $l \geq 3$. Write $H_i = v_1v_2\dots v_l$, and set $Y = \{v_1, v_3, \dots v_l\}$ and $Y' = \{v_2, v_4, \dots v_{l-1}\}$. Note that $\beta(V(H_i)) = \beta(Y) + \beta(Y')$. Our choice of $T_{i,j}$ depends on whether $\beta(Y) \leq \beta(Y') + 6$ or $\beta(Y) \geq \beta(Y') + 7$.

(D2-1) Assume that $\beta(Y) \leq \beta(Y') + 6$. In this case, we set $j_i = |Y|$, write $Y = \{z_1, \dots, z_{j_i}\}$, and set $T_{i,j} = \{z_j\}$ for each $1 \leq j \leq j_i$. We have $\beta(V(H_i)) \geq 2\beta(Y) - 6 = 2\sum_{1\leq j\leq j_i} \beta(T_{i,j}) - 6$.

(D2-2) Assume that $\beta(Y) \geq \beta(Y') + 7$. In this case, we set $j_i = |Y'|$, write $Y' = \{z_1, \dots, z_{j_i}\}$, and set $T_{i,j} = \{z_j\}$ for each $1 \leq j \leq j_i$. We have $\beta(V(H_i)) \geq 2\beta(Y') + 7 = 2\sum_{1\leq j\leq j_i} \beta(T_{i,j}) + 7$.

(D3) Assume that H_i is a cycle of order 3 or a path of order 1. In this case, we set $j_1 = 1$ and $T_{i,1} = V(H_i)$.

We estimate the lower bound of $\theta(S, T)$. For each $x \in S$, let $\mathcal{N}(x) = \{T_{i,j} \mid 1 \leq i \leq l, 1 \leq j \leq i_j, x \in N(T_{i,j})\}$. For each $1 \leq i \leq m$, set

$$\theta_i = \frac{4}{t-1} \sum_{1\leq j\leq j_i} \left|N(T_{i,j}) \cap (S \cup U)\right| + \frac{2}{3}\beta(V(H_i)) - \sum_{y\in V(H_i)} (4 - \deg_{H_i}(y)).$$

The following two claims are proved in [4], but we include their proofs for the convenience of the reader.

Claim 3.4. $(t-1)|S| \geq \sum_{1\leq i\leq m} \sum_{1\leq j\leq j_i} \left|N(T_{i,j}) \cap (S \cup U)\right|$

Proof. Since G is $K_{1,t}$-free, we have $|\mathcal{N}(x)| + |L(x)| \leq t-1$ for every $x \in S$. Note that $\sum_{x\in S} |\mathcal{N}(x)| = \sum_{1\leq i\leq m}\sum_{1\leq j\leq j_i} |N(T_{i,j}) \cap S|$, and that $\sum_{x\in S}|L(x)| = a = \sum_{1\leq i\leq m}\sum_{y\in V(H_i)} \alpha(y)$ by (3.1). Consequently

$$\begin{aligned}(t-1)|S| &\geq \sum_{1\leq i\leq m}\left(\sum_{1\leq j\leq j_i}\left|N(T_{i,j})\cap S\right| + \sum_{y\in V(H_i)}\alpha(y)\right)\\ &\geq \sum_{1\leq i\leq m}\Big(\sum_{1\leq j\leq j_i}(|N(T_{i,j})\cap S| + |N(T_{i.j})\cap U|)\Big),\end{aligned}$$

as desired. □

Claim 3.5. $\theta(S,T) \geq \sum_{1\leq i\leq m} \theta_i.$

Proof. Note that $h(S,T) = a + b \leq \sum_{1\leq j\leq m}\sum_{y\in V(H_i)}(\alpha(y)+\beta(y)/3)$ by Claim 3.1. Also $\deg_{G-S}(y) = \deg_{H_i}(y) + \alpha(y) + \beta(y)$ for every $y \in T$. Therefore it follows from Claim 3.4 that

$$\begin{aligned}\theta(S,T) \geq& \frac{4}{t-1}\sum_{1\leq i\leq m}\sum_{1\leq j\leq j_i}\left|N(T_{i,j})\cap(S\cup U)\right|\\ &+\sum_{1\leq i\leq m}\sum_{y\in V(H_i)}\left(\deg_{H_i}(y)+\alpha(y)+\beta(y)-4\right)\\ &-\sum_{1\leq i\leq m}\sum_{y\in V(H_i)}\left(\alpha(y)+\frac{1}{3}\beta(y)\right)\\ =&\sum_{1\leq i\leq m}\left\{\frac{4}{t-1}\sum_{1\leq j\leq j_i}\left(\left|N(T_{i,j})\cap(S\cup U)\right|\right)+\frac{2}{3}\beta(V(H_i))\right.\\ &\left.-\sum_{y\in V(H_i)}(4-\deg_{H_i}(y))\right\},\end{aligned}$$

as desired. □

4. Proof of Theorems 1.1 and 1.2

We continue with the notation of the proceeding section, and complete the proof of Theorems 1.1 and 1.2. In view of Claim 3.5, it suffices to show that $\theta_i \geq 0$ for each $1 \leq i \leq m$. Fix i with $1 \leq i \leq m$.

Case 1. H_i is a path of order l, where l is even, or H_i is a cycle of order l, where $l \geq 4$.

Let z_j be as in (D1). Thus $T_{i,j} = \{z_j\}$ for each $1 \leq j \leq j_i$. Set

$$p_i = \frac{4}{t-1}((t+2)j_i - \sum_{1\leq j\leq j_i} \deg_{H_i}(z_j)) - \sum_{y\in V(H_i)} (4 - \deg_{H_i}(y)).$$

By (D1), $\beta(V(H_i)) \geq 2\sum_{1\leq j\leq j_i} \beta(z_j) = 2\sum_{1\leq j\leq j_i} |N(z_j)\cap U'|$. Also note that $\delta(G) \geq t+2$. Hence

$$\begin{aligned}
\theta_i &= \frac{4}{t-1}\sum_{1\leq j\leq j_i} |N(z_j)\cap(S\cup U)| + \frac{2}{3}\beta(V(H_i)) - \sum_{y\in V(H_i)}(4-\deg_{H_i}(y))\\
&\geq \frac{4}{t-1}\sum_{1\leq j\leq j_i} \big|N(z_j)\cap(S\cup U)\big| + \frac{2}{3}\Big(2\sum_{1\leq j\leq j_i}|N(z_j)\cap U'|\Big) - \sum_{y\in V(H_i)}(4-\deg_{H_i}(y))\\
&\geq \frac{4}{t-1}\Big(\sum_{1\leq j\leq j_i}|N(z_j)\cap(S\cup U\cup U')|\Big) - \sum_{y\in V(H_i)}(4-\deg_{H_i}(y))\\
&= \frac{4}{t-1}\Big(\sum_{1\leq j\leq j_i}\deg_G(z_j) - \sum_{1\leq j\leq j_i}\deg_{H_i}(z_j)\Big) - \sum_{y\in V(H_i)}(4-\deg_{H_i}(y))\\
&\geq p_i.
\end{aligned}$$

Thus it suffices to show that $p_i \geq 0$.

First assume that H_i is a path of order l, where l is even. Then $j_i = l/2$, $\sum_{1\leq j\leq j_i}\deg_{H_i}(z_j) = 2j_i - 1$, and $\sum_{y\in V(H_i)}(4-\deg_{H_i}(y)) = 2l+2$. Hence

$$\begin{aligned}
p_i &= \frac{4}{t-1}\Big((t+2)j_i - (2j_i-1)\Big) - (2l+2)\\
&= \frac{2l}{t-1} - \frac{2t-6}{t-1}\\
&\geq \frac{4}{t-1} - \frac{2t-6}{t-1} \geq 0.
\end{aligned}$$

Next assume that H_i is a cycle of order l, where $l \geq 4$. Then $\sum_{1\leq j\leq j_i}\deg_{H_i}(z_j) = 2j_i$ and $\sum_{y\in V(H_i)}(4-\deg_{H_i}(z_j)) = 2l$. In the case where l is even, $j_i = l/2$, and hence

$$\begin{aligned}
p_i &= \frac{4}{t-1}\Big((t+2)j_i - 2j_i\Big) - 2l\\
&= \frac{2}{t-1}l > 0;
\end{aligned}$$

in the case where l is odd, $j_i = (l-1)/2$, and hence

$$\begin{aligned} p_i &= \frac{4}{t-1}\Big((t+2)j_i - 2j_i\Big) - 2l \\ &= \frac{2l}{t-1} - \frac{2t}{t-1} \\ &\geq \frac{10}{t-1} - \frac{2t}{t-1} \geq 0. \end{aligned}$$

This completes the discussion for Case 1.

Case 2. H_i is a path of order l, where l is odd.

Recall that $\delta(G) \geq t+2$.

First assume that the sets $T_{i,j}$ are difined as in (D2-1). Then $j_i = (l+1)/2$. Let z_j be as in (D2-1). Thus $T_{i,j} = \{z_j\}$ for each $1 \leq j \leq j_i$. By (D2-1), $\beta(V(H_i)) \geq 2\sum_{1\leq j\leq j_i} |N(z_j)\cap U'| - 6$. We also have $\sum_{1\leq j\leq j_i} \deg_{H_i}(z_j) = 2j_i - 2$ and $\sum_{y\in V(H_i)}(4-\deg_{H_i}(y)) = 2l+2$. Hence

$$\begin{aligned} \theta_i &= \frac{4}{t-1}\sum_{1\leq j\leq j_i} |N(z_j)\cap(S\cup U)| + \frac{2}{3}\beta(V(H_i)) - (2l+2) \\ &\geq \frac{4}{t-1}\sum_{1\leq j\leq j_i} \big|N(z_j)\cap(S\cup U)\big| + \frac{2}{3}\Big(2\sum_{1\leq j\leq j_i}|N(z_j)\cap U'| - 6\Big) - (2l+2) \\ &\geq \frac{4}{t-1}\sum_{1\leq j\leq j_i} \big|N(z_j)\cap(S\cup U\cup U')\big| - 4 - (2l+2) \\ &= \frac{4}{t-1}\sum_{1\leq j\leq j_i} \Big(\deg_G(z_j) - \deg_{H_i}(z_j)\Big) - 4 - (2l+2) \\ &\geq \frac{4}{t-1}((t+2)j_i - 2j_i + 2) - 4 - (2l+2) \\ &= \frac{2l}{t-1} - \frac{4t-14}{t-1} \\ &\geq \frac{6}{t-1} - \frac{4t-14}{t-1} \geq 0. \end{aligned}$$

Next assume that the sets $T_{i,j}$ are difined as in (D2-2). Then $j_i = (l-1)/2$. Let z_j be as in (D2-2). Thus $T_{i,j} = \{z_j\}$ for each $1 \leq j \leq j_i$. By (D2-2), $\beta(V(H_i)) \geq 2\sum_{1\leq j\leq j_i} |N(z_j)\cap$

$U'|+7$. We also have $\sum_{1\le j\le j_i}\deg_{H_i}(z_j)=2j_i$ and $\sum_{y\in V(H_i)}(4-\deg_{H_i}(y))=2l+2$. Hence

$$\begin{aligned}
\theta_i &= \frac{4}{t-1}\sum_{1\le j\le j_i}|N(z_j)\cap(S\cup U)|+\frac{2}{3}\beta(V(H_i))-(2l+2)\\
&\ge \frac{4}{t-1}\sum_{1\le j\le j_i}\big|N(z_j)\cap(S\cup U)\big|+\frac{2}{3}(2\sum_{1\le j\le j_i}|N(z_j)\cap U'|+7)-(2l+2)\\
&\ge \frac{4}{t-1}\sum_{1\le j\le j_i}\big|N(z_j)\cap(S\cup U\cup U')\big|+\frac{14}{3}-(2l+2)\\
&= \frac{4}{t-1}\sum_{1\le j\le j_i}\Big(\deg_G(z_j)-\deg_{H_i}(z_j)\Big)+\frac{14}{3}-(2l+2)\\
&\ge \frac{4}{t-1}((t+2)j_i-2j_i)+\frac{14}{3}-(2l+2)\\
&= \frac{2l}{t-1}+\frac{2t-8}{3(t-1)}>0.
\end{aligned}$$

Case 3. H_i is a cycle of order 3.

We have $j_i=1$ and $T_{i,1}=V(H_i)$ by (D3), and $\sum_{y\in V(H_i)}(4-\deg_{H_i}(y))=6$.

If $\beta(H_i)=0$, then $|N(V(H_i)\cap(S\cup U)|=|N(V(H_i))\cap(S\cup U\cup U')|\ge\kappa$ by the definition of β and Claim 3.3, and hence $\theta_i\ge 4/(t-1)\cdot\kappa-6>0$. Thus we may assume that $\beta(H_i)\ge 1$. Write $V(H_i)=\{z,z',z''\}$ with $\beta(z)\le\beta(z')\le\beta(z'')$. Then $\beta(z'')\ge 1$. Now we divide the proof into two subcases according as $t=5$ or $t=4$.

Subase 3.1. $t=5$.

First assume that $\beta(z')=0$. Then $\beta(z)=0$. By the definition of β and Claim 3.2, $|N(\{z,z'\})\cap(S\cup U))|=|N(\{z,z'\})\cap(S\cup U\cup U')|\ge\kappa-1=6$. Hence

$$\begin{aligned}
\theta_i =&|N(\{z,z',z''\})\cap(S\cup U)|+\frac{2}{3}\beta(z'')-6\\
\ge&\big|N(\{z,z'\})\cap(S\cup U)\big|-6\ge 0.
\end{aligned}$$

Next assume that $\beta(z')\ge 1$. Since $\delta(G)\ge 7$, we obtain

$$\begin{aligned}
\theta_i =&|N(\{z,z',z''\})\cap(S\cup U)|+\frac{2}{3}\beta(z)+\frac{2}{3}\beta(z')+\frac{2}{3}\beta(z'')-6\\
\ge&\big|N(z)\cap(S\cup U)\big|+\frac{2}{3}\beta(z)+\frac{2}{3}\beta(z')+\frac{2}{3}\beta(z'')-6\\
=&\big|N(z)\cap(S\cup U\cup U')\big|-\frac{1}{3}\beta(z)+\frac{2}{3}\beta(z')+\frac{2}{3}\beta(z'')-6\\
\ge&\big|N(z)\cap(S\cup U\cup U')\big|+\frac{1}{3}\beta(z')+\frac{2}{3}\beta(z'')-6
\end{aligned}$$

$$
\begin{aligned}
&\geq (\deg_G(z) - \deg_{H_i}(z)) + \frac{1}{3} + \frac{2}{3} - 6 \\
&\geq (7-2) + 1 - 6 = 0.
\end{aligned}
$$

Subase 3.2. $t = 4$.

Since $\delta(G) \geq 6$, we obtain

$$
\begin{aligned}
\theta_i =& \frac{4}{3}|N(\{z, z', z''\}) \cap (S \cup U)| + \frac{2}{3}\beta(z) + \frac{2}{3}\beta(z') + \frac{2}{3}\beta(z'') - 6 \\
\geq& \frac{4}{3}\big|N(z) \cap (S \cup U)\big| + \frac{2}{3}\beta(z) + \frac{2}{3}\beta(z') + \frac{2}{3}\beta(z'') - 6 \\
=& \frac{4}{3}\big|N(z) \cap (S \cup U \cup U')\big| - \frac{2}{3}\beta(z) + \frac{2}{3}\beta(z') + \frac{2}{3}\beta(z'') - 6 \\
\geq& \frac{4}{3}\big|N(z) \cap (S \cup U \cup U')\big| + \frac{2}{3}\beta(z'') - 6 \\
\geq& \frac{4}{3}(\deg_G(z) - \deg_{H_i}(z)) + \frac{2}{3} - 6 \\
\geq& \frac{4}{3}(6-2) + \frac{2}{3} - 6 = 0.
\end{aligned}
$$

Case 4. H_i is a path of order 1.

By (D3), $j_i = 1$. Write $V(H_i) = \{z\}$. Note that $\delta(G) \geq 6$. Hence

$$
\begin{aligned}
\theta_i &= \frac{4}{t-1}|N(z) \cap (S \cup U)| + \frac{2}{3}\beta(z) - (4 - \deg_{H_i}(z)) \\
&\geq \frac{2}{3}\big|N(z) \cap (S \cup U \cup U')\big| - (4 - \deg_{H_i}(z)) \\
&= \frac{2}{3}\deg_G(z) - 4 \\
&\geq \frac{2}{3} \cdot 6 - 4 = 0.
\end{aligned}
$$

This completes the proof of Theorems 1.1 and 1.2.

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