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Graph isomorphisms in discrete Morse theory

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Abstract

A discrete Morse function f on a graph G induces a sequence of subgraphs of G. In [1], the authors introduce a notion of equivalence between discrete Morse functions based on a sequence of homology groups of the corresponding subgraphs of G. In this paper, we use the homology sequence to study a new notion of equivalence between discrete Morse functions. This equivalence is based on the isomorphism type of the subgraphs of G. We count the number of equivalence classes on star graphs S_n and deduce an upper bound for the number of equivalence classes for a large collection of graphs.

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1. Introduction

Discrete Morse theory was invented by Robin Forman [4] as an analogue to "smooth" Morse theory popularized by Milnor [11]. Many classical results in Morse theory, such as the Morse inequalities, carry over into the discrete setting (See [6] for an excellent summary). Applications of discrete Morse theory are vast, ranging from applications in configuration spaces [12] to computer science search problems [5].

In its most general setting, discrete Morse theory is used to study an n-dimensional simplicial complex. In this paper, we utilize discrete Morse theory to study a 1-dimensional simplical complex i.e. a graph. Let f be a discrete Morse function (Definition 2.1) on a graph G. Then f induces a strictly increasing sequence of subgraphs of G (Definition 2.3). Inspired by Ayala et al. [1], we define a new notion (Definition 3.4) of equivalence, called *graph equivalence*, between discrete Morse functions based on the isomorphism type of the induced subgraphs. In Section 3 we compare this notion of graph equivalence with existing notions of equivalence. We then count the number of equivalence classes on a star S_n in Section 4. We introduce the property of a graph being $\ell\, b_0$-determined and $\ell\, b_1$-determined, two properties concerning the addition of edges yielding isomorphic subgraphs. This property is a kind of "inverse" of the edge reconstruction of G ([8, Section 2.3])- the former is interested in attaching edges to a subgraph to obtain a unique graph while the latter is concerned with removing an edge and obtaining unique graphs. This concept is used to compute an upper bound for the number of equivalence classes for graphs satisfying certain properties. In particular, we obtain an upper bound for the number of graph equivalence classes on n copies of K_3 joined at a single vertex, the so-called windmill graph W_3^n.

2. Preliminaries

Let $G = (V(G), E(G))$ be a finite, loopless graph without multi-edges. We will not distinguish between G as a graph and G as a 1-dimensional simplicial complex. We call an edge or a vertex of G a *simplex*. Let $H \subseteq G$ be a subgraph of G. We write $H \cup v$ for the subgraph of G whose edge set is $E(H)$ and whose vertex set is $V(H) \cup \{v\}$. If an edge $e = uv$ for vertices $u, v \in H$, we write $H \cup e$ for the subgraph of G whose edge set is $E(H) \cup \{e\}$ and whose vertex set is $V(H)$. We sometimes write $H \cup \sigma$ for σ an edge or a vertex. We say that $H \cup \sigma$ is *attaching* a vertex, edge, or simplex to H.

Definition 2.1. *A discrete Morse function f on a connected graph G is a function $f : G \to \mathbb{R}$ such that for every vertex $v \in G$*

$$|\{e : f(v) \geq f(e) \text{ for some edge } e \text{ incident to } v\}| \leq 1$$

and for every edge e

$$|\{v : f(v) \geq f(e) \text{ for some vertex } v \text{ incident to } e\}| \leq 1.$$

A vertex v of G is said to be a critical vertex with respect to a discrete Morse function f if

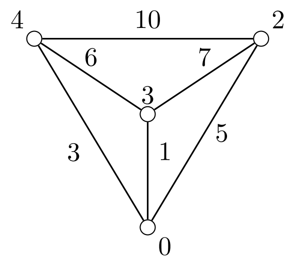
$$|\{e : f(v) \geq f(e) \text{ for some edge } e \text{ incident to } v\}| = 0.$$

In other words, a vertex v is critical if the value $f(v)$ assigned to it satisfies $f(v) < f(e)$ for every edge e incident with v. An edge e of G is said to be a critical edge with respect to a discrete Morse function f if

$$|\{v : f(v) \geq f(e) \text{ for some vertex } v \text{ incident to } e\}| = 0.$$

As in the case of a critical vertex, an edge e is critical if the value $f(e)$ satisfies $f(e) > f(v)$ for all vertices v incident with e. We call a critical edge or vertex a critical simplex. The number $f(\sigma)$ for σ a critical simplex is called a critical value.

Example 2.2. *Define the function f on $G = K_4$ as follows:*



Then f is a discrete Morse function. The critical vertices are $f^{-1}(0)$ and $f^{-1}(2)$ while the critical edges are $f^{-1}(5), f^{-1}(6), f^{-1}(7)$, and $f^{-1}(10)$.

We say that a discrete Morse function f is *excellent* if all its critical values are distinct. Thus if f has m critical simplices, we may write $c_0 < c_1 < ... < c_{m-1}$ for its critical values. *All discrete Morse functions throughout this paper will be assumed to be excellent.* We will occasionally note this hypothesis for emphasis.

Definition 2.3. *Let G be a connected graph. Given $a \in \mathbb{R}$ the level subcomplex $G(a)$ is defined to be the induced subcomplex of G consisting of all simplices σ with $f(\sigma) \leq a$. For each critical value $c_0 < \ldots < c_{m-1}$ of f, we consider the induced sequence of level subcomplexes $\{v\} = G(c_0) \subset G(c_1) \subset \ldots \subset G(c_{m-1})$. We say that $G(c_i)$ is a stage of f or the i^{th} stage of f.*

Recall that the j^{th} *Betti number of* G is defined by $b_j(G) = b_j = \text{rank}(H_j(G; \mathbb{Z}))$. The following Theorem shows that when studying Betti numbers of a graph, we only need to consider b_0 and b_1. Furthermore, if G is connected so that $b_0(G) = 1$, then $|V(G)|$ and $|E(G)|$ completely determine b_1.

Theorem 2.4. [10] *Let G be a graph. Then $H_i(G) = 0$ for $i > 1$ and $b_0(G) - b_1(G) = |V(G)| - |E(G)|$.*

Now to each level subcomplex $G(c_i)$, we consider the Betti numbers $b_0(G(c_i)) = b_0(c_i)$ and $b_1(G(c_i)) = b_1(c_i)$. The *homological sequences of f* are the two sequences $B_0, B_1 \colon \{0, 1, \ldots, m-1\} \to \mathbb{N}$ defined by $B_0(i) = b_0(c_i)$ and $B_1(i) = b_1(c_i)$. The following are easily verified for an excellent discrete Morse function. We will use them throughout without reference.

Proposition 2.5. [1] *The homological sequences of f satisfy $|B_0(i+1) - B_0(i)| = 0, 1$ and $B_1(i+1) - B_1(i) = 0, 1$. In addition, for all $i = 0, 1, \ldots, m-2$, exactly one of the following holds:*

1. $B_0(i) = B_0(i+1)$
2. $B_1(i) = B_1(i+1)$.

Since b_1 counts the number of independent cycles in a graph, $B_1(i) = 0$ for all stages i whenever the graph in question is a tree. It thus follows by the above Proposition that $|B_0(i+1) - B_0(i)| = 1$ for all i.

3. Equivalence of discrete Morse functions

We have two preexisting notions of equivalence of discrete Morse functions on graphs. The first is due to Foreman [7].

Definition 3.1. *Two discrete Morse functions f and g on G are said to be equivalent if for every vertex v and every edge e incident to v, $f(v) < f(e)$ if and only if $g(v) < g(e)$*

Ayala, Fernández, and Vilches have characterized equivalent discrete Morse functions in terms of their inducing the same gradient vector field [2].

The same authors along with Fernández-Ternero use homology to introduce a new notion of equivalence of discrete Morse functions in [1].

Definition 3.2. *Two excellent discrete Morse functions f and g defined on a graph G with critical values $a_0 < a_1 < \ldots < a_{m-1}$ and $c_0 < c_1 < \ldots < c_{m-1}$ respectively are homologically equivalent if $b_0(a_i) = b_0(c_i)$ and $b_1(a_i) = b_1(c_i)$ for all $0 \le i \le m-1$.*

It follows from the weak Morse inequalities [6, Theorem 2.11 (I)] that the number of critical values m on a connected graph G has the form $m = b_0 + b_1 + 2k$. Let $\bigvee^{n} S^1$ be a union of n cycles of any length joined at a common vertex, and write $C_k = \frac{1}{k+1}\binom{2k}{k}$ for the k^{th} Catalan number. We reference the following Theorem several times.

Theorem 3.3. [1] *The number of homology equivalence classes of excellent discrete Morse functions with $m = b_0 + b_1 + 2k$ critical elements on a graph G is:*

1. C_k *if* G *is a tree.*
2. $C_k \binom{m-2}{2k}$ *if* $G = \bigvee^{b_1} S^1$.

Since we will be comparing multiple discrete Morse functions defined on the same graph G, we may write c_i^f to denote the i^{th} critical value with respect to the function f. We now introduce a new notion of equivalence.

Definition 3.4. *Let* $f, g : G \to \mathbb{R}$ *be two discrete Morse functions on a graph* G *with critical values* $a_0, a_1, ..., a_{m-1}$ *and* $c_0, c_1, ..., c_{m-1}$ *respectively. The functions* f *and* g *are said to be graph equivalent if* $G(a_i) \cong G(c_i)$ *for every* $0 \leq i \leq m-1$.

For any excellent discrete Morse function f on G with critical values $c_0, c_1, ..., c_{m-1}$, we have $G(c_0) = \{v\}$ so we call the level subcomplex $G(c_0)$ *trivial.*

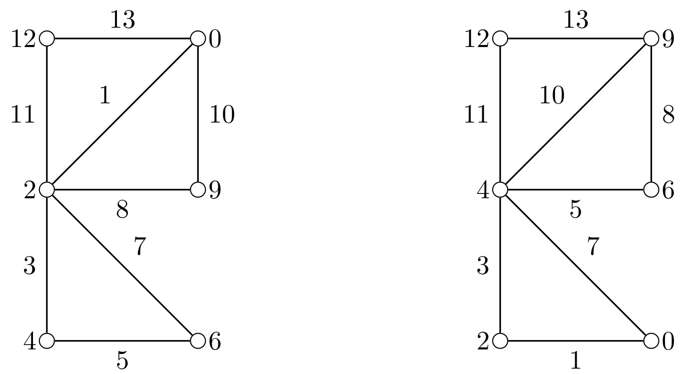
3.1. Relationships Between Types of Equivalence

Since the Betti number is an invariant of the isomorphism type, we have the following:

Proposition 3.5. *Let* f *and* g *be graph equivalent discrete Morse functions on* G*. Then* f *and* g *are homologically equivalent.*

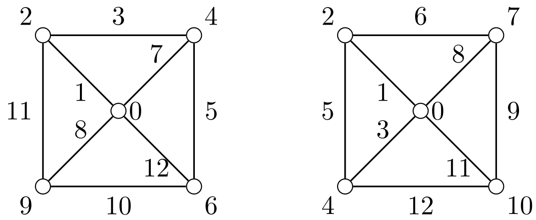
It is easy to see that if f and g are homologically equivalent, then they need not be graph equivalent. We now show that there are no implications between graph equivalent functions and equivalent functions.

Example 3.6. *Let* f *and* g *be defined by*



respectively. Let $f^{-1}(2) = g^{-1}(4) = v$ *and* $f^{-1}(3) = g^{-1}(3) = e$*. Since* $f(v) < f(e)$ *and* $g(v) > g(e)$*, we see that* f *and* g *are not equivalent. However,* f *and* g *are graph equivalent.*

Example 3.7. *Consider the functions f and g defined on a graph G by*



respectively. It is easy to check that f and g are equivalent discrete Morse functions. However, the level subcomplexes induced by f and g are not isomorphic for all critical values other than 0 *and* 12.

4. Counting on Star Graphs

Let $n \geq 2$ be a positive integer. Recall that the *star graph on n vertices* is defined by $S_n = K_{1,n-1}$ ([9, p. 17]). We call the unique vertex $c \in S_n$ of degree $n-1$ the *center of S_n* or *center vertex*.

We devote this section to counting the number of graph equivalence classes on S_n. First we prove a lemma that holds for all trees.

Lemma 4.1. *Let G be a tree and $f\colon G \to \mathbb{R}$ an excellent discrete Morse function with critical values $c_0 < c_1 < \ldots < c_{m-1}$. Then $G(c_{i+1})$ is obtained from $G(c_i)$ by attaching an odd number of simplices to $G(c_i)$ for $i = 0, 1, \ldots, m-2$.*

Proof. We first note that since G is a tree, for any two level subcomplexes $G(c_i), G(c_{i+1})$, we have $b_0(c_{i+1}) - b_0(c_i) = \pm 1$ by the remark immediately after the statement of Proposition 2.5. We consider the case $b_0(c_{i+1}) - b_0(c_i) = 1$. The case $b_0(c_{i+1}) - b_0(c_i) = -1$ is similar.

Hence assume that $b_0(c_{i+1}) - b_0(c_i) = 1$. Then a new component T_0 with critical simplex u is attached to $G(c_i)$. Now u must be a vertex since noncritical simplices come in disjoint pairs. That is, suppose vertex v is not critical because $f(v) > f(e)$ where e is an edge incident with v. But then e is also not critical. In addition, there cannot be another incident edge h to v which satisfies $f(v) > f(h)$ because that would violate the definition of being a discrete Morse function. Since we are adding a new component which is a tree and every tree has one more vertex than edge ([3, p. 82]), it follows that every tree has at least one critical vertex. But T_0 is made up of only the critical vertex u since $f(u) < f(e)$ for every edge e incident with u by definition of u being critical. If u is the only simplex attached at this stage, then this is an odd number of simplices and we are finished. Otherwise, any noncritical simplices must be attached to already existing components of G. Let T be a tree of order n in the forest $G(c_i)$ and write $T' = T \cup \bigcup_{i=1}^{\ell} \sigma_i$ for the tree in the forest $G(c_{i+1})$ of order n' obtained from T by attaching noncritical simplices. If T has e edges and T' has e' edges, then the expression $(n' + e') - (n + e)$ counts the total number of attached simplices from T to T'. Again, since the number of vertices is one more than the number of edges in any tree, we have $(n' + e') - (n + e) = (n' + (n' - 1)) - (n + (n - 1)) = 2(n' - n)$. We conclude that for any tree T in the forest $G(c_i)$, we will add an even number of simplices along with the critical simplex u which gives an odd number of simplices attached at each stage. □

A key property of S_n that will allow us to count the number of graph equivalence classes is that given any subgraph H, $H \cup \{e\}$ is uniquely determined for any $e \in E(G) - E(H)$.

Lemma 4.2. *Let H be a subgraph of S_n. Then*

- *$H \cup v$ is unique up to graph isomorphism for any vertex $v \in V(G) - V(H)$.*
- *$H \cup e$ is unique up to graph isomorphism for any edge $e \in E(G) - E(H)$.*

Proposition 4.3. *Let $S^k \subseteq S_m$ be a subgraph with k simplices, $1 \leq k \leq 2m - 1$. Then any attachment of $2l+1$ simplices to S^k which increases $b_0(S^k)$ by 1 is unique up to graph isomorphism for any k. Similarly, any attachment of $2l+1$ simplices to S^k which decreases $b_0(S^k)$ by 1 is unique up to graph isomorphism for any k.*

Proof. We show both claims simultaneously. Proceed by induction on l. For the initial case, let $l = 0$ so that we attach $2(0) + 1 = 1$ simplicies to S^k. Call the simplex we are attaching σ. By Lemma 4.2, $S^k \cup \sigma$ is unique up to graph isomorphism.

Let $l = n$ and assume that attaching any $2n + 1$ simplices to S^k resulting in the 0^{th} Betti number increasing or decreasing by 1 yields a unique graph. Consider $l = n + 1$ so that we are attaching $2(n + 1) + 1 = 2n + 3$ simplices to S^k, and suppose further that doing so increases or decreases the 0^{th} Betti number by 1. Write $S^k \cup \left[\bigcup_{i=1}^{2n+3} \tau_i\right]$ for τ_i some simplex in $S_m - S^k$. If $S^k \cup \left[\bigcup_{i=1}^{2n+3} \tau_i\right]$ contains no edges, then it was obtained by successive additions of a single vertex, hence an odd number of attachments per stage.

Otherwise, consider $\left(S^k \cup \left[\bigcup_{i=1}^{2n+3} \tau_i\right]\right) - \{\tau_j, \tau_h\} = \Lambda$ where $\tau_j \neq c$ is a vertex of degree 1 and τ_h is the edge incident with τ_j where $\tau_j, \tau_h \in \bigcup_{i=1}^{2n+3} \tau_i$. Since $b_0\left(S^k \cup \left[\bigcup_{i=1}^{2n+3} \tau_i\right]\right) = b_0(\Lambda)$ by the inductive hypothesis, Λ is obtained from S^k by attaching $2n + 1$ simplices. By Lemma 4.2, $\Lambda \cup \tau_j$ is a unique graph. Now $\Lambda \cup \tau_j$ can be obtained by the attachment of $2n + 2$ simplices to S^k. Then we can attach τ_h to $\Lambda \cup \tau_j$ and again by Lemma 4.2, $(\Lambda \cup \tau_j) \cup \tau_h$ is a unique graph with $2n + 3$ simplices attached to S^k. Thus the Proposition is proved. □

We are now able to prove our main result.

Theorem 4.4. *Let $G = S_n$ be a star graph and $m = b_0 + b_1 + 2k = 1 + 0 + 2k$ a fixed number of critical elements, $1 \leq m \leq 2n-1$, and suppose that $G(c_{m-1}) = G$. Then the number of graph equivalence classes for S_n with m critical elements is:*

$$\frac{1}{k+1}\binom{2k}{k}\sum_{y=0}^{x}\binom{x-1}{y-1}\binom{m-1}{y}$$

where

$$x = \frac{2n-m-1}{2}$$

$$k = \frac{m-1}{2}$$

where $\binom{h}{j} = 0$ if $j > h$ and $\binom{x-1}{-1} = 1$.

Proof. By Proposition 4.3, the number of possible graph isomorphism types at stage $l+1$ is completely determined by $b_0(G(c_l))$ and the number of attachments at stage l. We thus compute the total number of way to attach different numbers of simplices at different stages and multiply this by the total number of homological sequences. By Theorem 3.3, there are exactly $C_k = \frac{1}{k+1}\binom{2k}{k}$ homological sequences. Now the first attachment is always a single vertex, so we have $2n-2$ simplices to attach over $m-1$ stages. Since we must attach at least one simplex at each stage, we have $(2n-2)-(m-1) = 2n-m-1$ simplices left to distribute. By Lemma 4.1, each attachment must be an odd number of simplices. Hence we must distribute the remaining $2n-m-1$ simplices over at most $\frac{2n-m-1}{2}$ stages. Let y be an integer, $0 \leq y \leq x$ where x is defined in the statement of the Theorem. Since we have $m-1$ total stages, there are exactly $\binom{m-1}{y}$ choices of y stages. Among each stage, there are different distributions of the $2n-m-1$ remaining simplices. Because we have already counted one attachment per stage and Lemma 4.1, it follows that we must distribute an even number of simplices to each of the y stages. Write:

$$\underbrace{2\square 2\square 2\square \ldots \square 2}_{\text{x 2s}}$$

In order to count the number of ways to distribute the $2x$ simplices over y stages with an even number per stage, choose $y-1$ boxes to place a comma, and place a $+$ sign in the remaining boxes. In other words, we seek the composition number of x into y parts ([13, p. 14]). This gives us $\binom{x-1}{y-1}$ choices so that there are $\binom{x-1}{y-1}\binom{m-1}{y}$ attachment sequences, and thus there are at most $\frac{1}{k+1}\binom{2k}{k}\sum\limits_{y=0}^{x}\binom{x-1}{y-1}\binom{m-1}{y}$ graph equivalence classes for S_n.

Let $G = S_n$. We now show that there are exactly $\frac{1}{k+1}\binom{2k}{k}\sum_{y=0}^{x}\binom{x-1}{y-1}\binom{m-1}{y}$ graph equivalence classes of discrete Morse functions on S_n by constructing an excellent discrete Morse function with a specified sequence of isomorphism types. Our proof is in the spirit of [1, Theorem 6.1].

Fix $1 \leq m \leq 2n-1$ and write $m = 1+2k$. By Proposition 4.3, a subgraph S^{j-1} of S_n, $b_0(S^{j-1})$, and the number of simplices we attach to S^{j-1} determine a unique subgraph of S_n, so that the isomorphism type of G_j is completely determined by the number of components at stage j, denoted b_0^j, and the number of simplices attached at stage j, denoted d_j. Thus it suffices to define a discrete Morse function f on G which yields b_0^j components and d_j simplices attached at stage j, $0 \leq j \leq m-1$.

Let $\overline{V}$ be a set of some chosen $k+1$ critical vertices, one being the center. Let $\overline{E}$ be k critical edges which are incident with both the center and those in $\overline{V}$. We describe the construction of f by inducing on j.

Step j=0 Clearly $b_0^0 = 1$. Pick the center vertex $p_0 \in \overline{V}$. Define $f(p_0) = 0$. Then p_0 is a critical vertex.

Step j=1 We have exactly two components, so $b_0^1 = 2$. Pick another vertex $p_1 \in \overline{V}$ and set $f(p_1) = f(p_0)+1$. Note that whenever $d_j = 1$, no additional simplices need to be added. If $d_1 > 1$, then add $\frac{d_1-1}{2}$ vertices to the graph along with edges to attach the vertices to the center. Assign the $\frac{d_1-1}{2}$ vertices and $\frac{d_1-1}{2}$ edges the value $f(p_0)+1$.

Step j+1 Suppose that we have already defined f on the subgraph of G whose associated homological sequence is $B_0 = (b_0^0, b_0^1, ...b_0^j)$. Now we check if the number of connected components must increase or decrease.

- If $b_0^{j+1} - b_0^j = 1$ we take a vertex $p_{j+1} \in \overline{V}$ and define $f(p_{j+1}) = f(p_j)+1$. Add $\frac{d_{j+1}-1}{2}$ vertices as well as edges to attach these vertices to center. Assign the $\frac{d_{j+1}-1}{2}$ vertices and $\frac{d_{j+1}-1}{2}$ edges the value $f(p_j)+1$.
- If $b_0^{j+1} - b_0^j = -1$, pick $p_{j+1} \in \overline{E}$ so p_{j+1} connects a pre-existing 0 degree vertex with the center. Define $f(p_{j+1}) = f(p_j)+1$. Then add $\frac{d_{j+1}-1}{2}$ vertices to G and corresponding edges needed to attach them to the center. Assign these $\frac{d_{j+1}-1}{2}$ vertices and $\frac{d_{j+1}-1}{2}$ edges the value $f(p_j)+1$.

This yields the desired discrete Morse function f. □

5. Determined Isomorphism Type

A key property of S_n that allowed us to prove Theorem 4.4 was that given a level subcomplex, the number of simplices we wish to attach, and the number of components of the subgraph we wish to obtain, the isomorphism type is completely determined. However, for any graph G the attachment of even a single edge may yield multiple isomorphism types. Knowing how many isomorphism types are possible will yield an upper bound on the number of graph equivalence classes when all simplices are critical. We thus make the following definitions.

Definition 5.1. *Let G be a connected graph, H a nonempty proper subgraph of G with $b_0(H) = k$. If there are edges $e_1, e_2, ..., e_\ell$ such that for every $1 \leq i \leq \ell$:*

- $H \cup \{e_i\} \leq G$
- $b_0(H \cup \{e_i\}) = k - 1$
- $H \cup \{e_i\} \ncong H \cup \{e_j\}$ *for every* $i \neq j$

then we write $\ell \leq \delta_H^{b_0}(G)$. In addition, if there is no subgraph $H \subseteq G$ such that $\ell + 1 \leq \delta_H^{b_0}(G)$, we say that G is $\ell\, b_0$-determined and write $\delta^{b_0}(G) = \ell$

Definition 5.2. *Let G be a connected graph, H a nonempty proper subgraph of G with $b_1(H) = k$. If there are edges $e_1, e_2, ..., e_\ell$ such that for every $1 \leq i \leq \ell$:*

- $H \cup \{e_i\} \leq G$
- $b_1(H \cup \{e_i\}) = k + 1$
- $H \cup \{e_i\} \ncong H \cup \{e_j\}$ *for every* $i \neq j$

then we write $\ell \leq \delta_H^{b_1}(G)$. If there is no subgraph $H \subseteq G$ such that $\ell + 1 \leq \delta_H^{b_1}(G)$, we say that G is $\ell\, b_1$-determined and write $\delta^{b_1}(G) = \ell$

Example 5.3. *By Lemma 4.1, S_n is $1\, b_0$-determined.*

We will show a class of windmill graphs, W_3^n, is $4\, b_0$-determined and $2\, b_1$-determined in Proposition 5.6 and apply the following estimate to obtain a bound on the number of graph equivalence classes on W_3^n.

Theorem 5.4. *Let G be a connected graph with $m = |V(G)| + |E(G)| = 1 + b_1 + 2k$ (all simplices are critical) distinct critical values such that $\delta^{b_0}(G) \leq i$ and $\delta^{b_1}(G) \leq j$. If $G = \bigvee^{b_1} S^1$, then $C_k\binom{m-2}{2k}(2)(i^{k-2}j^{b_1-1})$ bounds above the number of graph equivalence classes. If $G = T$ is a tree, then $C_k(2)(i^{k-2})$ bounds above the number of graph equivalence classes.*

Proof. Let $G = \bigvee^{b_1} S^1$. By Theorem 3.3, the number of homological equivalence classes of excellent discrete morse functions on G is:

$$C_k\binom{m-2}{2k}.$$

Let H be a subgraph of G. We count the number of possible graph isomorphism types by considering the possible differences $b_0(c_l) - b_0(c_{l-1})$.

Suppose $b_0(c_l) - b_0(c_{l-1}) = 1$. Clearly there here is only one way to attach a vertex to H. Thus there is only one graph resulting from $H \cup v$ and the number of equivalence classes is 1.

Now suppose $b_0(c_l) - b_0(c_{l-1}) = -1$. Then an edge $e \in E(G) - E(H)$ was attached to H without completing a cycle. Since $\delta^{b_0}(G) \leq i$, there are at most i different graphs resulting from $H \cup e$. Because the existence of each such edge e implies the existence of a corresponding critical vertex, this occurs $\frac{m-b_0-b_1}{2} = k$ times in a fixed homological sequence. Clearly there is only one possibility up to graph isomorphism in the first stage such that $b_0(c_l) - b_0(c_{l-1}) = -1$. In the second stage such that $b_0(c_l) - b_0(c_{l-1}) = -1$, there are 2 possibilities; namely, attach the edge to two vertices of degree 0 or attach the edge to a vertex of degree 1. We thus see that there are at most $k-2$ stages that could yield i different graphs, and one stage yielding 2 possible graphs for a total estimate of giving $2 \cdot i^{k-2}$ possibilities.

Finally suppose $b_0(c_l) - b_0(c_{l-1}) = 0$. Then $b_1(c_l) - b_1(c_{l-1}) = 1$ and $(H \cup e) \subseteq G$. Since $\delta^{b_1}(G) \leq j$, there are at most j different graphs resulting from $H \cup e$. Since there is only one way to make the final attachment such that $b_0(c_l) - b_0(c_{l-1}) = 0$ is unique, we have j^{b_1-1} possibilities.

Combining these estimates together, we see that

$$C_k \binom{m-2}{2k} (2)(i^{k-2} j^{b_1-1})$$

bounds above the number of graph homological equivalence classes for G. The proof when $G = T$ is obtained by ignoring the b_1 estimate. □

5.1. Windmill Graphs

Definition 5.5. *For every $1 \leq n < \infty$, W_3^n is a graph with n copies of K_3 joined at a common vertex c. The $2n$ edges incident with c are called interior edges. The n edges which are not incident with c are called exterior edges.*

We note that the longest path from any vertex v to c is 2.

Proposition 5.6. *The graph W_3^n is $4\,b_0$-determined and $2\,b_1$-determined.*

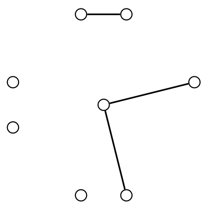
Proof. Let T be a subgraph of W_3^n. We first show that W_3^n is $4\,b_0$-determined. Consider $T \cup \{e\}$ for some edge $e \in E(W_3^n) - E(T)$ such that $b_0(T \cup \{e\}) = b_0(T) - 1$. Write $\overline{T}$ for the connected component of e in $T \cup \{e\}$.

Suppose c has the same degree in T and $T \cup \{e\}$. Suppose further $c \notin \overline{T}$. If $\overline{T}$ contains two or more edges, this implies that $\overline{T}$ contains a path to c contradicting the assumption that $c \notin \overline{T}$. Therefore the connected component contains one edge and there is only one graph that results from these assumptions.

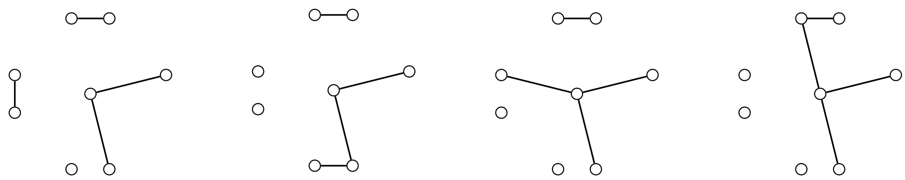
Now suppose $c \in \overline{T}$. Then an exterior edge must be added to a vertex, say v, attached to an existing interior edge. The shortest path from the vertex v to c is of length two. This results in exactly one graph.

Next suppose the degree of c increases from T to $T \cup \{e\}$. Considering the component attached to c, we observe that attaching a component of order one is different than attaching a component of order two. Each of these two attachments results in a unique graph. Attaching a component of order three or higher creates a path to c greater than length 2, which is impossible. Thus there are at most two possible graphs when the degree of c increases.

Therefore, $4 \leq \delta_T^{b_0}(W_3^n)$. We realize the lower bound by starting with the graph



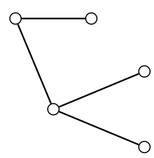
and observing that the four graphs below, no two of which are isomorphic, are obtained from this graph by the addition of a single edge.



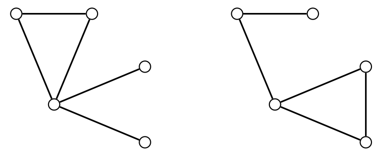
Therefore W_3^n is exactly $4\,b_0$-determined.

Now we show that W_3^n is $2\,b_1$-determined. Consider $T \cup \{e\}$. We can attach either an interior or exterior edge such that $T \cup \{e\}$ completes a cycle. Since W_3^n only contains interior and exterior edges, there is no other way to complete a cycle and $2 \leq \delta_T^{b_1}(W_3^n)$.

To see the lower bound, start with the graph



and observe that the two graphs resulting from attaching a single edge are not isomorphic.



Therefore W_3^n is $2\,b_1$-determined. □

Combining Proposition 5.6 with Theorem 5.4, we have

Corollary 5.7. *The number of graph equivalence classes of discrete Morse functions on* W_3^n *with* $m = 1 + 5n = 1 + n + 2[2n]$ *critical values is bounded above by* $C_{2n}\binom{5n-1}{4n}(2)(4^{2n-2})(2^{n-1})$.

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