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Secure Paired Domination in Graphs

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Abstract

We introduce a new strategy of domination, namely secure paired domination, which combines the advantages of both secure domination and paired domination. We propose different definitions of this concept and compare the definitions pairwise, obtaining properties of and inequalities between the secure paired domination numbers associated with the definitions. We determine the secure paired domination numbers of some classes of graphs, bound these parameters in terms of other domination-type parameters, and discuss extremal graphs for some of these bounds.

We conclude by revisiting the different definitions of secure dominating sets to narrow down the field of study to two particular definitions that we believe are worthy of further investigation.

Keywords:secure domination, paired domination, secure paired domination.

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1. Introduction

The placement of mobile guards on the vertices of a graph to protect it against attacks from outside the graph has its historical application in the ancient Roman empire and the military strategy of Constantine The Great, 274-337 AD. According to E. N. Luttwak, *The Grand Strategy of the Roman Empire*, as cited in [29], Constantine devised a strategy, called a *defense in depth* strategy, in which he deployed mobile Field Armies (FAs) to repel intruding enemies. An FA could deploy to protect a region only if it moved from an adjacent region where there was at least one other FA to help launch it. Constantine's strategy is known in domination theory as *Roman domination.*

Secure domination is a defense strategy that can be used when it is not possible or desirable to station two defense units at the same location. It was introduced in [12] and

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involves the placement of guards on the vertices of a graph, with at most one guard per vertex, as protection against single attacks. The vertices where the guards are stationed are called *occupied vertices*, while vertices without guards are *unoccupied vertices*. We assume that only unoccupied vertices are attacked, otherwise the guard at an occupied vertex defends the vertex without having to move.

For a graph $G = (V, E)$, the set $D \subseteq V$ is a *dominating set* if each vertex in $V - D$ is adjacent to a vertex in D. The set $X \subseteq V$ is a *secure dominating set (SDS)* of the graph $G = (V, E)$ if, for each $u \in V - X$, there exists a vertex $v \in X$ adjacent to u such that $(X - \{v\}) \cup \{u\}$ is a dominating set. Thus we place a guard on each vertex in X, and each unoccupied vertex u is adjacent to an occupied vertex v such that if the guard on v moves to u, the resulting set of occupied vertices is a dominating set. We say that v X-*defends* u, or simply v *defends* u if the set X is unimportant or clear from the context. The minimum cardinality of a dominating set is the *domination number* $\gamma(G)$, and the minimum cardinality of an SDS is the *secure domination number* $\gamma_s(G)$ of G. Results on secure domination can be found in [1, 2, 5, 6, 7, 10, 11, 12, 17, 25, 27].

Paired domination was introduced by Haynes and Slater in [20] and involves the placement of stationary guards on the vertices of a graph that work in pairs to act as back-up for each other. Thus a *paired dominating set (PDS)* of a graph G without isolated vertices is a dominating set $S \subseteq V$ such that the subgraph $\langle S \rangle$ of G induced by S contains a perfect matching; we say S *pairwise dominates* G. The minimum cardinality of a PDS is the *paired domination number* $\gamma_{\rm pr}(G)$. Paired domination is a well-studied concept in domination theory; other papers on this topic include [4, 13, 14, 15, 16, 19, 21, 22, 23, 28].

We introduce a new strategy of domination – *secure paired domination*, which combines the advantages of both secure domination and paired domination. The first question we consider is the following:

Question 1.1. *How do we define secure paired domination?*

We cannot simply require that a single guard runs to an adjacent vertex to defend it against an attack, because in this case not every graph without isolated vertices has a secure paired dominating set. Stars (i.e., $K_{1,n}$, $n \geq 2$) have only one type of PDS, and if only one guard moves along an edge, the resulting set of occupied vertices is not a PDS. But if the guard on the central vertex moves to a leaf, and the other guard moves to the central vertex, we do get a PDS of the same type.

We propose nine definitions of this concept in Section 2.1, comment on the existence of these sets in Section 2.2, and obtain inequalities between the secure paired domination numbers associated with the definitions in Section 2.3. In Section 3 we determine the secure paired domination numbers of some classes of graphs. In Section 4 we bound these parameters in terms of other domination parameters and discuss extremal graphs for some of the bounds. We revisit the nine definitions in Section 5 and highlight the two that, in our opinion, most deserve further study, and conclude with problems for future work.

2. Secure Paired Domination Numbers

We follow [9] for general graph theory terminology and [18] for domination terminology. Since only graphs without isolated vertices have a PDS, we assume henceforth that all our graphs are isolate-free. Let M be a matching of a graph. We denote the set of vertices that are matched under M by $V(M)$. If $uv \in M$, we say that u and v are *matched* by M and that v is the *M-partner* of u. We abbreviate the term "secure paired dominating set" (defined below) by *SPDS*.

A vertex of a graph of degree one is a *leaf* and its neighbour is a *stem*. A *branch vertex of* a tree T is a vertex of degree at least three. We denote the set of leaves of T by $L(T)$ and the set of stems by $S(T)$. For $v \in V(T)$ and $l \in L(T)$, a (v, l)*-endpath*, or v*-endpath* if l is unimportant, or *endpath* if neither v nor l is important, is a path P from v to l such that each internal vertex of P has degree two in T. A v-L path is any path from v to a leaf.

2.1. Definitions

We now state the nine definitions. While we show in Section 2.3 that these definitions indeed define different types of secure paired dominating sets, the concepts coincide for trees, as shown in Corollary 2.16. Since most of the results in Section 3 concern trees and other graphs for which the parameters coincide, the nine definitions do not result in nine different results for each class of graphs considered.

Definition 2.1. *A PDS D of a graph G is a* 1*-SPDS if, for each $v \in V - D$, there exists a vertex $u \in D$ adjacent to v such that $(D - \{u\}) \cup \{v\}$ is a PDS.*

Definition 2.2. *A PDS D of a graph G is a* 2*-SPDS if, for each $v \in V - D$, there exist a perfect matching M of $\langle D\rangle$ and an edge $uw \in M$ such that v is adjacent to w and $D' = (D - \{u\}) \cup \{v\}$ is a PDS.*

According to this definition, the guard on w moves along the edge wv to v, and the guard on u moves along uw to w to form a new PDS D', where $(M - \{uv\}) \cup \{vw\}$ is a perfect matching of $\langle D'\rangle$. The matching M need not be the same for all $v \in V - D$: let G be the graph obtained from the paths $u_1, ..., u_5$ and $v_1, ..., v_5$ by adding the edges u_2v_2 and u_4v_4, and let $D = \{u_2, u_3, u_4, v_2, v_3, v_4\}$. When u_1 is attacked, the matching $\{u_2u_3, v_2v_3, u_4v_4\}$ works, and when u_5 is attacked, the matching $\{u_2v_2, u_3u_4, v_3v_4\}$ works.

Definition 2.3. *A PDS D of a graph G is a* 3*-SPDS if D is a* 1*-SPDS or a* 2*-SPDS.*

Definition 2.4. *A PDS D of a graph G is a* 4*-SPDS if, for each $v \in V - D$, there exist a perfect matching M of $\langle D\rangle$, a vertex $x \neq v$, and an edge $uw \in M$ such that $\{vw, vx, ux\} \subseteq E(G)$ and $D' = (D - \{u, w\}) \cup \{v, x\}$ is a PDS.*

Here the guard on u moves along ux to x and the guard on w moves along wv to v to form D', and $M' = (M - uw) \cup \{vx\}$ is a perfect matching of $\langle D'\rangle$. If $x = w$, then we have the same move as in Definition 2.2, and if $x \neq w$, then u, w, v, x, u is a 4-cycle.

Definition 2.5. *A PDS D of a graph G is a 5-SPDS if D is a 1-SPDS or a 4-SPDS.*

Definition 2.6. *A PDS D of a graph G is a 6-SPDS if, for each $v \in V - D$, there exist a vertex $x \neq v$ and vertices $u, w \in D$ such that $uw, wv, vx, xu \in E(G)$ and $(D - \{u, w\}) \cup \{v, x\}$ is a PDS.*

Here the moving guards need not be on matched vertices of the PDS before or after the move, but their locations must be adjacent before and after the move.

Definition 2.7. *A PDS D of a graph G is a 7-SPDS if D is a 1-SPDS or a 6-SPDS.*

Definition 2.8. *A PDS D of a graph G is an 8-SPDS if, for each $v \in V - D$, there exist a vertex $x \neq v$ and vertices $u, w \in D$ such that $vw, ux \in E(G)$ and $(D - \{u, w\}) \cup \{v, x\}$ is a PDS.*

For this strategy the moving guards need not occupy adjacent vertices.

Definition 2.9. *A PDS D of a graph G is a 9-SPDS if D is a 1-SPDS or an 8-SPDS.*

The definitions above have a simple hierarchical relationship: Definition 2.2 is a special case of Definition 2.4, which is a special case of Definition 2.6, which is a special case of Definition 2.8. Definition 2.3 is a special case of Definition 2.5, which is a special case of Definition 2.7, which is a special case of Definition 2.9. In addition, each even numbered definition is a special case of the next odd numbered definition. A necessary (but not sufficient) condition for a graph G to have an i-SPDS, $i = 1, ..., 9$, of cardinality t is for every vertex of G to be contained in some PDS of cardinality t.

Definition 2.10. *The i^{th} secure paired domination number $\gamma_{\text{spr}}^{(i)}(G)$ is the smallest cardinality of an i-SPDS of G, $i = 1, \ldots, 9$.*

It is clear that Definitions 2.1 to 2.9 describe different guard movements. A configuration of guards that form an i-SPDS of G is said to *i-defend* G, or simply to *defend* G as before if i is clear. We now investigate the following questions.

Question 2.11. *For which of these definitions does $\gamma_{\text{spr}}^{(i)}(G)$ exist for all isolate-free graphs?*

Question 2.12. *Which of these definitions give equal secure paired domination numbers?*

2.2. Existence of Secure Paired Domination Numbers

As mentioned in the introduction, $\gamma_{\text{spr}}^{(1)}(G)$ does not exist for all graphs without isolated vertices (for example, P_3 does not have a 1-SPDS). Our first result is a necessary and sufficient condition for the existence of $\gamma_{\text{spr}}^{(1)}(G)$.

Proposition 2.13. *For a graph G without isolated vertices, $\gamma_{\text{spr}}^{(1)}(G)$ exists, and D is a 1-SPDS of G, if and only if G has a matching M such that $D = V(M)$ and any $v \in V - D$ is contained in an odd cycle $C = v, u_1, ..., u_{2r}, v$, where $\{u_1u_2, u_3u_4, ..., u_{2r-1}u_{2r}\} \subseteq M$.*

Proof. Suppose there exists a matching M of G satisfying the given conditions. We show that $D = V(M)$ is a 1-SPDS of G. If M is a perfect matching of G, then $D = V$ and we are done. Hence assume $D \neq V$. Consider any $v \in V - D$, which is contained in an odd cycle $C = v, u_1, ..., u_{2r}, v$ as described. Consider $D' = (D - \{u_{2r}\}) \cup \{v\}$ and $M' = (M - E(C)) \cup \{vu_1, u_2u_3, \ldots, u_{2r-2}u_{2r-1}\}$. Then M' is a perfect matching of $\langle D' \rangle$. Suppose u_{2r} is adjacent to a vertex $w \in V - V(C) - D$. By the hypothesis, w lies on an odd cycle C', all of whose other vertices are in D. Hence w is adjacent to at least one vertex of D other than u_{2r}. Therefore D' is a PDS of G; consequently, D is a 1-SPDS of G and $\gamma_{\text{spr}}^{(1)}(G)$ exists.

Conversely, suppose $\gamma_{\text{spr}}^{(1)}(G)$ exists, and let D be a 1-SPDS of G with M any perfect matching of $\langle D \rangle$. Place a guard on every vertex in D. If $|D| = |V|$, then there is nothing more to prove. Hence assume $D \neq V$ and let v be an arbitrary vertex in $V - D$. Let u_1 be a vertex in D that 1-defends v and define $D' = (D - \{u_1\}) \cup \{v\}$. Then D' is a PDS of G with matching M', formed after the guard on u_1 moves to v. Let u_2 be the M-partner of u_1 in D. Since $u_1 \notin D'$, u_2 has a new partner in D', say u_3. If $u_3 = v$, we are done, because then $C = v, u_1, u_2, v$ is a triangle that satisfies the statement of the proposition. If not, then by definition of D', $u_3 \in D$. Consider the M-partner u_4 of u_3 and note that $u_4 \notin \{u_1, u_2, u_3\}$. Now u_4 has an M'-partner, which we call u_5. If $u_5 = v$, we are done; otherwise, repeat the procedure to find distinct vertices $u_5, u_6 ..., u_{2r+1}$, where $u_{2r+1} = v$. Then $C = v, u_1, ..., u_{2r}, v$ is an odd cycle of G such that $\{u_1u_2, u_3u_4, ..., u_{2r-1}u_{2r}\} \subseteq M$ and the conditions are satisfied. □

We now show that $\gamma_{\text{spr}}^{(i)}(G)$, $2 \leq i \leq 9$, exists for all graphs G without isolated vertices.

Proposition 2.14. *For each $2 \leq i \leq 9$, $\gamma_{\text{spr}}^{(i)}(G)$ exists for all isolate-free graphs G.*

Proof. It suffices to show that $\gamma_{\text{spr}}^{(2)}(G)$ exists for any graph G without isolated vertices, because Definition 2.2 is a special case of all the other definitions. Let M be a maximum matching of G and let $D = V(M)$. Say $D = \{v_1, u_1, v_2, u_2, \ldots, v_i, u_i, ..., v_m, u_m\}$, where v_i and u_i are matched by M, and place a guard on each vertex in D. Since M is a maximum matching and G has no isolated vertices, D pairwise dominates V.

If $ab \in M$, then there do not exist distinct vertices $w, x \in V - D$ such that a is adjacent to w and b is adjacent to x, otherwise $M' = (M - \{ab\}) \cup \{aw, bx\}$ is a matching of G with $|M'| > |M|$, a contradiction. Suppose a vertex $v \notin D$ is attacked, where v is adjacent to (say) v_i. Move the guard on v_i to v, and the guard on u_i to v_i. Since v is the only possible vertex in $V - D$ adjacent to u_i, $(D - \{u_i\}) \cup \{v\}$ with the matching $(M - \{u_iv_i\}) \cup \{v_iv\}$ is a PDS of G. Thus D is a 2-SPDS of G and $\gamma_{\text{spr}}^{(i)}(G)$ exists. □

2.3. Comparison of Secure Paired Domination Numbers

The following corollary to Proposition 2.14 is obvious.

Corollary 2.15. *If the graph G has no isolated vertices, then*

(a) $\gamma_{\rm spr}^{(2)}(G) \geq \gamma_{\rm spr}^{(i)}(G)$ *for each* $3 \leq i \leq 9$

(b) $\gamma_{\rm spr}^{(i)}(G) \geq \gamma_{\rm spr}^{(i+1)}(G)$ *for each* $i = 2, 4, 6, 8$

(c) $\gamma_{\rm spr}^{(2)}(G) \geq \gamma_{\rm spr}^{(4)}(G) \geq \gamma_{\rm spr}^{(6)}(G) \geq \gamma_{\rm spr}^{(8)}(G)$ *and* $\gamma_{\rm spr}^{(3)}(G) \geq \gamma_{\rm spr}^{(5)}(G) \geq \gamma_{\rm spr}^{(7)}(G) \geq \gamma_{\rm spr}^{(9)}(G)$.

Corollary 2.15 and Propositions 2.13 and 2.14 also have the following immediate corollary.

Corollary 2.16.

(a) *If G is bipartite, then* $\gamma_{\rm spr}^{(i)}(G) = \gamma_{\rm spr}^{(i+1)}(G)$ *for each* $i = 2, 4, 6, 8$.

(b) *If T is a tree, then* $\gamma_{\rm spr}^{(2)}(T) = \cdots = \gamma_{\rm spr}^{(9)}(T)$.

We show that all inequalities in Corollary 2.15 can be strict. Consider the graphs in Fig. 1.

2.3.1. $\gamma_{\rm spr}^{(2)}(F) > \gamma_{\rm spr}^{(3)}(F)$

The set $\{u, v, x, z\}$ is a 3-SPDS of F. If y is attacked, the guard at x moves to y, and $\{u, v, y, z\}$ with matching $\{uv, yz\}$ is a PDS of F. If the attack is at (say) t, the guard at v moves to t while the guard at x moves to v, and $\{t, u, v, z\}$ with matching $\{vt, uz\}$ is a PDS. Since no single pair of vertices dominates F, $\gamma_{\rm spr}^{(3)}(F) = 4$. Up to isomorphism, F has three PDS's of cardinality four:

1. $D = \{u, v, x, z\}$ with matching $\{uz, vx\}$: if y is attacked, then, up to symmetry, $D' = \{u, z, x, y\}$ is the only set that can be formed by the movement of guards according to Definition 2.2. But D' does not dominate t.
2. $D = \{u, v, y, z\}$ with matching $\{uv, yz\}$: if t is attacked, then $D' = \{t, v, y, z\}$ is the only set that can be formed by the movement of guards according to Definition 2.2, but D' does not dominate s.
3. $D = \{t, v, u, z\}$ with matching $\{vt, uz\}$, which does not protect y or s.

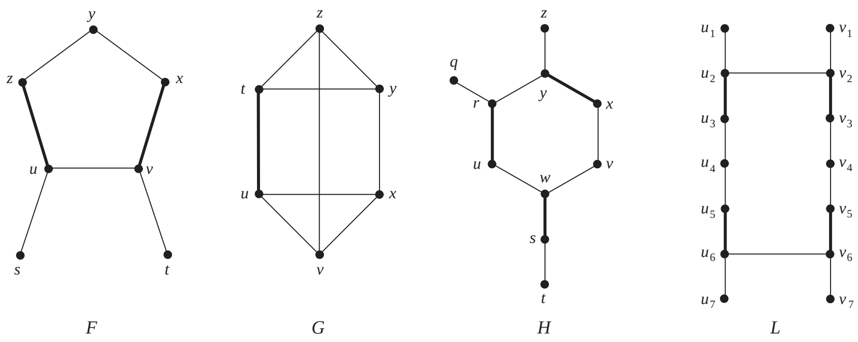

Figure 1: Graphs with different secure paired domination numbers

Hence $\gamma_{\mathrm{spr}}^{(2)}(F) \geq 6$. Since F clearly can be securely pairwise dominated with six vertices, $\gamma_{\mathrm{spr}}^{(2)}(F) = 6$. Similarly, $\gamma_{\mathrm{spr}}^{(i)}(F) > \gamma_{\mathrm{spr}}^{(i+1)}(F)$ for each $i = 4, 6, 8$ and thus $\gamma_{\mathrm{spr}}^{(2)}(F) > \gamma_{\mathrm{spr}}^{(i)}(F)$ for each odd $i \geq 3$.

2.3.2. $\gamma_{\mathrm{spr}}^{(2)}(G) > \gamma_{\mathrm{spr}}^{(4)}(G)$ and $\gamma_{\mathrm{spr}}^{(3)}(G) > \gamma_{\mathrm{spr}}^{(5)}(G)$

The set $D = \{t, u\}$ is a 4-SPDS of G, but not a 2-SPDS. For example, if y is attacked, the guard on t moves to y while the guard on u moves to x, according to Definition 2.4. But if the guard on t moves to y while the guard on u moves to t, according to Definition 2.2, then v is not dominated. Since $\{t, u\}$, $\{v, z\}$ and $\{x, y\}$ are the only PDS's of G of cardinality two, $\gamma_{\mathrm{spr}}^{(2)}(G) \geq 4$, and it is easy to see that $\gamma_{\mathrm{spr}}^{(2)}(G) = 4$. It follows that

$$\gamma_{\mathrm{spr}}^{(2)}(G) > \gamma_{\mathrm{spr}}^{(i)}(G) \text{ for each even } i \geq 4.$$

As in the proof of Proposition 2.13, no pair of vertices protect G with guard moves as in Definition 2.1. Hence $\gamma_{\mathrm{spr}}^{(3)}(G) = \gamma_{\mathrm{spr}}^{(2)}(G) = 4$, while $\gamma_{\mathrm{spr}}^{(5)}(G) = \gamma_{\mathrm{spr}}^{(4)}(G) = 2$, and so

$$\gamma_{\mathrm{spr}}^{(3)}(G) > \gamma_{\mathrm{spr}}^{(i)}(G) \text{ for each odd } i \geq 5.$$

2.3.3. $\gamma_{\mathrm{spr}}^{(4)}(H) > \gamma_{\mathrm{spr}}^{(6)}(H)$ and $\gamma_{\mathrm{spr}}^{(5)}(H) > \gamma_{\mathrm{spr}}^{(7)}(H)$

The set $D = \{s, w, u, r, x, y\}$ with the unique perfect matching $M = \{sw, ur, xy\}$ is a 6-SPDS: It is easy to see how to 2-defend and thus 6-defend q, t and z. To defend v, the adjacent but unpaired guards on w and u move to v and w, respectively to form the set $D' = (D - \{u\}) \cup \{v\}$ of occupied vertices. Since $\{sw, vx, ry\}$ is a perfect matching of $\langle D' \rangle$, D is a 6-SPDS. No PDS consisting of four vertices defends u or v, hence $\gamma_{\mathrm{spr}}^{(6)}(H) = 6$. Suppose S is a 4-SPDS of H such that $|S| = 6$. Since r, s and y are stems, $\{r, s, y\} \subseteq S$. Then $\{t, w\} \cap S \neq \varnothing$. If r and y are paired, then neither q nor z is defended. Thus

$\{q,u\} \cap S \neq \varnothing$ and $\{z,x\} \cap S \neq \varnothing$. None of the possible sets defends v. Therefore $\gamma_{\rm spr}^{(4)}(H) \geq 8$ and in fact $\gamma_{\rm spr}^{(4)}(H) = 8$. We now have that

$$\gamma_{\rm spr}^{(4)}(H) > \gamma_{\rm spr}^{(i)}(H) \text{ for } i = 6, 8.$$

Since H is bipartite, Corollary 2.16 implies that $\gamma_{\rm spr}^{(5)}(H) = \gamma_{\rm spr}^{(4)}(H)$ and $\gamma_{\rm spr}^{(7)}(H) = \gamma_{\rm spr}^{(6)}(H)$, that is,

$$\gamma_{\rm spr}^{(5)}(H) > \gamma_{\rm spr}^{(i)}(H) \text{ for } i = 7, 9.$$

2.3.4. $\gamma_{\rm spr}^{(6)}(L) > \gamma_{\rm spr}^{(8)}(L)$ and $\gamma_{\rm spr}^{(7)}(L) > \gamma_{\rm spr}^{(9)}(L)$:

The set $D = \{v_2, v_3, u_2, u_3, v_5, v_6, u_5, u_6\}$ with the unique perfect matching $M = \{v_2v_3, v_5v_6, u_2u_3, u_5u_6\}$ is an 8-SPDS of L: If (say) v_4 is attacked, then $(D - \{v_3, u_3\}) \cup \{v_4, u_4\}$ with the matching $\{v_2u_2, v_4v_5, u_4u_5, v_6u_6\}$ is a PDS of L, and if (say) u_1 is attacked, then $(D - \{u_3\}) \cup \{u_1\}$ with the matching $\{u_1u_2, v_2v_3, v_5v_6, u_5u_6\}$ is a PDS of L. Since L does not have a PDS of cardinality 6, $\gamma_{\rm spr}^{(8)}(L) = 8$.

To see that L does not have a 6-SPDS of cardinality 8, observe that since u_2, v_2, u_6, v_6 are stems, they are in any PDS of L; furthermore, u_2 and v_2 are not matched in any 6-SPDS, for otherwise, if (say) v_1 is attacked, then u_1 is not guarded by the set formed by moving the guard on v_2 to v_1 and the guard on u_2 to v_2. Thus any 6-SPDS of L has at least four pairs of guards, each containing exactly one of u_2, v_2, u_6, v_6. But then neither u_4 nor v_4 can be protected by the moves defined in Definition 2.6, yielding $\gamma_{\rm spr}^{(6)}(L) \geq 10$. (In fact, one can show that $\gamma_{\rm spr}^{(6)}(L) = 12$.) Since L is bipartite, Corollary 2.16 implies that $\gamma_{\rm spr}^{(7)}(L) > \gamma_{\rm spr}^{(9)}(L)$.

2.3.5. Comparing $\gamma_{\rm spr}^{(i)}$ and $\gamma_{\rm spr}^{(i+1)}$ for $i = 3, 5, 7$

We show that there is no inequality between $\gamma_{\rm spr}^{(i)}$ and $\gamma_{\rm spr}^{(i+1)}$, $i = 3, 5, 7$, that holds for all graphs. First, as shown in 2.3.1, $\gamma_{\rm spr}^{(3)}(F) = 4$ and $\gamma_{\rm spr}^{(2)}(F) = 6$. Since F has no 4-cycle, any guard movement according to Definition 2.4 corresponds to the movement in Definition 2.2, hence $\gamma_{\rm spr}^{(4)}(F) = \gamma_{\rm spr}^{(2)}(F) = 6$, that is, $\gamma_{\rm spr}^{(4)}(F) > \gamma_{\rm spr}^{(3)}(F)$. On the other hand, as shown in 2.3.2, $\gamma_{\rm spr}^{(2)}(G) = 4 = \gamma_{\rm spr}^{(3)}(G)$ and $\gamma_{\rm spr}^{(4)}(G) = 2$, that is, $\gamma_{\rm spr}^{(4)}(G) < \gamma_{\rm spr}^{(3)}(G)$.

As shown in 2.3.1, $\gamma_{\rm spr}^{(3)}(F) = 4$, and it follows that $\gamma_{\rm spr}^{(5)}(F) = \gamma_{\rm spr}^{(3)}(F) = 4$. Again as in 2.3.1, $\gamma_{\rm spr}^{(6)}(F) = \gamma_{\rm spr}^{(2)}(F) = 6$, and so $\gamma_{\rm spr}^{(5)}(F) < \gamma_{\rm spr}^{(6)}(F)$. On the other hand, as shown in 2.3.3, $\gamma_{\rm spr}^{(6)}(H) = 6 < 8 = \gamma_{\rm spr}^{(4)}(H) = \gamma_{\rm spr}^{(5)}(H)$.

Finally, from 2.3.4 it follows similarly that $\gamma_{\rm spr}^{(7)}(F) = 4 < 6 = \gamma_{\rm spr}^{(8)}(F)$ and $\gamma_{\rm spr}^{(7)}(L) = \gamma_{\rm spr}^{(6)}(L) \geq 10 > 8 = \gamma_{\rm spr}^{(8)}(L)$.

3. Secure Paired Domination Numbers for Some Graph Classes

We consider four classes graphs, namely paths, cycles, spiders and ladders, and determine their secure paired domination numbers according to moves defined in each of Definitions 2.2 to 2.9. Since paths and spiders are trees, Corollary 2.16(b) implies that $\gamma_{\rm spr}^{(2)} = \cdots = \gamma_{\rm spr}^{(9)}$ for each of these graphs, and we write $\gamma_{\rm spr}$ instead of $\gamma_{\rm spr}^{(i)}$ for each of these graphs. We also refer to an i-SPDS of a tree, $i = 2, ..., 9$, simply as an SPDS.

3.1. Paths and Cycles

We start with the secure domination number of paths and cycles, as stated in the following proposition.

Proposition 3.1. [12] *For any integer n, $\gamma_s(P_n) = \left\lceil \frac{3n}{7} \right\rceil$. If $n = 3$, then $\gamma_s(C_3) = 1$, otherwise $\gamma_s(C_n) = \gamma_s(P_n)$.*

Proposition 3.2. *For any $n \geq 2$, $\gamma_{\rm spr}(P_n) = 2\left\lceil \frac{3n}{10} \right\rceil$.*

Proof. For each integer $n \in \{2, ..., 10\}$, an SPDS of P_n of cardinality $2\left\lceil \frac{3n}{10} \right\rceil$ is shown in Fig. 2 and it is easy to check that each set is a minimum SPDS.

Let $k \geq 1$ and consider P_{7k}. By Proposition 3.1, P_{7k} has a minimum SDS X with $|X| = 3k$. Replace each vertex $x \in X$ by two adjacent vertices x_1 and x_2, joining x_1 to the predecessor and x_2 to the successor of x on P_{7k}, to form the path P_{10k}. Let $D' = \bigcup_{x \in D}\{x_1, x_2\}$. It is easy to see that D' is an SPDS of P_{10k} of cardinality $6k$, hence $\gamma_{\rm spr}(P_{10k}) \leq 6k = 2\left\lceil \frac{3 \cdot 10k}{10} \right\rceil$. Note that D' does not contain either leaf of P_{10k}.

Now let $n = 10k + m$, where $k \geq 1$ and $m \in \{1, ..., 9\}$ and consider $P_n = v_1, ..., v_{10k}, ..., v_{10k+m}$. If $m = 1$, let $D = D' \cup \{v_{10k}, v_{10k+1}\}$, and if $m \in \{2, ..., 9\}$, let D be the union of D' and the minimum SPDS of P_m in Fig. 2. Then D is an SPDS of P_n of cardinality $6k + 2\left\lceil \frac{3m}{10} \right\rceil$, hence $\gamma_{\rm spr}(P_n) \leq 2\left\lceil \frac{3n}{10} \right\rceil$.

Suppose P_n has an SPDS with fewer than $6k + 2\left\lceil \frac{3m}{10} \right\rceil$ vertices. Then P_n also has an SPDS Y such that $|Y| = 2t = 6k+2\left\lceil \frac{3m}{10} \right\rceil - 2$. Contract each pair of vertices in Y to a single vertex, thus forming a vertex subset Y' of the path P_r, where $r = n-t = 7k+m-\left\lceil \frac{3m}{10} \right\rceil + 1$ and $|Y'| = t$. Since guards on vertices in Y move as defined in Definition 2.2 to defend P_n, guards on vertices in Y' defend P_r. Hence Y' is an SDS of P_r. But as can be seen from Table 1, $\gamma_s(P_r) > t$ for all $m \in \{1, ..., 9\}$, a contradiction. □

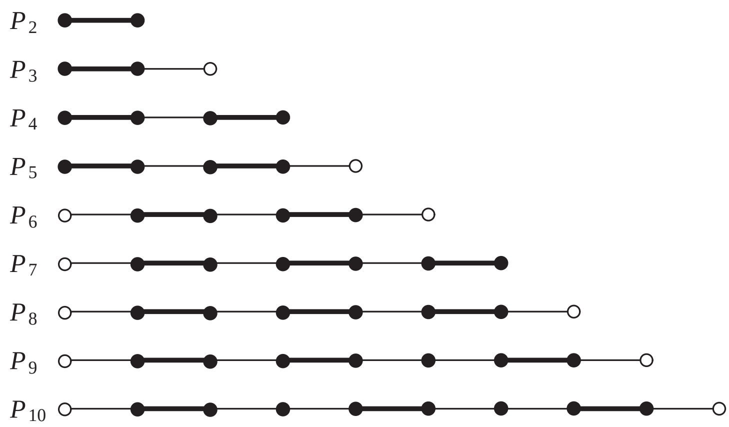

Figure 2: Minimum SPDS's for P_2 to P_{10}

m	1	2	3	4	5	6	7	8	9
r	$7k+1$	$7k+2$	$7k+3$	$7k+3$	$7k+4$	$7k+5$	$7k+5$	$7k+6$	$7k+7$
t	$3k$	$3k$	$3k$	$3k+1$	$3k+1$	$3k+1$	$3k+2$	$3k+2$	$3k+2$
$\gamma_s(P_r)$	$3k+1$	$3k+1$	$3k+2$	$3k+2$	$3k+2$	$3k+3$	$3k+3$	$3k+3$	$3k+3$

Table 1: The values of $r = 7k + m - \lceil \frac{3m}{10} \rceil + 1$, $t = 3k + \lceil \frac{3m}{10} \rceil - 1$ and $\gamma_s(P_r) = \lceil \frac{3r}{7} \rceil$

The strategy used to obtain $\gamma_{\text{spr}}(P_n)$ can also be applied to find $\gamma_{\text{spr}}(C_n)$. The proof is omitted here but can be found in [24].

Proposition 3.3. *For each $n \geq 3$ and $i = 2, \ldots, 9$, $\gamma_{\text{spr}}^{(i)}(C_n) = 2$ if $n = 4$ and $\gamma_{\text{spr}}^{(i)}(C_n) = \gamma_{\text{spr}}^{(i)}(P_n)$ otherwise.*

3.2. Spiders

A *spider* $S(q_1, ..., q_p)$ is a tree with exactly one branch vertex v and $p = \deg v$ v-endpaths, called the *legs*, of lengths $1 \leq q_1 \leq \cdots \leq q_p$. If $q_1 = q_p = q$, we write $S(p; q)$ instead of $S(q_1, ..., q_p)$. Let Q_i, $i = 1, ..., p$, be the legs of $S(q_1, ..., q_p)$. We label the vertices of $Q_i - \{v\}$ by $u_{i,j}$, $j = 1, 2, ...q_i$, where $d(u_{i,j}, v) = j$. We call the path $u_{i,1}, \ldots, u_{i,q_i}$ the i^{th} *proper leg* and denote it by L_i. In this section we first determine $\gamma_{\text{spr}}(S(p; q))$ and then generalize the result to $S(q_1, ..., q_p)$. Fig. 3 illustrates minimum SPDS's for $S(3; q)$, $q = 1, \ldots, 10$.

The following result by Burger, Henning and Van Vuuren [8] will be used in the proof of Proposition 3.5.

Proposition 3.4. [8] If $(V_1, ..., V_r)$ *is a partition of* $V(G)$ *and* $G_i = \langle V_i \rangle$, $i = 1, ..., r$, *then* $\gamma_s(G) \leq \sum_{i=1}^{r} \gamma_s(G_i)$.

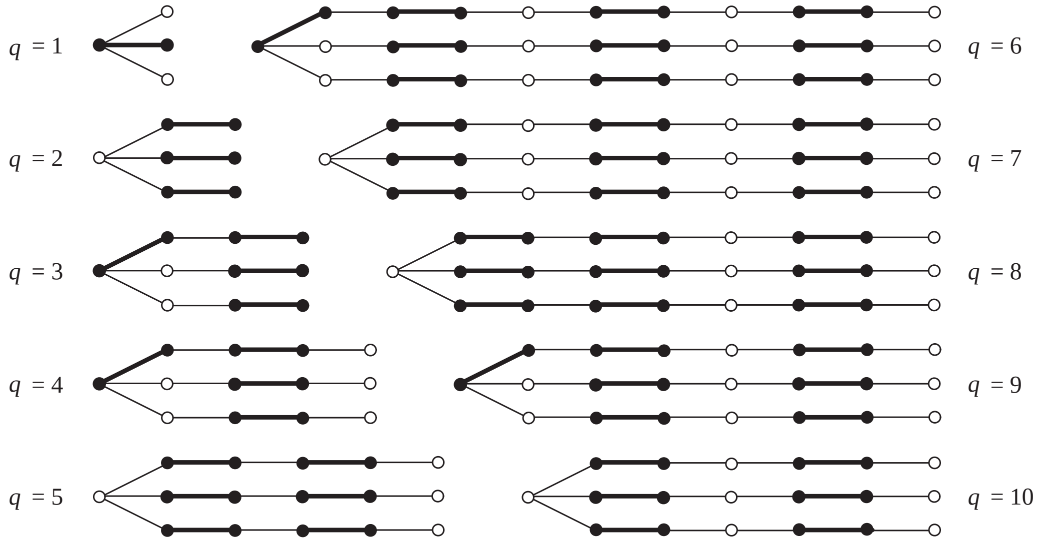

Figure 3: Minimum SPDS's for $S(3;q)$, $q = 1, ..., 10$

Proposition 3.5. *For $p \geq 3$, $\gamma_{\rm spr}(S(p;1)) = 2$ and for $q \geq 2$,*

$$\gamma_{\rm spr}(S(p;q)) = p\gamma_{\rm spr}(P_{q-1}) + \eta,$$

where $\eta = 2$ when $q \equiv 0, 1, 3, 4, 7 \pmod{10}$ and $\eta = 0$ otherwise.

Proof. The result is obvious for $q = 1, 2$, hence assume $q \geq 3$. Let D be a minimum SPDS of $G = S(p;q)$ and let v be the branch vertex of G. Let (s,t) be a pair of vertices that D-defends v. Notice that $(s,t) = (v, u_{i,1})$ or $(s,t) = (u_{i,1}, u_{i,2})$ for some i; say $i = 1$. There are two cases to consider: either (s,t) defends some vertex $u_{j,1}$, $j \geq 2$, or it does not.

If (s,t) defends some vertex $u_{j,1}$, $j \geq 2$, then it defends $u_{j,1}$ for all $j = 2, \ldots, p$. Decompose G into p subgraphs as follows. Let $H_1 = \langle L_1 \cup \{v, u_{2,1}, u_{3,1}, ..., u_{p,1}\}\rangle$ and $H_i = L_i - \{u_{i,1}\}$ for $i = 2, ..., p$. Since any SPDS of P_{q+2} is an SPDS of H_1 and vice versa, $\gamma_{\rm spr}(H_1) = \gamma_{\rm spr}(P_{q+2})$. Let D_i be a minimum SPDS of H_i, $i = 1, 2, ..., p$. By Proposition 3.4,

$$|D| \leq \sum_{i=1}^{p} |D_i| = \gamma_{\rm spr}(P_{q+2}) + (p-1)\gamma_{\rm spr}(P_{q-1}).$$

On the other hand, if (s,t) defends no vertex from any proper leg other than L_1, then, by decomposing G into $H_1 = \langle L_1 \cup \{v\}\rangle$ and $H_i = L_i$, $i = 2, 3, ..., p$, Proposition 3.4 gives

$$|D| \leq \sum_{i=1}^{p} |D_i| = \gamma_{\rm spr}(P_{q+1}) + (p-1)\gamma_{\rm spr}(P_q),$$

where D_i is a minimum SPDS of H_i, $i = 1, 2, \ldots, p$. Thus

$$\gamma_{\rm spr}(G) \leq \min\{\gamma_{\rm spr}(P_{q+1}) + (p-1)\gamma_{\rm spr}(P_q),\ \gamma_{\rm spr}(P_{q+2}) + (p-1)\gamma_{\rm spr}(P_{q-1})\}. \quad (1)$$

Notice that when $q \equiv 6 \pmod{10}$, we can place the guards on G so that $u_{i,1}$ is occupied for all $i = 1, \ldots, p$, all defending the branch vertex v, while its partner $u_{i,2}$ defends $u_{i,3}$. In this case we only need $\gamma_{\mathrm{spr}}(P_q) = \gamma_{\mathrm{spr}}(P_{q+1}) - 2$ guards on $L_1 \cup \{v\}$. It can be shown by exhaustively examining all congruence classes modulo 10 that this is the only case when strict inequality in (1) is achieved. Hence we conclude that

$$\gamma_{\mathrm{spr}}(S(p;q)) = \begin{cases} \gamma_{\mathrm{spr}}(P_{q+2}) + (p-1)\gamma_{\mathrm{spr}}(P_{q-1}) & \text{if} \quad q \equiv 0,1,3,4,7 \pmod{10} \\ p\gamma_{\mathrm{spr}}(P_q) & \text{if} \quad q \equiv 6 \pmod{10} \\ \gamma_{\mathrm{spr}}(P_{q+1}) + (p-1)\gamma_{\mathrm{spr}}(P_q) & \text{if} \quad q \equiv 2,\ 5,\ 8,\ 9 \pmod{10} \end{cases},$$

which simplifies to the desired result. □

The idea of decomposing spiders used in the proof of Proposition 3.5 provides a good strategy to find the SPD numbers for general spiders. In what follows we use the notational convention that if $q_j = 1$, then $\gamma_{\mathrm{spr}}(P_{q_j-1}) = 0$, even though P_{q_j-1} is not defined, and if $q_j = 2$, then $\gamma_{\mathrm{spr}}(P_{q_j-1}) = 2$, even though P_1 has no PDS.

Proposition 3.6. *For positive integers $q_1, \ldots, q_p$,*

$$\gamma_{\mathrm{spr}}(S(q_1, \ldots, q_p)) = \sum_{j=1}^{p} \gamma_{\mathrm{spr}}(P_{q_j-1}) + \eta,$$

where $\eta = 0$ if

(a) *there exists t such that $q_t \equiv 8 \pmod{10}$, or*

(b) *there does not exist r such that $q_r \equiv 1,4,7,8 \pmod{10}$ and there exists t such that $q_t \equiv 2,5,6,9 \pmod{10}$,*

and $\eta = 2$ otherwise.

Proof. Let $G = S(q_1, \ldots, q_p)$. By following the proof method of Proposition 3.5, we obtain $\gamma_{\mathrm{spr}}(G) = \sum_{j=1}^{p} \gamma_{\mathrm{spr}}(P_{q_j-1}) + \eta$, where $\eta = 0$ or 2, depending on the value of q_j, $j = 1, \ldots, p$. To find the exact value of η, we consider the following cases.

Case 1. There exists t such that $q_t \equiv 8 \pmod{10}$. Then $q_t + 2 \equiv 0 \pmod{10}$, and as in the proof of Proposition 3.5 we can securely pairwise dominate $\langle L_t \cup \{v, u_{1,1}, \ldots, u_{p,1}\}\rangle$ by $\gamma_{\mathrm{spr}}(P_{q_t+2}) = \gamma_{\mathrm{spr}}(P_{q_t-1})$ guards, as Proposition 3.2 yields $\gamma_{\mathrm{spr}}(P_n) = \gamma_{\mathrm{spr}}(P_{n+3})$ if $n \equiv 7 \pmod{10}$. Hence we need at most, and in fact exactly, $\gamma_{\mathrm{spr}}(P_{q_j-1})$ guards for L_j, $j \neq t$. Thus $\eta = 0$.

Case 2. There does not exist r such that $q_r \equiv 8 \pmod{10}$, and there exists t such that $q_t \equiv 1,4,7 \pmod{10}$. If $q_t \neq 1$, then $\gamma_{\mathrm{spr}}(P_{q_t}) = \gamma_{\mathrm{spr}}(P_{q_t-1}) + 2$. We only consider $q_t \equiv 1 \pmod{10}$, as the other two cases follow similarly. Let D_t be a minimum SPDS of $L_t - \{u_{t,1}\}$ and consider $u_{t,1} \in V(L_t)$. It is defended by a pair of guards from D_t or $\langle \{u_{t,1}, v\} \cup L_j \rangle$ for some $j \neq t$, which is a path of order $q_j + 2$. Since $\gamma_{\mathrm{spr}}(P_{q_t}) = \gamma_{\mathrm{spr}}(P_{q_t-1}) + 2$, $u_{t,2}$

does not defend $u_{t,1}$, so we need $\gamma_{\mathrm{spr}}(P_{q_t-1})+2$ guards in the former case. In the latter case, we need $\gamma_{\mathrm{spr}}(P_{q_j-1})+2$ guards on $\langle\{u_{t,1},v\}\cup L_j\rangle$ because $q_j-1\not\equiv 7 \pmod{10}$ and $\gamma_{\mathrm{spr}}(P_{n+3})=\gamma_{\mathrm{spr}}(P_n)+2$ unless $n\equiv 7 \pmod{10}$. Therefore $\eta=2$ in this case.

Case 3. There does not exist r such that $q_r\equiv 1,4,7,8 \pmod{10}$, and there exists t such that $q_t\equiv 2,5,6,9 \pmod{10}$. We claim $\eta=0$. For $q_t\equiv 2,5,9$, $\gamma_{\mathrm{spr}}(P_{q_t+1})=\gamma_{\mathrm{spr}}(P_{q_t-1})$, hence we need only $\gamma_{\mathrm{spr}}(P_{q_t-1})$ guards to defend $\langle L_t\cup\{v\}\rangle$. For any j such that $q_j\equiv 0,2,3,5,6,9 \pmod{10}$ we need $\gamma_{\mathrm{spr}}(P_{q_j-1})=\gamma_{\mathrm{spr}}(P_{q_j})$ guards to defend L_j; the guards can be placed so that one of them is on $u_{j,1}$. Thus if $q_t\equiv 6 \pmod{10}$, $\gamma_{\mathrm{spr}}(P_{q_t-1})$ guards can also defend $\langle L_t\cup\{v\}\rangle$.

Case 4. For all $j=1,\dots,p$, $q_j\equiv 0,3 \pmod{10}$. Then it is an easy consequence of Proposition 3.5 that $\eta=2$. ☐

3.3. Ladders

The *ladder* H_n is the Cartesian product $K_2\Box P_n$. Let $u_1,\dots,u_n$ and $v_1,\dots,v_n$ be the two n-paths and let $V(H_n)=\{u_1,v_1,u_2,v_2,\dots,u_n,v_n\}$, where $u_iv_i\in E(H_n)$ for all $i=1,\dots,n$. In order to determine $\gamma^{(i)}_{\mathrm{spr}}(H_n)$ we first prove four lemmas.

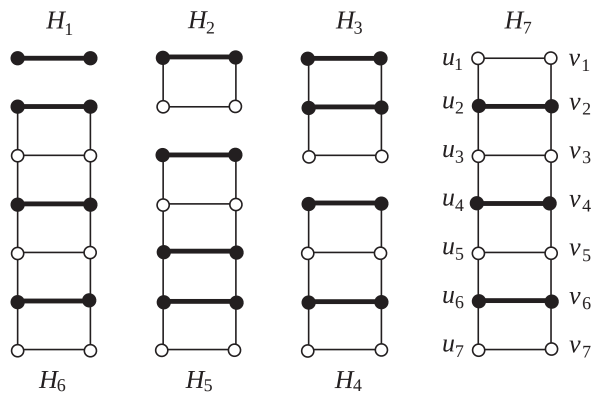

Figure 4: Minimum SPDS's for H_i, $i=1,\dots,7$

Lemma 3.7. *If J is the graph obtained by deleting a vertex of H_5 of degree two, then $\gamma^{(i)}_{\mathrm{spr}}(J)=6$, $i=2,\dots,9$.*

Proof. Label the vertices of J by $u_1,\dots,u_5,v_1,\dots,v_4$, that is, $J=H_5-v_5$. Since J has no PDS of cardinality four that contains v_3, $\gamma^{(i)}_{\mathrm{spr}}(J)\ge 6$ for each i. It is routine to verify that the PDS of H_5 in Fig. 4 2-defends J and the result follows. ☐

Lemma 3.8. *For $i=2,\dots,9$, $\gamma^{(i)}_{\mathrm{spr}}(H_1)=\gamma^{(i)}_{\mathrm{spr}}(H_2)=2$, $\gamma^{(i)}_{\mathrm{spr}}(H_3)=\gamma^{(i)}_{\mathrm{spr}}(H_4)=4$ and $\gamma^{(i)}_{\mathrm{spr}}(H_5)=\gamma^{(i)}_{\mathrm{spr}}(H_6)=\gamma^{(i)}_{\mathrm{spr}}(H_7)=6$.*

Proof. The statement is trivial for $n = 1, 2$. For $n = 3, ..., 7$, let D_n be the PDS of H_n illustrated in Fig. 4. By examining these sets it is clear that D_n is a 2-SPDS of H_n, hence $\gamma_{\rm spr}^{(9)}(H_n) \leq \cdots \leq \gamma_{\rm spr}^{(2)}(H_n) \leq |D_n|$.

The set $\{u_2, v_2\}$ is the unique PDS of H_3 of cardinality two and clearly does not 9-defend H_3, hence $\gamma_{\rm spr}^{(9)}(H_3) \geq 4$. Obviously, then, $\gamma_{\rm spr}^{(9)}(H_4) \geq 4$. It follows from Lemma 3.7 that $\gamma_{\rm spr}^{(9)}(H_5) \geq 6$ and consequently $\gamma_{\rm spr}^{(9)}(H_7) \geq \gamma_{\rm spr}^{(9)}(H_6) \geq 6$. The result follows by combining the inequalities. □

Lemma 3.9. *The set* $X = \{u_2, v_2, u_4, v_4, u_6, v_6\}$ *is the unique* 9*-SPDS of* H_7*.*

Proof. Suppose D is a minimum SPDS of H_7 that does not contain v_2. To dominate v_1, $|\{u_1, v_1, u_2\}| \cap D \geq 2$.

Suppose $v_3 \in D$. Then either $u_3, v_3 \in D$ or $v_3, v_4 \in D$. In the former case the copy of H_3 induced by $\{u_5, v_5, ., ., u_7, v_7\}$ must be defended by two guards, which is impossible. In the latter case the only way to 9-defend u_3 is for the guard on v_3 to move to u_3, and the guard on v_4 to move to v_3 or u_4. To ensure that v_5, u_7 and v_7 remain dominated after this move, either $u_6, v_6 \in D$ or $v_6, v_7 \in D$. But then D does not defend u_5 (either D does not dominate u_5 or a guard movement to u_5 leaves v_7 undominated).

Hence $v_3 \notin D$. After defending v_3, there is a pair of guards on v_3 and $x \in \{v_2, u_3, v_4\}$. If $x \in \{v_2, u_3\}$, then this leaves only two guards to defend $\langle\{u_5, v_5, ..., u_7, v_7\}\rangle \cong H_3$, which is impossible. Thus $x = v_4$. For guards to have moved to v_3 and v_4, D contains one of the following sets: (c) $\{u_3, u_4\}$, (d) $\{u_4, v_4\}$, (e) $\{v_4, v_5\}$, or (f) $\{u_3, u_4, u_5, v_5\}$, where the guard on u_3 moves to v_3 and the guard on v_5 moves to v_4. To ensure that u_6, v_6, u_7, v_7 remain dominated when v_3 is defended, two of these vertices are occupied. Thus (f) is impossible. If (c) holds, then a guard movement to v_5 leaves u_7 undominated, if (d) holds, then defending u_5 leaves v_3 or v_7 undominated, and if (e) holds, then $u_1, u_2 \in D$ to dominate u_3, and defending u_3 leaves v_1 undominated. This shows that v_2 is in every minimum 9-SPDS of H_7. By symmetry, u_2, u_6 and v_6 are in every minimum 9-SPDS of H_7.

Suppose S is a minimum 9-SPDS of H_7 such that $|S \cap \{u_4, v_4\}| < 2$. Then $S \cap \{u_3, v_3, u_5, v_5\} \neq \varnothing$; without loss of generality assume $u_3 \in S$. Then u_3 is paired in S with u_2, u_4 or v_3. In each case defending v_5 leaves u_7 undominated. We conclude that X is the only minimum 9-SPDS of H_7. □

Lemma 3.9 forms the basis step of an inductive proof of the next lemma. For $U \subseteq V$ or a subgraph U of G, we use the notation $g(U)$ to denote the set of guards stationed on vertices of U; when these guards move we still refer to them as being from $g(U)$ even though they may no longer be occupying vertices of U. When we say a guard *dominates* a vertex v we mean that the vertex occupied by the guard at that instant dominates v.

Lemma 3.10. *The set* $W_n = \bigcup_{i \equiv 2,4,6 \pmod 7} \{u_i, v_i\}$ *is the unique* 9*-SPDS of* H_{7n}*.*

Proof. Since W_n obviously 4-defends and thus 9-defends H_{7n} we only need to prove uniqueness and that it is a *minimum* 9-SPDS.

Assume that for some integer $k \geq 2$, $W_{k-1} = \bigcup_{i\equiv 2,4,6 (\text{mod } 7)} \{u_i, v_i\}$ is the unique minimum 9-SPDS of $H_{7(k-1)}$, let Z be a minimum 9-SPDS of H_{7k} and place a guard on each vertex in Z. Then $|Z| \leq 6k$. For $j = 1, ..., k$, let $F_j \cong H_7$ be the subgraph of H_{7k} induced by $\{u_{7j-6}, v_{7j-6}, ..., u_{7j}, v_{7j}\}$. We also abbreviate F_k to F and $H_{7k} - F \cong H_{7(k-1)}$ to G. Let $Z_j = Z \cap V(F_j)$. We prove that $|Z_j| = 6$ for each j.

Suppose $|Z_j| \leq 4$ for some j. We assume that $j \neq 1, k$; these cases can be proved similarly. Since the vertices in $Z_{j-1} \cup Z_{j+1}$ do not dominate any vertex in $B = \{u_{7j-5}, v_{7j-5}, ..., u_{7j-1}, v_{7j-1}\}$, there are four vertices available to defend $\langle B\rangle \cong H_5$. If $Z_j \subseteq B$, this is impossible because $\gamma^{(9)}_{\text{spr}}(H_5) = 6$ (Lemma 3.8), and if $Z_j \nsubseteq B$, this can be shown similarly to be impossible.

Suppose next that $|Z_j| = 5$. Then an odd number of vertices in Z_j are paired with vertices in $Z_{j-1} \cup Z_{j+1}$. Assume without loss of generality that u_{7j-6} is paired with u_{7j-7} and that v_{7j-6} is not paired with v_{7j-7}. Since the vertices in $Z_{j-1} \cup Z_{j+1} \cup \{u_{7j-6}\}$ do not dominate any vertex in $B' = \{u_{7j-4}, ..., u_{7j-1}\} \cup \{v_{7j-5}, ..., v_{7j-1}\}$, there are four vertices available to defend $\langle B'\rangle \cong J$ (as defined in Lemma 3.7). If $Z_j \subseteq B'$, then this is impossible by Lemma 3.7. If $Z_j \nsubseteq B'$ it is easy to show (similar to the proof of Lemma 3.7) that Z_j does not defend all vertices in B'. Therefore $|Z_j| \geq 6$ for each j. Since $|Z| \leq 6k$, equality holds throughout, as required.

Consider the subgraphs G and F of H_{7k} as defined above and let $Z' = \bigcup_{j=1}^{k-1} Z_j$. If Z_k does not dominate any vertices of G, then Z' 9-defends G. By the induction hypothesis, $Z' = W_{k-1}$. Then Z' does not dominate any vertex of F, thus Z_k 9-defends F. By Lemma 3.9, $Z_k = \{u_{7(k-1)+2}, ..., v_{7(k-1)+6}\}$, thus $Z = W_k$ and we are done.

Hence assume Z_k dominates some vertices of G. Then $Z_k \neq \{u_{7(k-1)+2}, ..., v_{7(k-1)+6}\}$ and again by Lemma 3.9, Z_k does not defend all of F. Therefore Z' dominates some vertices of F. It follows that $\{u_{7k-7}, v_{7k-7}\} \cap Z' \neq \varnothing$ and $\{u_{7k-6}, v_{7k-6}\} \cap Z_k \neq \varnothing$. Let $A = g(\{u_{7k-7}, v_{7k-7}\} \cap Z')$ and $B = g(\{u_{7k-6}, v_{7k-6}\} \cap Z_k)$. Before and/or after a guards movement to defend some vertex of F, some guard in A dominates at least one vertex of F that is not dominated by a guard in $g(Z_k)$, because Z_k does not defend all of F. Similarly, before and/or after a guards movement to defend some vertex of G, some guard in B dominates a vertex of G that is not dominated by a guard in $g(G)$.

If $\{u_{7k-7}, v_{7k-7}\} \subseteq Z'$, then regardless of the movements of the guards in A, u_{7k-7} and v_{7k-7} remain dominated by $g(G)$ after the moves. But then the only vertices of G dominated by B are also dominated by $g(G)$, which is not the case. Therefore $|\{u_{7k-7}, v_{7k-7}\} \cap Z'| = 1$. Similarly, $|\{u_{7k-6}, v_{7k-6}\} \cap Z_k| = 1$. Let $z \in \{u_{7k-7}, v_{7k-7}\} \cap Z'$ and $y \in \{u_{7k-6}, v_{7k-6}\} \cap Z_k$.

Now, with any choice of $z \in \{u_{7k-7}, v_{7k-7}\} \cap Z'$, let $\{u_{7k-6}, v_{7k-6}\} \cap Z_k = \{u_{7k-6}\}$. Since $|Z_k|$ is even, u_{7k-5} (and not u_{7k-7}) is the Z-partner of u_{7k-6} in F. Since no vertex in $Q = \{u_{7k-3}, ...u_{7k}, v_{7k-4}, ..., v_{7k}\}$ is dominated by $\{z, u_{7k-6}, u_{7k-5}\}$, there are four vertices in Z_k available to 9-defend $\langle Q \rangle$. But $\langle Q \rangle \cong J$ and we obtain a contradiction as above. Therefore $Z = W_k$ and the result follows by the principle of induction. □

We are now ready to determine $\gamma_{\text{spr}}^{(i)}(H_n)$ for all values of n.

Theorem 3.11. *For $i = 2, ..., 9$ and $n \geq 1$, $\gamma_{\text{spr}}^{(i)}(H_n) = 2\gamma_s(P_n)$.*

Proof. For $n \leq 7$ the result follows from Lemma 3.8, hence assume $n \geq 8$. Let $n = 7k+m$, where $k \geq 1$ and $0 \leq m < 7$ and label the vertices of H_n as described above. Let $G \cong H_{7k}$ be the subgraph of H_n induced by $\{u_1, v_1, \ldots, u_{7k}, v_{7k}\}$ and define the PDS W_k of G as in the proof of Lemma 3.10. If $m \neq 0$, let H_m be the subgraph of H_n induced by $\{u_{7k+1}, v_{7k+1}, , \ldots, u_n, v_n\}$ and let D_m be the PDS of H_m illustrated in Fig. 4.

To 2-defend u_i in H_{7k}, where $i \equiv 1 \pmod 7$, the guard on u_{i+1} moves to u_i while the guard on v_{i+1} moves to u_{i+1}. The resulting configuration of guards is clearly a PDS of H_{7k}. The vertices $v_i, u_{i+2}, v_{i+2}, u_{i+4}, v_{i+4}, u_{i+6}$ and v_{i+6} are similarly 2-defended. Hence D_{7k} is a 2-SPDS of H_{7k}. By Lemma 3.8, D_m is a 2-SPDS of H_m, $m = 1, ..., 6$. Thus $D_{7k} \cup D_m$ is a 2-SPDS of H_n and $\gamma_{\text{spr}}^{(2)}(H_n) \leq 2\gamma_s(P_n)$.

Suppose Z is a 9-SPDS of H_n of cardinality at most $2\gamma_s(P_n) - 2$ and let $Z' = Z \cap V$ and $Z_m = Z \cap V(H_m)$. If $m = 0$ the result follows from Lemma 3.10, hence assume $m \geq 1$.

As in the proof of Lemma 3.10 it follows that $|Z'| \geq 6k = 2\gamma_s(P_k)$. If $m \in \{1, 2\}$, then $2\gamma_s(P_n) - 2 = 6k$, so that $Z_m = \varnothing$. Then Z' defends all of G, hence $Z' = Z = W_k$, which does not dominate H_m, a contradiction. If $m \in \{3, 4\}$, then $2\gamma_s(P_n) - 2 = 6k + 2$, so that $|Z_m| \leq 2$. Since Z_m dominates u_{7k+3} and v_{7k+3}, $Z_m \cap \{u_{7k+1}, v_{7k+1}\} = \varnothing$. Hence Z_m does not dominate any vertex of G. By Lemma 3.10, $Z' = W_k$. Therefore Z' does not dominate any vertex of H_m, which implies that Z_m is a 9-SPDS of H_3 or H_4, contradicting Lemma 3.8. If $m = 5$, then $2\gamma_s(P_n) - 2 = 6k + 4$, so that $|Z_m| \leq 4$. Arguing as in the proof of Lemma 3.10 we deduce that for Z to 9-defend G and H_m, respectively, $|Z_m \cap \{u_{7k+1}, v_{7k+1}\}| = 1$ and $|Z' \cap \{u_{7k}, v_{7k}\}| = 1$. Without loss of generality assume $Z_m \cap \{u_{7k+1}, v_{7k+1}\} = \{u_{7k+1}\}$ and say $Z' \cap \{u_{7k}, v_{7k}\} = \{z\}$. Then u_{7k+1} is paired with z or with u_{7k+2}. Since $\{z, u_{7k+1}, u_{7k+2}\}$ does not dominate any vertex in $A = \{u_{7k+4}, u_{7k+5}, v_{7k+3}, v_{7k+4}, v_{7k+5}\}$, there are two vertices in Z_m available to defend $\langle A \rangle$. But $\langle A \rangle$ has no PDS of cardinality two that contains v_{7k+3}, a contradiction. Similar contradictions follow if $m \in \{6, 7\}$. Therefore any 9-SPDS of H_n has cardinality at least $2\gamma_s(P_n)$, that is, $\gamma_{\text{spr}}^{(9)}(H_n) \geq 2\gamma_s(P_n)$, and the proof is complete. □

Upper bounds for the general grid graphs $P_m \,\square\, P_n$ and circular grid graphs $C_m \,\square\, C_n$, as well as proofs of the exact values for the small cases mentioned below, can be found in [24].

Proposition 3.12. [24]

$$\gamma^{(2)}_{\text{spr}}(P_m \,\square\, P_m) = \begin{cases} 6 & \textit{if } m = 3 \\ 8 & \textit{if } m = 4 \\ 12 & \textit{if } m = 5 \\ 16 & \textit{if } m = 6. \end{cases}$$

4. Bounds

We compare the secure paired domination number of a graph G to various other parameters and provide upper bounds for $\gamma_{\text{spr}}(G)$. A *clique cover* of G is a partition of V such that each subset induces a clique, and the *clique covering number* $\theta(G)$ is the minimum number of sets in a clique cover of G. The *independence number* $\beta(G)$ is the cardinality of a largest independent set of G. A set $S \subseteq V$ is called a *vertex cover* of G if every edge of G has at least one endpoint in S. The *vertex covering number* $\alpha(G)$ is the cardinality of a minimum vertex cover of G. It is well known that $\alpha(G) + \beta(G) = |V|$ for any graph G.

For any $X \subseteq V$ and $x \in X$, we say that $v \in V - X$ is an *external private neighbour of x (relative to X)* if v is adjacent to x but to no other vertex of X. We denote the set of all external private neighbours of x by $\text{epn}(x, X)$. It is well known that any graph without isolated vertices has a minimum dominating set X such that $\text{epn}(x, X) \neq \varnothing$ for each $x \in X$ [3].

Theorem 4.1. *For any isolate-free graph G and $i = 2, ..., 9$, $\gamma(G) \leq \gamma^{(i)}_{\text{spr}}(G) \leq 2\gamma(G)$.*

Proof. The first inequality is trivial. We prove the second inequality for $i = 2$; the results for $i = 3, ..., 9$ will follow immediately. Let $D = \{v_1, ..., v_\gamma\}$ be a minimum dominating set of G such that $\text{epn}(v_i, D) \neq \varnothing$ for each i; say $u_i \in \text{epn}(v_i, D)$, $i = 1, ..., \gamma$. Define $S = \{v_1, u_1, ..., v_\gamma, u_\gamma\}$ and let M be the matching $\{v_1u_1, ..., v_\gamma u_\gamma\}$ of $\langle S \rangle$. Clearly, S is a PDS of G and $|S| = 2\gamma(G)$. We show that S is also a 2-SPDS of G.

Place a guard on each vertex in S and consider arbitrary $w \in V - S$. Since D is a dominating set there exists a vertex $v_i \in D$ adjacent to w. Move the guard on v_i to w and the guard on u_i to v_i. Let S_w denote the new set of occupied vertices and let $M_w = (M - \{v_iu_i\}) \cup \{v_iw\}$. Then M_w is a matching of $\langle S_w \rangle$ and thus, since $D \subseteq S_w$, S_w is a PDS of G. Thus S is an SPDS of G. □

Since $\gamma(G)$ is bounded above by each of the parameters $\gamma_s(G), \gamma_{\text{pr}}(G), \alpha(G), \beta(G), \theta(G)$, the next corollary follows directly from Theorem 4.1.

Corollary 4.2. *For any isolate-free graph G and $i = 2, ..., 9$,*

$$\gamma^{(i)}_{\text{spr}}(G) \leq 2 \min\{\gamma_s(G), \gamma_{\text{pr}}(G), \alpha(G), \beta(G), \theta(G)\}.$$

The 4-cycle is the smallest graph for which the first inequality in Theorem 4.1 is an equality. For each positive integer k we construct a class $\mathcal{G}_k$ of connected graphs such that $\gamma_{\text{spr}}^{(i)}(G_k) = \gamma(G_k) = 2k$ for each $G_k \in \mathcal{G}_k$ and $i = 2, ..., 9$.

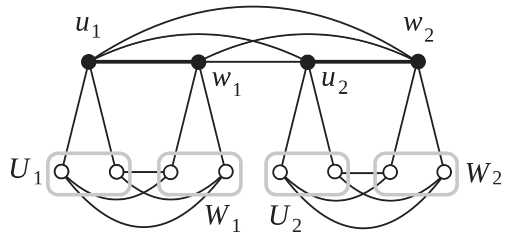

Figure 5: A graph $G_2 \in \mathcal{G}_2$; $\gamma_{\text{spr}}^{(i)}(G_2) = \gamma(G_2) = 4$

We first construct a class $\mathcal{A}_k$ of graphs A_k with k components. We begin with the set $D_k = \{u_1, w_1, ..., u_k, w_k\}$ and the matching $M = \{u_1w_1, ..., u_kw_k\}$. Let $U_1, W_1, ..., U_k, W_k$ be disjoint sets of vertices such that $|U_i|, |W_i| \geq 2$ for each i. Join u_i and w_i to each vertex in U_i and W_i, respectively, $i = 1, ..., k$. Add all edges between U_i and W_i so that $\langle U_i \cup W_i \rangle \cong K_{|U_i|,|W_i|}$. The graphs thus obtained form the class $\mathcal{A}_k$. Now form the class $\mathcal{G}_k$ by adding any additional edges between vertices in D_k that ensure that the graph is connected. The graph $G_2 \in \mathcal{G}_2$ in Fig. 5 is obtained by adding all possible edges between vertices in D_2 and where $|U_i| = |W_i| = 2$, $i = 1, 2$.

The set D_k dominates any $G_k \in \mathcal{G}_k$. Suppose X is a dominating set of G_k such that $|X| < 2k$. By the pigeonhole principle, $|X \cap (\{u_i, w_i\} \cup U_i \cup W_i)| \leq 1$ for some i. Since no vertex in $V(G_k) - (\{u_i, w_i\} \cup U_i \cup W_i)$ dominates a vertex in $U_i \cup W_i$, and no single vertex in $(\{u_i, w_i\} \cup U_i \cup W_i)$ dominates $U_i \cup W_i$, this is impossible. Hence $\gamma(G_k) = 2k$. Clearly, D_k is also a PDS of G_k. Place a guard on each vertex in D_k. If a vertex $v \in U_i$ is attacked, move the guard on u_i to v and the guard on w_i to u_i. The resulting set of occupied vertices is a PDS of G_k. Defend a vertex in W_i similarly. We conclude that D_k is a 2-SPDS of G_k and hence $\gamma_{\text{spr}}^{(2)}(G_k) = \gamma_{\text{spr}}^{(i)}(G_k) = \gamma(G_k) = 2k$.

On the other hand, the class $\mathcal{F}_k$ of graphs constructed just like $\mathcal{G}_k$, except that $\langle U_i \cup W_i \rangle \cong \overline{K_{|U_i|,|W_i|}}$, provides connected extremal graphs for the second inequality in Theorem 4.1.

5. Future Work

5.1. The Nine Definitions Revisited

An issue that needs to be addressed is the question of which definition is the "right" definition of an SPDS, or perhaps whether there *is* a "right" definition, and if so, which one. We feel (although the reader may disagree) that there are two situations that deserve consideration. It is either necessary that the same guards are paired before *and* after they move in response to an attack, or it isn't. In the former case the most suitable definition is Definition 2.4, because the two types of moves allowed here (one guard following the other along adjacent edges, or two paired guards moving along independent edges, ending as a pair again) are the only moves where the same guards are paired before and after the move, and in the latter case Definition 2.9 (where any one or two guards may move, as long as the guards form a PDS before and after the move) is the most suitable.

5.2. Other Problems

Problem 5.1. *Find (or characterize) extremal graphs (or trees) for the bounds in Theorem 4.1 and Corollary 4.2.*

Problem 5.2. *Find exact values of secure paired domination numbers for infinitely many grid graphs other than ladders.*

Problem 5.3. *Characterize graphs for which* $\gamma_{\text{spr}}^{(4)} = \gamma_{\text{spr}}^{(9)}$*. In particular, determine the role* 4*-cycles play in these two types of secure paired domination.*

Problem 5.4. *Study criticality and stability concepts for secure paired domination, especially for* $\gamma_{\text{spr}}^{(4)}$ *and* $\gamma_{\text{spr}}^{(9)}$*.*

Problem 5.5. *Consider the* eternal *version of secure paired domination, especially for* $\gamma_{\text{spr}}^{(4)}$ *and* $\gamma_{\text{spr}}^{(9)}$*, where the guards must defend the graph against an infinite sequence of attacks without returning to their original positions between attacks. Two possible models for guard movements are*

(a) *two guards (at most two guards, respectively) move at any given time, according to the moves for an i-SPDS, where i is even (odd, respectively).*

(b) *any number of guards may move, according to the respective rules.*

A survey of eternal domination, eternal total domination, eternal vertex covering, etc. is given in [26].

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