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On Heterogeneous Decompositions of Uniform Complete Multigraphs into Spanning Trees

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Abstract

Let $K_n^{(r)}$ be the order n uniform complete multigraph with edge multiplicity r. A spanning tree decomposition of $K_n^{(r)}$ partitions its edge set into a family $\mathcal{T}$ of edge-induced spanning trees. In a purely heterogeneous decomposition $\mathcal{T}$ no trees are isomorphic. Every order n tree occurs in a fully heterogeneous decomposition $\mathcal{T}$. All trees have equal multiplicity in a balanced decomposition $\mathcal{T}$. We show: (1) $K_5^{(2)}$ has 16 decomposition classes, four of which contain the 24 fully heterogeneous decompositions; (2) $K_5^{(4)}$ has 34 fully heterogeneous decomposition classes; exactly one lacks decompositions reducible to two $K_5^{(2)}$ decompositions; (3) when $k \geq 1$, many balanced fully heterogeneous decompositions of $K_5^{(6k)}$ reduce to "smaller" decompositions, but one such decomposition of $K_5^{(6)}$ is irreducible.

Keywords: graph decomposition, spanning tree.

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1. Heterogeneous Decompositions into Spanning Trees

This paper examines some rather natural graph decomposition questions involving heterogeneous families of spanning trees. This is a continuation of the work done by Eggleton in [3]. Typically the graphs we decompose have at least one edge between each pair of vertices, so they are "over-complete" multigraphs, the multiplicity of each adjacency being at least one. The focus is on decompositions of *uniform* complete multigraphs, that is, complete multigraphs in which all adjacencies have the same multiplicity.

A spanning tree decomposition of a multigraph K is a family $\mathcal{T}$ of edge-induced spanning trees that partitions the edges of K. A given multigraph K need not have any such decomposition. The *purely heterogeneous* case is when $\mathcal{T}$ is a set, so no two members of $\mathcal{T}$ are isomorphic. Typically $\mathcal{T}$ is a multiset; it is a *homogeneous* decomposition of K if all members are isomorphic, or a *heterogeneous* decomposition if at least two members are non-isomorphic. We denote by $\mathcal{T}(n)$ the family of all unlabeled trees of order n. If K has order n, a spanning tree decomposition $\mathcal{T}$ of K is *fully heterogeneous* if $\mathcal{T}(n) \subseteq \mathcal{T}$. Two spanning tree decompositions $\mathcal{T}$ and $\mathcal{T}'$ of K are *similar* if each tree in $\mathcal{T}(n)$ has the same multiplicity in $\mathcal{T}$ and in $\mathcal{T}'$, and they are *equivalent* if $\mathcal{T}$ can be transformed into $\mathcal{T}'$ by some automorphism of K.

Explanation of several of our notational conventions is appropriate. If $\mathcal{T}$ is a set or multiset and k is a positive integer, then $k\mathcal{T}$ is the multiset in which the multiplicity of each member is k times its multiplicity in $\mathcal{T}$. If G and H are two simple graphs or multigraphs, the *union* $G \cup H$ comprises one copy of each, and the copies are vertex-disjoint. In contrast, the *sum* $G + H$ comprises one copy of each, and the copies are edge-disjoint but the vertices of H are identified with distinct vertices of G (assuming the order of G is not less than the order of H). Usually the sum notation is ambiguous without further specification, because the vertex identifications can be achieved in various ways, but there is no ambiguity if G is a complete graph or, more generally, a uniform complete multigraph. In particular, if $G = H$ we write $2G$ for the union and $G^{(2)}$ for the sum with corresponding vertices identified. More generally, if k is any positive integer, then kG is the union of k copies of G, and $G^{(k)}$ is the sum of k copies of G with corresponding vertices identified. (We read $G^{(k)}$ as "G, k-fold".) Again, the *difference* $G - H$ is defined when G and H are simple graphs or multigraphs, with H a subgraph or submultigraph of G: the vertices of $G - H$ are those of G, and the multiplicity of any adjacency in $G - H$ is its multiplicity in G reduced by the corresponding multiplicity in H.

The complete multigraph $K_n^{(r)}$ is the order n uniform complete multigraph with all adjacencies of multiplicity r. The following two fully heterogeneous decompositions involving particular complete multigraphs are presented in [3], together with some oriented analogs:

Theorem 1.1. *The complete multigraph* $K_6^{(2)}$ *can be decomposed into* $\mathcal{T}(6)$*, one copy of each of the six trees of order* 6.

Theorem 1.2. *The complete multigraph* $K_4^{(2)}$ *can be decomposed into* $2\mathcal{T}(4)$*, two copies of each of the two trees of order* 4.

These are the motivating paradigms for the present paper. Here we investigate the possibility of various decompositions like these two, for the most part involving uniform complete multigraphs of order 5, but subsequently looking at some order n general decomposition results. We are planning sequel papers to discuss the rich decomposition results in the oriented and monochromatic cases [1, 2].

2. Fully Heterogeneous Spanning Tree Decompositions of $K_5^{(2)}$

The family of trees of order 5 is $\mathcal{T}(5) = \{S_5, P_5, Y\}$, where S_5 is the star, P_5 is the path and Y is the order 5 tree with maximum degree 3. (We call the unique vertex of degree 3 in Y the *high* vertex.) Since each tree in $\mathcal{T}(5)$ has 4 edges, any order 5 multigraph K with a spanning tree decomposition must have a multiple of 4 edges. Suppose $\mathcal{T}$ is a spanning tree decomposition of K and a, b, c are the respective multiplicities of the trees S_5, P_5, Y in $\mathcal{T}$; then K has $4(a+b+c)$ edges, and the *similarity class* of $\mathcal{T}$ is uniquely specified by the ordered triple (a, b, c). As a coarser classification, it is convenient to say $\mathcal{T}$ is of *type* $[r, s, t]$ when r, s, t is the permutation of a, b, c such that $r \geq s \geq t$. In particular, decompositions $\mathcal{T}$ of type $[r, s, t]$ with multiplicities $r \geq s \geq t \geq 1$ are the *fully heterogeneous* decompositions of order 5, containing each of the trees S_5, P_5, Y.

Since the complete graph K_5 has 10 edges while $\mathcal{T}(5)$ contains a total of 12 edges, there is no decomposition of K_5 into $\mathcal{T}(5)$. However, the complete multigraph $K_5^{(2)}$ has 20 edges, equal to the size of five trees of order 5, so it is natural to ask whether $K_5^{(2)}$ has a fully heterogeneous decomposition into five spanning trees. Such a decomposition $\mathcal{T}$ would be of type $[2, 2, 1]$ or $[3, 1, 1]$. The first has the form $\mathcal{T} = 2\mathcal{T}(5)\backslash\{T\}$ for some $T \in \mathcal{T}(5)$, so is "as heterogeneous as possible", while the second has the form $\mathcal{T} = \mathcal{T}(5) \cup 2\{T\}$ for some $T \in \mathcal{T}(5)$, so could be considered "moderately heterogeneous".

A relevant easier question is whether two edges can be added to K_5 so that the resulting multigraph has a purely heterogeneous decomposition into $\mathcal{T}(5)$, that is, a type $[1, 1, 1]$ decomposition. Consider this question first. Let $K_5 + G$ be the multigraph formed by adding two edges to K_5: evidently, there are just three possible choices for the edge-induced multigraph G of size 2.

Lemma 2.1. *Let G be an edge-induced multigraph of size 2. The multigraph $K_5 + G$ is uniquely decomposable into $\mathcal{T}(5)$ if $G \in \{P_2^{(2)}, P_3\}$, but it has no such decomposition if $G = 2P_2$.*

Proof. Note that $K_5 - S_5 = K_4 \cup K_1$. If two edges can be added to $K_4 \cup K_1$ to form a graph that decomposes into $\{P_5, Y\}$, the new edges must be incident with the isolated vertex v so that all vertices have at least two incident edges. Then v must be a leaf of P_5 and Y, so $P_5 - v$ and $Y - v$ are complementary trees of order 4. Consider the two order 4 trees: the path P_4 is self-complementary, but the complement of the star S_4 is $C_3 \cup K_1$, which is not even a tree. Hence $P_5 - v$ and $Y - v$ must be complementary copies of P_4.

If we extend this decomposition to $\{P_5, Y\}$ by adding two edges incident with v, the edge vx forming P_5 must be incident with an end vertex x of one copy of P_4, while the edge vy forming Y must be incident with an internal vertex y of the other copy of P_4. Since the internal vertices of P_4 are the end vertices of the complementary P_4, it follows that both x and y are end vertices of one copy of P_4. Hence, either (i) $x = y$ and the added edges form $G = P_2^{(2)}$, or (ii) x and y are the two end vertices of one copy of P_4, and the added edges form $G = P_3$. Thus, in precisely these cases a decomposition into $\mathcal{T}(5)$ exists and is unique. □

We label the vertices of K_5 with the elements of $\mathbb{Z}_5$, the integers modulo 5. The proof of Lemma 2.1 shows that each choice of $G \in \{P_2^{(2)}, P_3\}$ yields a unique equivalence class of decompositions of $K_5 + G$ into $\mathcal{T}(5)$. To record these decompositions canonically, we choose the lexicographically earliest sequence of edge labels corresponding to the edges of the component trees taken in the order S_5, P_5, Y, as follows

The class (1,1,1) decompositions $K_5 + G \to S_5 + P_5 + Y$

(A) $G = P_2^{(2)}$ S_5, P_5, Y: 01, 02, 03, 04; 04, 12, 23, 34; 04, 13, 14, 24.

(B) $G = P_3$ S_5, P_5, Y: 01, 02, 03, 04; 01, 12, 23, 34; 04, 13, 14, 24.

A spanning tree decomposition $\mathcal{T}$ of a multigraph K can be viewed from an analytic or a synthetic perspective. The analytic viewpoint ("discovering the parts") takes K as its starting point, and shows how K can be dismantled into the trees in $\mathcal{T}$. The synthetic viewpoint ("some assembly required") takes the trees in $\mathcal{T}$ as its starting point, and specifies how K can be assembled from them.

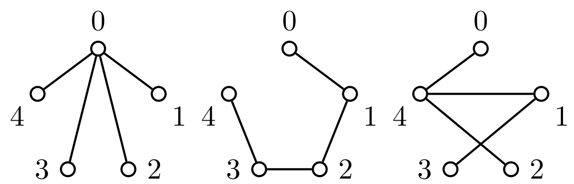

Figure 1: The analytic viewpoint: $K_5 + P_3$ decomposes into $\mathcal{T}(5)$.

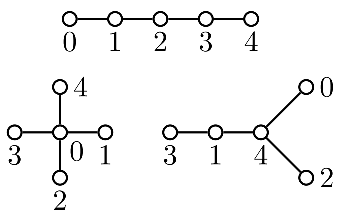

Figure 2: The synthetic viewpoint: $\mathcal{T}(5)$ assembles into $K_5 + P_3$.

To illustrate, Figure 1 shows the $\mathcal{T}(5)$ decomposition of K_5+P_3 from the analytic viewpoint, while Figure 2 shows it from the synthetic viewpoint. Each viewpoint is informative, and for visual purposes both are helpful. However, to save space, we shall standardize on the synthetic viewpoint and subsequently we represent decompositions visually just from that perspective.

We return now to the question about decomposition of $K_5^{(2)}$ into a family $\mathcal{T}$ of five spanning trees with $\mathcal{T}(5) \subset \mathcal{T}$. Note that K_5+P_3 is a subgraph of $K_5^{(2)}$, whereas $K_5+P_2^{(2)}$ is not. Thus decomposition of $K_5^{(2)}$ can contain (B), but not (A). With Lemma 2.1, if K_5-P_3 can be decomposed into two trees of order 5, this will give us a fully heterogeneous spanning tree decomposition of $K_5^{(2)}$. In fact this approach yields decompositions of both types $[2,2,1]$ and $[3,1,1]$. For completeness we shall give a representative of each equivalence class of decompositions.

Lemma 2.2. *The complete multigraph $K_5^{(2)}$ has exactly 11 equivalence classes of spanning tree decompositions with a subfamily $\mathcal{T}(5)$ that assembles into K_5+P_3. Each such decomposition contains just one S_5. The similarity classes $(1,2,2)$, $(1,3,1)$ and $(1,1,3)$ comprise three, four and four of the equivalence classes, respectively.*

Proof. We begin the decomposition of $K_5^{(2)}$ by partitioning the edges of K_5+P_3. Lemma 2.1 ensures that the vertices of K_5+P_3 can be labeled by $\mathbb{Z}_5$ so that its decomposition into $\mathcal{T}(5)$ is the canonical decomposition (B).

To complete the decomposition of $K_5^{(2)}$, we can partition the edges of K_5-P_3 into one of the families $\{P_5, Y\}$, $2\{P_5\}$ or $2\{Y\}$. All three cases arise, and the families form the similarity classes $(1,2,2)$, $(1,3,1)$ and $(1,1,3)$ respectively.

Using the same labeling of $K_5^{(2)}-(K_5+P_3)=K_5-P_3$ from Lemma 2.1 results in four distinct labelings of each decomposition family, so each of the decomposition families $2\mathcal{T}(5)\backslash\{S_5\}$, $\mathcal{T}(5)\cup 2\{P_5\}$ and $\mathcal{T}(5)\cup 2\{Y\}$ for $K_5^{(2)}$ has four distinct labelings.

It turns out that there is a nontrivial automorphism of $K_5^{(2)}$ under which two of the labelings of $2\mathcal{T}(5)\backslash\{S_5\}$ are interchanged, and each of the other labelings of this family has a nontrivial automorphism. The two unrepeated members of each of the other families ensure that no nontrivial automorphism of $K_5^{(2)}$ either preserves the labeling or maps it onto one of the other labelings.

In different words, for each decomposition in the similarity class $(1,2,2)$, there are two copies each of P_5 and Y. One can verify that two of the four different decompositions are isomorphic. As a result, there are only three nonisomorphic decompositions of K_5-P_3 into a copy of P_5 and a copy of Y, and thus there are only three nonisomorphic decompositions of $K_5^{(2)}$ in the similarity class $(1,2,2)$. For the similarity classes $(1,3,1)$ and $(1,1,3)$, the copies of P_5 and Y from (B) ensure that there are four nonisomorphic decompositions of

$K_5^{(2)}$ in their similarity class.

The canonical edge sequences of the eleven equivalence classes of spanning tree decompositions are listed below:

The class (1,2,2) decompositions $K_5^{(2)} \to S_5 + 2P_5 + 2Y$ containing $K_5 + P_3 \to S_5 + P_5 + Y$

(1) $K_5 + P_3$, P_5, Y: (B) & 02, 12, 14, 34; 03, 13, 23, 24.

(2) $K_5 + P_3$, P_5, Y: (B) & 02, 13, 14, 24; 03, 12, 23, 34.

(3) $K_5 + P_3$, P_5, Y: (B) & 03, 12, 14, 34; 02, 13, 23, 24.

The class (1,3,1) decompositions $K_5^{(2)} \to S_5 + 3P_5 + Y$ containing $K_5 + P_3 \to S_5 + P_5 + Y$

(4) $K_5 + P_3$, P_5, P_5: (B) & 02, 12, 13, 34; 03, 14, 23, 24.

(5) $K_5 + P_3$, P_5, P_5: (B) & 02, 13, 14, 23; 03, 12, 24, 34.

(6) $K_5 + P_3$, P_5, P_5: (B) & 02, 13, 24, 34; 03, 12, 14, 23.

(7) $K_5 + P_3$, P_5, P_5: (B) & 02, 14, 23, 34; 03, 12, 13, 24.

The class (1,1,3) decompositions $K_5^{(2)} \to S_5 + P_5 + 3Y$ containing $K_5 + P_3 \to S_5 + P_5 + Y$

(8) $K_5 + P_3$, Y, Y: (B) & 02, 12, 13, 24; 03, 14, 23, 34.

(9) $K_5 + P_3$, Y, Y: (B) & 02, 12, 14, 23; 03, 13, 24, 34.

(10) $K_5 + P_3$, Y, Y: (B) & 02, 12, 24, 34; 03, 13, 14, 23.

(11) $K_5 + P_3$, Y, Y: (B) & 02, 14, 23, 24; 03, 12, 13, 34.

□

Figure 3 shows decomposition (1) has automorphism (0)(12)(34). Decomposition (3), with automorphism (0)(13)(24), is the only other one of these 11 decompositions with a nontrivial automorphism. It is noteworthy that each of (1), (2) and (3) has two subfamilies $\mathcal{T}(5)$ that assemble into $K_5 + P_3$.

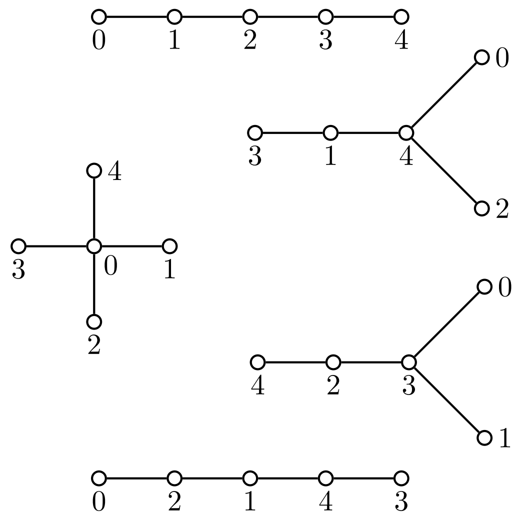

Figure 3: A symmetric decomposition of $K_5^{(2)}$ into $2\mathcal{T}(5)\backslash\{S_5\}$, extending a decomposition of $K_5 + P_3$ into $\mathcal{T}(5)$.

The following observation generalizes a feature of the decompositions described in Lemma 2.2:

Lemma 2.3. *If $\mathcal{T}$ is any decomposition of the complete multigraph $K_5^{(2)}$ into spanning trees with $S_5 \in \mathcal{T}$, then each subfamily $\{S_5, T\} \subset \mathcal{T}$ has exactly one common adjacency.*

Proof. The central vertex v of S_5 in $K_5^{(2)} - S_5$ has four incident edges, and $\mathcal{T}\backslash\{S_5\}$ comprises four spanning trees, so every $T \in \mathcal{T}\backslash\{S_5\}$ must contain exactly one of the edges incident with v. □

Lemma 2.4. *Let $\mathcal{T}$ be any decomposition of the complete multigraph $K_5^{(2)}$ into spanning trees, of which exactly s are copies of S_5. If $\{S_5, Y\} \subset \mathcal{T}$, then $1 \le s \le 3$; if $\{S_5, P_5\} \subset \mathcal{T}$ then $1 \le s \le 2$.*

Proof. If $\mathcal{T}$ contains more than one copy of S_5, any two copies must have distinct central vertices, otherwise they would have more than one common adjacency, contradicting Lemma 2.3. If $Y \in \mathcal{T}$ then $\mathcal{T}$ contains at most three copies of S_5, since Lemma 2.3 requires that the central vertex of each copy of S_5 must be a leaf of any copy of Y in $\mathcal{T}$. Similarly, if $P_5 \in \mathcal{T}$ then $\mathcal{T}$ contains at most two copies of S_5, since the central vertex of each must be a leaf of any copy of P_5. □

It follows from Lemma 2.4 that no type $[3, 1, 1]$ decomposition of $K_5^{(2)}$ can have three copies of S_5, so the similarity class $(3, 1, 1)$ is empty. Since Lemma 2.2 shows the existence of four decompositions of $K_5^{(2)}$ in each of the similarity classes $(1,3,1)$ and $(1,1,3)$, this settles the existence question for all similarity classes within the type $[3,1,1]$. Though our focus is on fully heterogeneous decompositions, let us note in passing that Lemma 2.4 also has immediate implications for decompositions of types $[3,2,0]$ and $[4,1,0]$. It shows that $K_5^{(2)}$ has no decomposition into three copies of S_5 and two copies of P_5, and no decomposition into four copies of S_5 and one copy of either P_5 or Y; thus the similarity classes $(3,2,0)$, $(4,1,0)$ and $(4,0,1)$ are empty.

Turning now to type $[2,2,1]$ decompositions of $K_5^{(2)}$, Lemma 2.2 shows there are three decompositions in the similarity class $(1,2,2)$, namely $2\mathcal{T}(5)\backslash\{S_5\}$. Let us next decide the existence question for $2\mathcal{T}(5)\backslash\{T\}$ when $T \in \{P_5, Y\}$. Lemma 2.2 implies that if such decompositions exist, they have no subfamily $\mathcal{T}(5)$ that assembles into $K_5 + G$, for any edge-induced multigraph G of size 2.

Lemma 2.5. *The complete multigraph $K_5^{(2)}$ has exactly two equivalence classes of decompositions into a family of spanning trees with a subfamily $2\{S_5\}$ and a subfamily $\mathcal{T}(5)$. Both are in the similarity class $(2,1,2)$, and neither has a subfamily $\mathcal{T}(5)$ that assembles into $K_5 + P_3$.*

Proof. Suppose $K_5^{(2)}$ has a spanning tree decomposition $\mathcal{T}$ that contains at least two copies of S_5. Lemma 2.3 requires that any two copies of S_5 in $\mathcal{T}$ have distinct central vertices, so when we delete their edges from $K_5^{(2)}$ the remaining multigraph (Figure 4) is $M = K_5 + C_3 - P_2$, where the P_2 is uniquely placed by the requirement that it be vertex-disjoint from the added 3-cycle C_3. Lemma 2.3 also implies that any P_5 in $\mathcal{T}$ has its end vertices at the centers of these two stars.

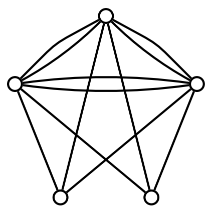

Figure 4: The multigraph M obtained from $K_5^{(2)}$ by deleting two copies of S_5 with distinct central vertices.

Since we seek all fully heterogeneous decompositions of $K_5^{(2)}$, let us determine the possible decompositions $\mathcal{T}^*$ of M into three spanning trees such that $\{P_5, Y\} \subset \mathcal{T}^*$. Note that Lemma 2.4 implies M has no $\mathcal{T}(5)$ decomposition, since that would yield a similarity class $(3,1,1)$ decomposition of $K_5^{(2)}$.

For $P_5 \in \mathcal{T}^*$, the multigraph $M - P_5$ is uniquely determined because it must have vertices of degree 2, so the P_5 must have both end vertices at the vertices of degree 3.

It has two inequivalent spanning tree decompositions that include a copy of Y. Indeed, both are of the form $2\{Y\}$. Correspondingly, $K_5^{(2)}$ has two equivalence classes of fully heterogeneous decompositions, both in the similarity class $(2,1,2)$. □

Thus, $K_5^{(2)}$ has exactly two equivalence classes of decomposition within the similarity class $(2,1,2)$ and none in the similarity class $(2,2,1)$. For canonicity, we list the members of each decomposition in Lemma 2.5 in the order S_5, S_5, P_5, Y, Y so each canonical edge sequence begins:

(C) S_5, S_5: 01, 02, 03, 04; 01, 12, 13, 14

Thus the decompositions in Lemma 2.5 have the following canonical representatives:

The class (2,1,2) decompositions $K_5^{(2)} \to 2S_5 + P_5 + 2Y$ avoiding $K_5 + P_3 \to S_5 + P_5 + Y$

(12) S_5, S_5, P_5, Y, Y: (C) & 02, 13, 24, 34; 03, 12, 23, 34; 04, 14, 23, 24.

(13) S_5, S_5, P_5, Y, Y: (C) & 02, 13, 24, 34; 03, 14, 23, 34; 04, 12, 23, 24.

Decomposition (12) has no nontrivial automorphism, but (13) is symmetric, with automorphism (01)(23)(4). In (12) the high vertices of the two copies of Y are at adjacent internal vertices of the P_5, while in (13) they are at nonadjacent internal vertices of the P_5 (Figure 5).

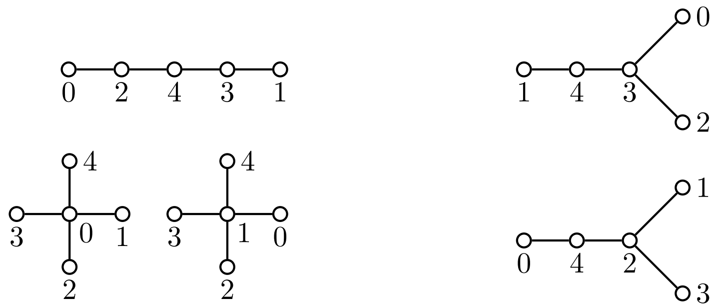

Figure 5: A symmetric decomposition of $K_5^{(2)}$ into $2\mathcal{T}(5)\backslash\{P_5\}$.

We cannot yet claim to know all decompositions $\mathcal{T}$ of $K_5^{(2)}$ that have $\mathcal{T}(5)$ as a subfamily, that is, all decompositions of types $[2,2,1]$ and $[3,1,1]$. The reason is that there could be such decompositions $\mathcal{T}$ which contain just one copy of S_5 (so $\mathcal{T}$ would be excluded from Lemma 2.5) but such that no subfamily $\mathcal{T}(5)$ assembles into $K_5 + P_3$ (so $\mathcal{T}$ would also be excluded from Lemma 2.2). In the following three lemmas we settle these remaining possibilities, beginning with the similarity class $(1,2,2)$:

Lemma 2.6. *The complete multigraph* $K_5^{(2)}$ *has exactly six inequivalent decompositions into a family* $\mathcal{T}$ *of spanning trees of the form* $2\mathcal{T}(5)\backslash\{S_5\}$ *with no subfamily* $\mathcal{T}(5)$ *that assembles into* $K_5 + P_3$.

Proof. Let $\mathcal{T}$ be any decomposition of $K_5^{(2)}$ into $2\mathcal{T}(5)\backslash\{S_5\}$, that is, a $(1, 2, 2)$ decomposition. Let v be the central vertex of the S_5 in $\mathcal{T}$. By Lemma 2.3, every other tree in $\mathcal{T}$ has a leaf at v. No vertex can be the high vertex in both copies of Y, since its degree in the other three trees must be positive, yet its degree in $K_5^{(2)}$ is only 8. Hence there are two distinct vertices w and x that are high vertices of copies of Y. Each has degree 1 in the S_5, so has degrees 2, 1, 1 in the other three trees. Thus each of w and x is an end vertex of some copy of P_5, so one P_5 is a v, w-path and the other is a v, x-path. We call such paths *linked* and denote their sum $P_5 + P_5$ by L.

Systematic investigation shows there are twelve nonisomorphic sums of two linked paths, but only nine possibilities for L since the two paths in $\mathcal{T}$ cannot share the same adjacency with S_5. Label the vertices of $K_5^{(2)}$ by $\mathbb{Z}_5$ so that the labeling of S_5 and one of the linked paths is the same as in the canonical decomposition (B). Decomposing the complement $K_5^{(2)} - L$ into two copies of Y in all possible ways yields a total of nine inequivalent decompositions. Three of these are the canonical decompositions (1), (2) and (3), each containing (B) as a subfamily. The remaining six have no subfamily $\mathcal{T}(5)$ that assembles into $K_5 + P_3$ so these are the decompositions required. □

For the canonical forms of the decompositions determined in Lemma 2.6 we list members in the order S_5, P_5, P_5, Y, Y so the canonical edge sequences begin

(D) S_5, P_5: 01, 02, 03, 04; 01, 12, 23, 34.

Thus the decompositions in Lemma 2.6 have the following canonical representatives:

The class (1,2,2) decompositions $K_5^{(2)} \to S_5 + 2P_5 + 2Y$ avoiding $K_5 + P_3 \to S_5 + P_5 + Y$

(14) S_5, P_5, P_5, Y, Y: (D) & 02, 13, 14, 24; 03, 12, 13, 34; 04, 14, 23, 24.

(15) S_5, P_5, P_5, Y, Y: (D) & 02, 13, 14, 24; 03, 13, 14, 23; 04, 12, 24, 34.

(16) S_5, P_5, P_5, Y, Y: (D) & 02, 13, 14, 24; 03, 13, 23, 24; 04, 12, 14, 34.

(17) S_5, P_5, P_5, Y, Y: (D) & 02, 13, 24, 34; 03, 12, 13, 14; 04, 14, 23, 24.

(18) S_5, P_5, P_5, Y, Y: (D) & 03, 14, 23, 24; 02, 12, 13, 14; 04, 13, 24, 34.

(19) S_5, P_5, P_5, Y, Y: (D) & 04, 13, 23, 24; 02, 12, 13, 14; 03, 14, 24, 34.

None of these decompositions has a nontrivial automorphism.

Lemma 2.7. *The complete multigraph* $K_5^{(2)}$ *has exactly two inequivalent decompositions into a family* $\mathcal{T}$ *of spanning trees of the form* $\mathcal{T}(5) \cup 2\{P_5\}$ *with no subfamily* $\mathcal{T}(5)$ *that assembles into* $K_5 + P_3$.

Proof. Let $\mathcal{T}$ be any decomposition of $K_5^{(2)}$ into $\mathcal{T}(5) \cup 2\{P_5\}$, that is, a $(1, 3, 1)$ decomposition. Let v be the central vertex of the S_5 in $\mathcal{T}$, and w be the high vertex of the Y. By Lemma 2.3, in the complement $K_5^{(2)} - (S_5 + Y)$ the degree of v is 3 and the degree of w is 4, so v is an end vertex of all three copies of P_5 in $\mathcal{T}$, and w is an end vertex of exactly two of them, which we shall call *twin* paths. We denote the sum of the twin paths $P_5 + P_5$ by W.

Label the vertices of $K_5^{(2)}$ by $\mathbb{Z}_5$ so that the S_5 and one of the twin paths are labeled as in the canonical decomposition (B). Now 0 and 4 are the end vertices of the second path, and 01 is not one of its edges. By systematic checking, there are four inequivalent labelings of the second path, so four possibilities for W. In each case $K_5^{(2)} - (S_5 + W)$ has either two or three decompositions $\{S_5, Y\}$, for a total of 10 decompositions. Since $(4) - (7)$ each occur twice among them, just the remaining two, which are inequivalent, have no subfamily $\mathcal{T}(5)$ that assembles into $K_5 + P_3$. □

We give the canonical description of the two decompositions in Lemma 2.7 in the form S_5, P_5, Y, P_5, P_5, for easy comparison with $(4) - (7)$. Each canonical edge sequence begins

(D) S_5, P_5: 01, 02, 03, 04; 01, 12, 23, 34

Thus the decompositions in Lemma 2.7 have the following canonical representatives:

The class (1,3,1) decompositions $K_5^{(2)} \to S_5 + 3P_5 + Y$ avoiding $K_5 + P_3 \to S_5 + P_5 + Y$

(20) S_5, P_5, Y, P_5, P_5: (D) & 02, 14, 24, 34; 03, 12, 13, 24; 04, 13, 14, 23.

(21) S_5, P_5, Y, P_5, P_5: (D) & 03, 14, 24, 34; 02, 13, 14, 23; 04, 12, 13, 24.

Neither of these decompositions has a nontrivial automormorphism.

Lemma 2.8. *The complete multigraph* $K_5^{(2)}$ *has exactly three inequivalent decompositions into a family* $\mathcal{T}$ *of spanning trees of the form* $\mathcal{T}(5) \cup 2\{Y\}$ *with no subfamily* $\mathcal{T}(5)$ *that assembles into* $K_5 + P_3$.

Proof. Let $\mathcal{T}$ be any decomposition of $K_5^{(2)}$ into $\mathcal{T}(5) \cup 2\{Y\}$, in the similarity class $(1, 1, 3)$, with no subfamily $\mathcal{T}(5)$ that assembles into $K_5 + P_3$. Label the vertices of $K_5^{(2)}$ by $\mathbb{Z}_5$ so that the labeling of S_5 and P_5 is the same as in the canonical decomposition (B). If $\mathcal{T}$ contains a copy of Y with high vertex at 4, it must be different from the Y in (B), leaving precisely five possibilities for this copy of Y. Let X be the sum of this Y with $S_5 + P_5$. The complement $K_5^{(2)} - X$ has a decomposition into $2\{Y\}$ in just two of the five cases.

As shown in the proof of Lemma 2.6, no vertex can be the high vertex in two copies of Y. By Lemma 2.3, the center of the S_5 must be a leaf of each copy of Y, so if $\mathcal{T}$ contains no copy of Y with high vertex at 4, then its three copies of Y must have high vertices at 1, 2 and 3. In particular, since $S_5 + P_5$ contains both the 01 adjacencies, there are just three possibilities for the Y with high vertex at 1. Let X^* be the sum of this Y with $S_5 + P_5$. In just one of the three cases does $K_5^{(2)} - X^*$ have a decomposition into $2\{Y\}$. □

We give the canonical description of the three decompositions found in Lemma 2.8 in the form S_5, P_5, Y, Y, Y. Each canonical edge sequence begins

(D) S_5, P_5: 01, 02, 03, 04; 01, 12, 23, 34

Thus the decompositions in Lemma 2.8 have the following canonical representatives:

The class (1,1,3) decompositions $K_5^{(2)} \to S_5 + P_5 + 3Y$ avoiding $K_5 + P_3 \to S_5 + P_5 + Y$

(22) S_5, P_5, Y, Y, Y: (D) & 02, 12, 13, 14; 03, 13, 24, 34; 04, 14, 23, 24.

(23) S_5, P_5, Y, Y, Y: (D) & 02, 14, 23, 24; 03, 12, 13, 14; 04, 13, 24, 34.

(24) S_5, P_5, Y, Y, Y: (D) & 02, 14, 23, 24; 03, 13, 24, 34; 04, 12, 13, 14.

None of these decompositions has a nontrivial automormorphism.

The results in Lemmas 2.1–2.8, and our observations on (1)–(24), give us the full picture regarding spanning tree decompositions of $K_5^{(2)}$ which contain at least one copy of each of the three trees of order 5:

Theorem 2.9. *The complete multigraph $K_5^{(2)}$ has exactly 24 inequivalent fully heterogeneous spanning tree decompositions. There are 11 of type $[2,2,1]$, and 13 of type $[3,1,1]$. Those of type $[2,2,1]$ comprise two in the similarity class $(2,1,2)$, and nine in the similarity class $(1,2,2)$. Those of type $[3,1,1]$ comprise six in the similarity class $(1,3,1)$, and seven in the similarity class $(1,1,3)$.*

Remark 2.10. *Of the 24 inequivalent fully heterogeneous spanning tree decompositions of $K_5^{(2)}$ just three have a nontrivial automorphism, and in each case its order is 2. One of these symmetric decompositions is in the similarity class $(2,1,2)$, the other two are in $(1,2,2)$.*

Before we conclude this section, it is convenient to discuss another property of the decompositions which recalls our starting point decomposing K_5+G, where G is an edge-induced multigraph of size 2.

Lemma 2.11. *Suppose $\mathcal{T}$ is a fully heterogeneous spanning tree decomposition of $K_5^{(2)}$ with a subfamily $\mathcal{T}^* \subset \mathcal{T}$ that assembles into K_5+G, where G is an edge-induced multigraph of size 2. Then $S_5 \in \mathcal{T}^*$ and $G = P_3$.*

Proof. Since $\mathcal{T}^*$ assembles into $K_5 + G$ for some edge-induced multigraph G of size 2, and $K_5 + G \subset K_5^{(2)}$, it follows that G has no adjacency of multiplicity greater than 1, so $G \in \{2P_2, P_3\}$. Now $S_5 \in \mathcal{T}$ since $\mathcal{T}$ is fully heterogeneous. Since $K_5^{(2)} - (K_5+G) = K_5 - G$, if $S_5 \notin \mathcal{T}^*$ then $K_5 - G - S_5 = (K_4 \cup K_1) - G$ is a tree of order 5, which is clearly false since there is an isolated vertex. Hence $S_5 \in \mathcal{T}^*$ and $K_5 + G - S_5 = (K_4 \cup K_1) + G$ is the sum of two trees of order 5, so every vertex has degree at least 2: this is impossible if $G = 2P_2$, so $G = P_3$. □

Let $\mathcal{T}$ be any spanning tree decomposition of $K_5^{(2)}$. We call $\mathcal{T}$ *subreducible* if it has a subfamily $\mathcal{T}^* \subset \mathcal{T}$ which assembles into $K_5 + G$, where G is an edge-induced multigraph of size 2. (In Section 4 we shall introduce reducible decompositions.) Lemma 2.11 shows that $\mathcal{T}^*$ and G are both constrained when $\mathcal{T}$ is fully heterogeneous and subreducible: then we say that $\mathcal{T}$ is *fully subreducible* if it has a subfamily $\mathcal{T}(5)$ that assembles into $K_5 + P_3$.

Theorem 2.12. *The 24 inequivalent fully heterogeneous spanning tree decompositions of the complete multigraph $K_5^{(2)}$ comprise 14 which are subreducible and 10 which are not. The similarity class $(1,2,2)$ includes six subreducible decompositions, three of them fully subreducible. Each of the similarity classes $(1,3,1)$ and $(1,1,3)$ includes four subreducible decompositions, and all are fully subreducible. Of the 14 subreducible decompositions, 12 have a subfamily different from $\mathcal{T}(5)$ that assembles into $K_5 + P_3$; the two exceptions are in $(1,2,2)$.*

Proof. Our derivation of (1)–(24) has automatically identified all decompositions with a subfamily $\mathcal{T}(5)$ that assembles into $K_5 + P_3$. With Lemma 2.11, to complete the classification it only remains to identify all subfamilies that assemble into $K_5 + P_3$.

A decomposition $\mathcal{T}$ of $K_5^{(2)}$ has a subfamily $\mathcal{T}^*$ that assembles into $K_5 + P_3$ if and only if the subfamily $\mathcal{T}\backslash\mathcal{T}^*$ assembles into $K_5 - P_3$. Since $\mathcal{T}\backslash\mathcal{T}^*$ has only two members, and no adjacency in $K_5 - P_3$ has multiplicity greater than one, the task becomes a simple inspection of $\mathcal{T}$ to determine all pairs of trees with no common adjacency, that is, all edge-disjoint pairs in $\mathcal{T}$. We omit the remaining details, which can be readily reconstructed: the theorem summarizes the results. □

3. Other Spanning Tree Decompositions of $K_5^{(2)}$

There are a number of other decomposition questions we wish to examine, the first being to investigate the spanning tree decompositions of $K_5^{(2)}$ that are not fully heterogeneous. However, the full catalogue of such decompositions, as with others of later interest, appears to be quite extensive. So, beginning with this section, for the remainder of the paper we shall simply give existence results rather than a comprehensive determination of all solutions.

Any spanning tree decomposition of $K_5^{(2)}$ that is not fully heterogeneous must be of one of three types: $[3, 2, 0]$, $[4, 1, 0]$ or $[5, 0, 0]$. In Section 2 we deduced from Lemma 2.4 that the similarity classes $(3, 2, 0)$, $(4, 1, 0)$ and $(4, 0, 1)$ are empty. We now give an example of every other similarity class. First, those with at least two copies of S_5 can be obtained by decomposing $M = K_5^{(2)} - (S_5 + S_5)$, as in Fig. 4. They have canonical specifications beginning with

(C) S_5, S_5: 01, 02, 03, 04; 01, 12, 13, 14

In the similarity classes $(3, 0, 2)$, $(2, 3, 0)$, $(2, 0, 3)$ and $(5, 0, 0)$ we have:

(25) S_5, S_5, S_5, Y, Y: (C) & 02, 12, 23, 24; 03, 13, 24, 34; 04, 14, 23, 34.

(26) S_5, S_5, P_5, P_5, P_5: (C) & 02, 13, 24, 34; 03, 14, 23, 24; 04, 12, 23, 34.

(27) S_5, S_5, Y, Y, Y: (C) & 02, 12, 23, 34; 03, 13, 24, 34; 04, 14, 23, 24.

(28) S_5, S_5, S_5, S_5, S_5: (C) & 02, 12, 23, 24; 03, 13, 23, 34; 04, 14, 24, 34.

Canonical specification of any member of $(1, 4, 0)$ begins

(D) S_5, P_5: 01, 02, 03, 04; 01, 12, 23, 34.

Our example continues:

(29) S_5, P_5, P_5, P_5, P_5: (D) & 02, 13, 14, 24; 03, 13, 14, 24; 04, 12, 23, 34.

Canonical specification of any member of $(1, 0, 4)$ begins

(E) S_5, Y: 01, 02, 03, 04; 01, 12, 13, 24.

Our example continues:

(30) S_5, Y, Y, Y, Y: (E) & 02, 12, 13, 24; 03, 14, 23, 34; 04, 14, 23, 34.

Next, consider decompositions with no S_5 but at least two copies of P_5. Their canonical specifications begin

(F) P_5, P_5: 01, 12, 23, 34; 01, 04, 13, 23.

In the similarity classes $(0, 2, 3)$, $(0, 3, 2)$, $(0, 4, 1)$ and $(0, 5, 0)$ we have:

(31) P_5, P_5, Y, Y, Y: (F) & 02, 03, 04, 14; 02, 12, 13, 24; 03, 14, 24, 34.

(32) P_5, P_5, P_5, Y, Y: (F) & 03, 13, 14, 24; 02, 03, 04, 12; 02, 14, 24, 34.

(33) P_5, P_5, P_5, P_5, Y: (F) & 02, 03, 14, 24; 02, 12, 13, 34; 03, 04, 14, 24.

(34) P_5, P_5, P_5, P_5, P_5: (F) & 02, 03, 14, 24; 02, 03, 14, 34; 04, 12, 13, 24.

Canonical specification of any member of $(0, 1, 4)$ begins

(G) P_5, Y: 01, 12, 23, 34; 01, 12, 13, 24.

Our example continues:

(35) P_5, Y, Y, Y, Y: (G) & 02, 03, 04, 14; 02, 13, 23, 34; 03, 04, 14, 24.

Finally, canonical specification of any member of $(0, 0, 5)$ begins

(H) Y, Y: 01, 02, 03, 14; 01, 04, 12, 13.

Note that in this case we are again decomposing $M = K_5^{(2)} - (Y + Y)$, as in Fig. 4. Our example continues:

(36) Y, Y, Y, Y, Y: (H) & 02, 13, 24, 23; 03, 14, 23, 34; 04, 12, 24, 34.

Summarizing, we have

Lemma 3.1. *There is a spanning tree decomposition of the complete multigraph* $K_5^{(2)}$ *for each of the types* $[3, 2, 0]$, $[4, 1, 0]$ *and* $[5, 0, 0]$, *and all potential similarity classes except* $(3, 2, 0)$, $(4, 0, 1)$ *and* $(4, 1, 0)$ *are realized.*

Combining this with Theorem 2.9 yields

Theorem 3.2. *The five possible types of spanning tree decomposition of the complete multigraph* $K_5^{(2)}$ *are all realized. In all,* 16 *of the* 21 *potential similarity classes are realized; the five exceptions are* $(2, 2, 1)$, $(3, 1, 1)$, $(3, 2, 0)$, $(4, 0, 1)$ *and* $(4, 1, 0)$.

4. Spanning Tree Decompositions of $K_5^{(4)}$

The complete multigraph $K_5^{(4)}$ has size 40, so it is the next natural candidate for decomposition into spanning trees. Note that for heterogeneous decomposition of $K_5^{(4)}$, we can just use two copies of the heterogeneous decomposition of $K_5^{(2)}$. A spanning tree decomposition $\mathcal{T}$ of $K_5^{(4)}$ is *fully heterogeneous* if $\mathcal{T}(5) \subset \mathcal{T}$. Any such decomposition $\mathcal{T}$ of $K_5^{(4)}$ has type $[a, b, c]$ with $a \geq b \geq c \geq 1$ and $a + b + c = 10$: there are eight possible types. Since there are abc subfamilies $\mathcal{T}(5)$ in any member of $[a, b, c]$, the number of such subfamilies ranges from 36 for $[4, 3, 3]$, down to 8 for $[8, 1, 1]$.

We shall focus on the natural question: Which types and similarity classes are realized by the fully heterogeneous spanning tree decompositions of $K_5^{(4)}$? We begin by looking at those decompositions equivalent to two decompositions of $K_5^{(2)}$. Let us say that a given spanning tree decomposition $\mathcal{T}$ of $K_5^{(4)}$ is *reducible* if it has a subfamily $\mathcal{T}^*$ that assembles into $K_5^{(2)}$. In this case the subfamily $\mathcal{T}\backslash\mathcal{T}^*$ also assembles into $K_5^{(2)}$, so effectively $\mathcal{T}$ is a decomposition of two copies of $K_5^{(2)}$.

Lemma 4.1. *The eight possible types of fully heterogeneous spanning tree decomposition of the complete multigraph $K_5^{(4)}$ are all realized. Moreover,* 33 *of the* 36 *potential similarity classes contain a reducible decomposition, the three exceptions being* $(4, 5, 1)$, $(7, 2, 1)$ *and* $(8, 1, 1)$.

Proof. First consider type $[4, 3, 3]$ decompositions. Theorem 3.2 shows $K_5^{(2)}$ has decompositions in $(1, 2, 2)$, $(2, 1, 2)$, $(1, 3, 1)$ and $(3, 0, 2)$. Note that

$$\begin{aligned}(3,\ 3,\ 4) &= (1,\ 2,\ 2) + (2,\ 1,\ 2)\\ (3,\ 4,\ 3) &= (1,\ 3,\ 1) + (2,\ 1,\ 2)\\ (4,\ 3,\ 3) &= (1,\ 3,\ 1) + (3,\ 0,\ 2).\end{aligned}$$

It follows that all three similarity classes of type $[4, 3, 3]$ are realized, and each class contains a heterogeneously reducible decomposition. Proceeding in this way for all 36 potential similarity classes for $K_5^{(4)}$, we use Theorem 3.2 to determine (Table 1) when a decomposition can be achieved by combining members of two similarity classes for $K_5^{(2)}$. If Theorem 3.2 shows a particular similarity class has no such decomposition, then $K_5^{(4)}$ has no reducible decomposition in that similarity class. It turns out that $(4, 5, 1)$, $(7, 2, 1)$ and $(8, 1, 1)$ are the only similarity classes with this property. □

Suppose $\mathcal{T}$ is a reducible spanning tree decomposition of $K_5^{(4)}$, and $\mathcal{T}^* \subset \mathcal{T}$ is a subfamily that assembles into $K_5^{(2)}$. Then $\mathcal{T}$ is *heterogeneously reducible* if $\mathcal{T}(5) \subset \mathcal{T}^*$, and $\mathcal{T}$ is *bi-heterogeneously reducible* if $\mathcal{T}(5) \subset \mathcal{T} \backslash \mathcal{T}^*$ also holds. In compiling Table 1, wherever possible we have specified a bi-heterogeneous decomposition, and if Theorem 3.2 implies that there is no such decomposition, then wherever possible we have specified a heterogeneous decomposition. By Theorem 3.2 and Table 1, we have

Lemma 4.2. *Of the* 33 *similarity classes of fully heterogeneous spanning tree decompositions of* $K_5^{(4)}$ *that contain a reducible decomposition,* 30 *have a heterogeneously reducible member, and nine have a bi-heterogeneously reducible member. The three exceptions are* $(5, 2, 3)$*,* $(5, 3, 2)$ *and* $(5, 4, 1)$*.*

Table 1. Reducibility of fully heterogeneous decompositions of $K_5^{(4)}$.

Type [**4, 3, 3**]			**Type** [**5, 4, 1**]			**Type** [**6, 2, 2**]		
3 3 4	1 2 2	2 1 2	1 4 5	1 2 2	0 2 3	2 2 6	1 1 3	1 1 3
3 4 3	1 3 1	2 1 2	1 5 4	1 2 2	0 3 2	2 6 2	1 3 1	1 3 1
4 3 3	1 3 1	3 0 2	4 1 5	2 1 2	2 0 3	6 2 2	1 2 2	5 0 0
Type [**4, 4, 2**]			4 5 1	—	—	**Type** [**7, 2, 1**]		
2 4 4	1 2 2	1 2 2	5 1 4	2 1 2	3 0 2	1 2 7	1 2 2	0 0 5
4 2 4	2 1 2	2 1 2	5 4 1	0 4 1	5 0 0	1 7 2	1 2 2	0 5 0
4 4 2	2 1 2	2 3 0	**Type** [**6, 3, 1**]			2 1 7	2 1 2	0 0 5
Type [**5, 3, 2**]			1 3 6	1 3 1	0 0 5	2 7 1	1 3 1	1 4 0
2 3 5	1 2 2	1 1 3	1 6 3	1 3 1	0 3 2	7 1 2	2 1 2	5 0 0
2 5 3	1 2 2	1 3 1	3 1 6	1 1 3	2 0 3	7 2 1	—	—
3 2 5	1 1 3	2 1 2	3 6 1	1 3 1	2 3 0	**Type** [**8, 1, 1**]		
3 5 2	1 2 2	2 3 0	6 1 3	1 1 3	5 0 0	1 1 8	1 1 3	0 0 5
5 2 3	0 2 3	5 0 0	6 3 1	1 3 1	5 0 0	1 8 1	1 3 1	0 5 0
5 3 2	0 3 2	5 0 0				8 1 1	—	—

To complete the picture, we now examine the three possible similarity classes which Lemma 4.1 shows contain no reducible decompositions.

Lemma 4.3. *The complete multigraph* $K_5^{(4)}$ *has fully heterogeneous spanning tree decompositions in the similarity class* $(4, 5, 1)$*, but the potential similarity classes* $(7, 2, 1)$ *and* $(8, 1, 1)$ *are not realized.*

Proof. Suppose $K_5^{(4)}$ has a spanning tree decomposition $\mathcal{T}$ in one of the similarity classes $(7, 2, 1)$ and $(8, 1, 1)$. Then $\mathcal{T}$ has at least seven copies of S_5. If a vertex is the center of s copies of S_5, the remaining $10 - s$ trees in $\mathcal{T}$ each require at least one incidence with that vertex, so $4s + (10 - s) \leq 16$. Therefore $s \leq 2$ and no vertex is the center of more than two copies of S_5. Thus two vertices must be centers of two copies of S_5, and either (i) each of the other three vertices is the center of at least one copy of S_5, or else (ii) a third vertex is the center of two copies of S_5 and a fourth is the center of at least one copy of S_5.

In case (i), the complement of the seven copies of S_5 is the multigraph M (Fig. 4). Any spanning tree decomposition of M which contains both P_5 and Y belongs to the similarity class $(0,1,2)$, so $\mathcal{T}$ is in the similarity class $(7,1,2)$, contradicting our initial choice of $\mathcal{T}$. In case (ii), the complement of the seven copies of S_5 is a multigraph $N = S_5^{(2)} + S_5$, where the center x of $S_5^{(2)}$ is distinct from the center y of S_5. As x has degree 9 in N, any spanning tree decomposition of N which contains both P_5 and Y must also contain S_5, and all three trees must attain their maximum degree at x. Deleting an S_5 and a Y with their maximum degree at x leaves another Y, not a P_5. Hence, no fully heterogeneous $\mathcal{T}$ satisfies case (ii). Thus neither of the similarity classes $(7,2,1)$ and $(8,1,1)$ is realized.

Let us now seek a decomposition $\mathcal{T}$ in the similarity class $(4,5,1)$, with distinct vertices as the centers of all four copies of S_5. The complement of the four copies of S_5 is $K_5^{(2)} + S_5$. This can also be viewed as $S_5^{(3)} + K_4^{(2)}$, where the four vertices of the $K_4^{(2)}$ are the end vertices of $S_5^{(3)}$. In order to decompose this multigraph into five copies of P_5 and one copy of Y, we partition the 12 edges in $K_4^{(2)}$ into six pairs, none with a repeated adjacency, and extend each pair by adding two suitable adjacencies with the central vertex of $S_5^{(3)}$. At least two decompositions result from this construction: in one the five copies of P_5 have three distinct central vertices, while in the other they have four distinct central vertices. This completes the proof. □

To specify any fully heterogeneous decomposition of $K_5^{(4)}$ in the similarity class $(4,5,1)$ we list the members in the order S_5, S_5, S_5, S_5, P_5, P_5, P_5, P_5, P_5, Y. Our two examples of irreducible decompositions for Lemma 4.3 have a canonical edge sequence which begins:

(I) S_5, S_5, S_5, S_5: 01, 02, 03, 04; 01, 12, 13, 14; 02, 12, 23, 24; 03, 13, 23, 34.

The continuations also agree in the next two members:

(J) P_5, P_5: 01, 12, 24, 34; 01, 14, 23, 24.

Finally, our two completions P_5, P_5, P_5, Y are

(37) (I) & (J) & 02, 04, 12, 34; 02, 04, 13, 14; 03, 04, 13, 24; 03, 14, 23, 34.

(38) (I) & (J) & 04, 12, 23, 34; 04, 13, 02, 34; 02, 03, 13, 14; 04, 03, 24, 14.

Summarizing the results of this section, we have shown:

Theorem 4.4. *The complete multigraph* $K_5^{(4)}$ *has* 34 *similarity classes of fully heterogeneous spanning tree decompositions; the only potential classes not realized are* $(7,2,1)$ *and* $(8,1,1)$. *With* $(4,5,1)$ *as the sole exception,* 33 *of the similarity classes contain reducible decompositions. Of these,* 30 *contain heterogeneously reducible decompositions,* *and exactly nine of those contain bi-heterogeneously reducible decompositions. The three similarity classes which contain reducible decompositions, but none that is heterogeneously reducible, are* $(5, 2, 3)$, $(5, 3, 2)$ *and* $(5, 4, 1)$.

5. Balanced Spanning Tree Decompositions of $K_5^{(6k)}$

Let $\mathcal{T}$ be a fully heterogeneous spanning tree decomposition of type $[a, b, c]$ for an order 5 multigraph K. We call $\mathcal{T}$ a *balanced* spanning tree decomposition if $a = b = c$. Since $\mathcal{T}(5)$ has total size 12 and $K_5^{(r)}$ has size $10r$, it follows that $K_5^{(r)}$ can have a balanced fully heterogeneous spanning tree decomposition only when $r = 6k$, for some positive integer k. If $K_5^{(6k)}$ has a balanced decomposition $\mathcal{T}$, its type must be $[5k, 5k, 5k]$, and $\mathcal{T} = 5k\mathcal{T}(5)$. Here we shall consider the existence of such decompositions $K_5^{(6k)}$. Note that the existence question for balanced spanning tree decompositions of $K_5^{(6)}$ and $K_7^{(22)}$ was posed in [3], and answered affirmatively by Riskin [4], who added some natural extensions.

Generalizing earlier terminology, we shall say that a given spanning tree decomposition $\mathcal{T}$ of $K_5^{(2r)}$ is *reducible* if it has a subfamily $\mathcal{T}^*$ that assembles into $K_5^{(2s)}$, for some integers $r > s > 0$. In that case $\mathcal{T}\backslash\mathcal{T}^*$ is a decomposition of $K_5^{(2t)}$, where $t = r - s$, so $\mathcal{T}$ is the union of decompositions of $K_5^{(2s)}$ and $K_5^{(2t)}$. We say $\mathcal{T}$ is *heterogeneously reducible* if $\mathcal{T}(5) \subset \mathcal{T}^*$, and $\mathcal{T}$ is *bi-heterogeneously reducible* if $\mathcal{T}(5) \subset \mathcal{T}\backslash\mathcal{T}^*$ also holds.

Building on our results from Sections 2–4, it is easy to see that $K_5^{(6)}$ has balanced decompositions into $5\mathcal{T}(5)$. As a first result, we have

Lemma 5.1. *For any given spanning tree decomposition* $\mathcal{T}^*$ *of* $K_5^{(2)}$, *there is a reducible balanced spanning tree decomposition* $\mathcal{T}$ *of the complete multigraph* $K_5^{(6)}$ *such that* $\mathcal{T}^* \subset \mathcal{T}$.

Proof. By Theorem 3.2, any spanning tree decomposition $\mathcal{T}^*$ of $K_5^{(2)}$ belongs to one of 16 similarity classes. The combination of any three homogeneous decompositions, one from each of the similarity classes $(5, 0, 0)$, $(0, 5, 0)$ and $(0, 0, 5)$, justifies our claim when $\mathcal{T}^*$ is homogeneous. In all other cases $\mathcal{T}^*$ belongs to one of 13 similarity classes, and the complement of each of these similarity classes relative to $(5, 5, 5)$ corresponds to one of the similarity classes of fully heterogeneous spanning tree decompositions realized for $K_5^{(4)}$, by Theorem 4.4. □

Without attempting to be comprehensive regarding numbers of distinct balanced spanning tree decompositions, let us define a characteristic that will readily enable us to distinguish certain examples as inequivalent. We choose a descriptor of the configuration of stars in each spanning tree decomposition $\mathcal{T}$ of $K_5^{(2r)}$. If $\mathcal{T}$ contains exactly s stars, let $(a_0, a_1, a_2, a_3, a_4) \in \mathbb{Z}^5$ be the ordered partition of s in which a_i is the number of stars in $\mathcal{T}$

centered at the vertex i. The *star type* of $\mathcal{T}$ is the non-increasing partition $\alpha = [a, b, c, d, e]$ of s in which $a \geq b \geq c \geq d \geq e \geq 0$ is a permutation of a_0, a_1, a_2, a_3, a_4. As an example of the use of star types, we prove

Lemma 5.2. *The complete multigraph $K_5^{(6)}$ has at least five inequivalent bi-heterogeneously reducible balanced spanning tree decompositions formed from three spanning tree decompositions of $K_5^{(2)}$, two of subtype $(2,1,2)$ and one of subtype $(1,3,1)$.*

Proof. Theorem 3.2 guarantees $K_5^{(2)}$ has decompositions in the similarity classes $(2,1,2)$ and $(1,3,1)$. Their star types are $[1,1,0,0,0]$ and $[1,0,0,0,0]$, respectively. In combining two of the first and one of the second into a $(5,5,5)$ decomposition of $K_5^{(6)}$, any desired vertex matching of the underlying copies of $K_5^{(2)}$ can be used. Thus configurations can be produced with the centers of the five stars all distinct, or variously coinciding. The five star types $[1,1,1,1,1]$, $[2,1,1,1,0]$, $[2,2,1,0,0]$, $[3,1,1,0,0]$ and $[3,2,0,0,0]$ can all be realized. Different star types guarantee inequivalence, so the lemma follows. □

Likewise, at least four inequivalent bi-heterogeneously reducible $(5,5,5)$ decompositions of $K_5^{(6)}$ can be formed by combining three decompositions of $K_5^{(2)}$, one from each of the similarity classes $(1,2,2)$, $(1,3,1)$ and $(3,0,2)$.

Lemma 5.3. *For any integer $k \geq 1$, there is a balanced spanning tree decomposition of $K_5^{(6k)}$ with star type α if and only if $\alpha \in \mathbb{Z}^5$ is a non-increasing partition of $5k$ with no summand exceeding $3k$.*

Proof. Any balanced spanning tree decomposition $\mathcal{T}$ of $K_5^{(6k)}$ has $5k$ stars. Suppose s of those stars have a common central vertex v. The remaining $15k - s$ trees in $\mathcal{T}$ are spanning, so each has at least one adjacency with v. The degree of each vertex in $K_5^{(6k)}$ is $24k$, and the stars centered on v use a total of $4s$ adjacencies, so $4s + (15k - s) \leq 24k$, whence $s \leq 3k$.

We claim that every non-increasing partition of $5k$ with length 5 and no summand exceeding $3k$ is the star type of some balanced spanning tree decomposition of $K_5^{(6k)}$. The claim holds when $k = 1$, since there are precisely five non-increasing partitions of length 5 with no summand exceeding 3, and the proof of Lemma 5.2 shows each is the star type of a balanced spanning tree decomposition $\mathcal{T}$ of $K_5^{(6)}$. Now fix some $k \geq 1$, and suppose the claim holds for k. Let $\alpha = [a, b, c, d, e]$ be any non-increasing partition of $5(k+1)$ with length 5 and no summand exceeding $3(k+1)$. First suppose $k = 1$. If $\alpha = [2,2,2,2,2]$, note that two decompositions of $K_5^{(6)}$ with star type $[1,1,1,1,1]$ assemble into a decomposition of $K_5^{(12)}$ with star type α. If $\alpha = [6,1,1,1,1]$, a suitable combination of two decompositions of $K_5^{(6)}$ with star type $[3,1,1,0,0]$ yields a decomposition of $K_5^{(12)}$ with star type α. If $k = 1$ and $3 \leq a \leq 5$, then $2 \leq b \leq 5$ and $c \leq 3$, so suitably permuting

$(a-3, b-2, c, d, e)$ yields a length 5 non-increasing partition β of 5 with no summand exceeding 3. Then $[3,2,0,0,0]$ and β are star types of two decompositions of $K_5^{(6)}$ that have a suitable combination yielding a decomposition of $K_5^{(12)}$ with star type α. Finally if $k \geq 2$ then $3 \leq a \leq 3k+3$, so $2 \leq b \leq 2k+2$, permuting $(a-3, b-2, c, d, e)$ as before yields a length 5 non-increasing partition β of $5k$ with no summand exceeding $3k$. By hypothesis, β is the star type of some balanced spanning tree decomposition of $K_5^{(6k)}$, and suitably combining it with a decomposition of $K_5^{(6)}$ with star type $[3,2,0,0,0]$ yields a decomposition of $K_5^{(6k+6)}$ with star type α. The claim now follows by induction on k. □

For any integer $k \geq 1$, let $p(k)$ be the number of non-increasing length 5 partitions of $5k$ with no summand exceeding $3k$. In particular, $p(1) = 5$ and $p(2) = 23$.

Theorem 5.4. *For any integer $k \geq 1$, there are at least $p(k)$ reducible balanced spanning tree decompositions of $K_5^{(6k)}$, each the union of $3k$ fully heterogeneous decompositions of $K_5^{(2)}$.*

Proof. In the proof of Lemma 5.2 we used decompositions of $K_5^{(2)}$ from the similarity classes $(2,1,2)$ and $(1,3,1)$ to produce balanced decompositions of $K_5^{(6)}$ of all five possible star types. The constructions used in the proof of Lemma 5.3 were combinations of the decompositions used in Lemma 5.2, so ultimately each decomposition of $K_5^{(6k)}$ produced is reducible to a combination of $2k$ from the similarity class $(2,1,2)$ and k from the similarity class $(1,3,1)$. The theorem follows. □

It has been relatively simple to use our earlier results to show that $K_5^{(6k)}$ has a plethora of reducible balanced decompositions. Could it be that every balanced decomposition of $K_5^{(6k)}$ is reducible? As it turns out, the answer is no! Consider the following balanced decomposition $\mathcal{T}$ of $K_5^{(6)}$, produced by combining three homogeneous cyclic decompositions of $K_5^{(2)}$. As starter trees, take

$$S_5 = 01, 02, 03, 04;\ Y = 02, 03, 04, 12;\ P_5 = 02, 03, 12, 34$$

and from these form the decomposition

$$\mathcal{T} = \{S_5 + i \pmod 5, Y + i \pmod 5, P_5 + i \pmod 5 : i \in \mathbb{Z}_5\}.$$

This is the union of cyclic decompositions from the similarity classes $(5,0,0)$, $(0,0,5)$ and $(0,5,0)$ respectively, so $\mathcal{T}$ is a reducible balanced decomposition of $K_5^{(6)}$. Now we modify this decomposition to produce a new balanced decomposition $\mathcal{T}^* = (\mathcal{T} \backslash S) \cup S^*$, by replacing the subfamily

$$\begin{aligned} S = & \ \{S_5 + 1 \pmod 5, Y + 3 \pmod 5, P_5 + 0 \pmod 5, P_5 + 2 \pmod 5\} \\ = & \ \{01, 12, 13, 14; 03, 04, 13, 23; 02, 03, 12, 34; 01, 02, 24, 34\} \end{aligned}$$

with the isomorphic new subfamily

$$S^* = \{03, 13, 23, 34; 01, 02, 13, 14; 02, 04, 12, 34; 01, 03, 12, 24\}.$$

Both S and S^* comprise a star, a Y and two paths, and they constitute the same multiset of 16 adjacencies, so $\mathcal{T}^*$ is a balanced decomposition of $K_5^{(6)}$. This modified balanced decomposition contains no homogeneous decomposition of $K_5^{(2)}$, because the trees in S^* "spoil" their homogeneous subfamilies. In fact, a routine computer computation verifies that the 10×15 matrix representing the adjacencies in the 15 trees of $\mathcal{T}^*$ has no 10×5 submatrix in which all row sums are 2, corresponding to a decomposition of $K_5^{(2)}$. Hence

Theorem 5.5. *The balanced spanning tree decomposition $\mathcal{T}^*$ of $K_5^{(6)}$ is irreducible.*

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References

[1] Atif Abueida, Andrew Blinco, Sally Clark, Mike Daven and Roger B. Eggleton, Reducible and Purely Heterogeneous oriented decompositions into rooted spanning trees, *Australas. J. Combin.*, **55** (2013), 145–152.

[2] Atif Abueida, Andrew Blinco, Sally Clark, Mike Daven and Roger B. Eggleton, *Monochromatic Heterogeneous decompositions into rooted spanning trees*, Manuscript.

[3] Roger B. Eggleton, Special heterogeneous decompositions into spanning trees, *Bull. Inst. Combinatorics and its Applications*, **43** (2005), 33–36.

[4] Adrian Riskin, A note on heterogeneous decompositions into spanning trees, *Bull. Inst. Combinatorics and its Applications*, **51** (2007), 69–71.