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Distance in Graphs - Taking the Long View

Gary Chartrand[1] and Ping Zhang[2]
Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA
[1]*e-mail: chartrand@wmich.edu,* [2]*e-mail:* ping.zhang@wmich.edu

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Abstract

The detour distance between two vertices u and v in a connected graph G is the length of a longest $u-v$ path in G. We survey results and some open questions on detour distance, including connections of this distance to domination, coloring and Hamiltonian properties of graphs.

Key Words: distance, detour distance, domination, coloring, Hamiltonian.

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1 Introduction

The standard *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. Although this concept has been known for a very long time, it is only in recent decades that distance has received considerable attention as a subject of its own. In 1990 Buckley and Harary [2] wrote the book *Distance in Graphs*. The 2004 *Handbook of Graph Theory*, edited by Gross and Yellen [18], contains a section devoted exclusively to distance in graphs.

A number of results on distance come from the fact that two vertices u and v are adjacent if and only if $d(u,v)=1$ and two distinct vertices u and v are nonadjacent if and only if $d(u,v)\geq 2$. Hence any concept whose definition relies on the adjacency or non-adjacency of two vertices in a graph can be restated in terms of distance, thereby giving rise to a natural generalization of the concept.

This distance d is a metric on the vertex set of any connected graph G, that is,

(1) $d(u,v)\geq 0$ for all vertices u and v of G;
(2) $d(u,v)=0$ if and only if $u=v$;
(3) $d(u,v)=d(v,u)$ for all vertices u and v of G; and
(4) $d(u,v)+d(v,w)\geq d(u,w)$ for all vertices, u,v and w of G.

This is not the only way, however, that distance has been defined on the vertex set of a connected graph. The length of a longest $u-v$ path between two vertices u and v in a connected graph is called the *detour distance* $D(u,v)$ between u and v. As with standard distance, detour distance is a metric on the vertex set of any connected graph. A $u-v$ path of length $D(u,v)$ is a $u-v$ *detour*. For the vertices u,v and w in the graph of Figure 1, $d(u,v)=2$ and $d(u,w)=1$; while $D(u,v)=4$ and $D(u,w)=5$.

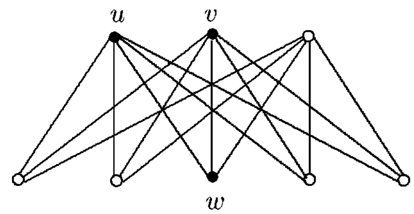

Figure 1: Illustrating distance and detour distance

Several results on detour distance are surveyed and are compared to the corresponding results for standard distance. Connections of detour distance to domination, coloring and Hamiltonian properties of graphs are also surveyed.

2 A Few Facts Involving Detour Distance

For vertices *u* and *v* in a connected graph *G* of order *n*, $0 \le d(u,v) \le D(u,v) \le n-1$, where $D(u,v)=0$ if and only if $d(u,v)=0$ if and only if $u=v$, $D(u,v)=1$ if and only if *uv* is a bridge of *G*, and $D(u,v)=n-1$ if and only if *G* contains a Hamiltonian $u-v$ path. Furthermore, $d(u,v)=D(u,v)$ for every two vertices *u* and *v* of *G* if and only if *G* is a tree. It is possible, however, that $d(u,v)=D(u,v)$ for some pairs *u*,*v* of distinct vertices in a graph that contains no bridges. For example, if *u* and *v* are antipodal vertices in the even cycle C_{2k}, $k \ge 2$, then $D(u,v)=d(u,v)=k$. Indeed, even more can be said.

Proposition 2.1[4] *Let G be a 2-connected graph. If u and v are two vertices of G for which* $D(u,v)=d(u,v)$*, then u and v are antipodal vertices of G.*

It is a simple observation that if *G* is a connected graph of order *n* for which there is a constant *k* such that $d(u,v)=k$ for every two distinct vertices *u* and *v* of *G*, then $k=1$ and $G \cong K_n$. There is a corresponding statement for detour distance.

Proposition 2.2 [4] *Let G be a connected graph of order* $n \ge 2$. *If there exists an integer k such that* $D(u,v)=k$ *for every pair u,v of distinct vertices of G, then* $k=n-1$ *and G is Hamiltonian-connected.*

The eccentricity $e(v)$ of a vertex *v* in a connected graph *G* is the distance from *v* to a vertex farthest from *v* in *G*. As expected, the detour eccentricity $e_D(v)$ of a vertex *v* is the detour distance from *v* to a vertex farthest from *v*. If *u* and *v* are distinct vertices in a connected graph *G*, then it is known that $|e(u)-e(v)| \le d(u,v)$. Again, there is a corresponding statement for detour distance.

Proposition 2.3 [4] *If u and v are distinct vertices in a connected graph, then* $|e_D(u)-e_D(v)| \le D(u,v)$.

The *radius rad* (*G*) of a connected graph *G* is defined as the minimum eccentricity among the vertices of *G* and the *diameter diam*(*G*) is a maximum eccentricity among the vertices of *G*. Similarly, the *detour radius* $rad_D(G)$ of a connected graph *G* is the minimum detour eccentricity among the vertices of *G* and the *detour diameter*

$diam_D(G)$ is the maximum detour eccentricity among the vertices of G. Since $d(x,y) \le D(x,y)$ for every two vertices x and y in a connected graph G, it follows that $e(v) \le e_D(v)$ for every vertex v in a connected graph G. Therefore, $rad(G) \le rad_D(G)$ and $diam(G) \le diam_D(G)$ for every connected graph G. In any connected graph, the detour radius and detour diameter are related by the following inequalities.

Theorem 2.4 [4] *For every connected graph G,*

$$rad_D(G) \le diam_D(G) \le 2rad_D(G).$$

Moreover, for each pair a , b of positive integer with $a \le b \le 2a$, there exists a connected graph G with $rad_D(G) = a$ and $diam_D(G) = b$.

The center $C(G)$ of a connected graph G is the subgraph of G induced by those vertices of G having eccentricity $rad(G)$. As expected, the *detour center* $C_D(G)$ of G is the subgraph of G induced by those vertices of G whose detour eccentricity is $rad_D(G)$. Harary and Norman[21] proved, for standard distance in graphs, that the center of every connected graph G lies in a single block of G. This is true for detour distance as well.

Proposition 2.5 [4] *The detour center $C_D(G)$ of every connected graph G lies in a single block of G.*

Hedetniemi (see [3]) showed that every graph is the center of some connected graph. This too is true for detour centers.

Theorem 2.6[4] *Every graph is the detour center of some connected graph.*

For example, the graph G is the detour center of the graph H in Figure 2. In general, a graph G of order n is the center of $H \cong G + \overline{K}_{n+1}$.

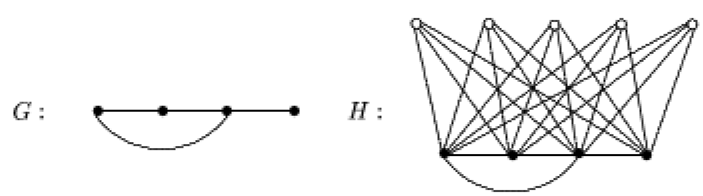

Figure 2: Illustrating the detour center of a graph

A connected graph G is detour self-centered if $rad_D(G) = diam_D(G)$, that is, if G is its own detour center. The detour eccentricity of a vertex in a detour self-centered graph of sufficiently large order cannot be extremely small.

Theorem 2.7 [4] *Let G be a connected graph of order 6 or more. If G is detour self-centered, then $e_D(v) \ge 5$ for every vertex v in G.*

It is thought, however, that there is a considerably stronger result than Theorem 2.7.

Conjecture 2.8 [4] *If G is a detour self-centered graph of order n, then* $e_D(v) = n-1$ *for every vertex v of G.*

The *periphery* $P(G)$ of a connected graph G is the subgraph of G induced by the vertices of G having eccentricity $diam(G)$ and similarly the *detour periphery* $P_D(G)$ of G is the subgraph of G induced by the vertices of G having detour eccentricity $diam_D(G)$. Bielak and Syslo [1] showed that a nontrivial graph G is the periphery of some connected graph if and only if every vertex of G has eccentricity 1 or no vertex of G has eccentricity 1. This suggests a natural question.

Problem 2.9 *Which graphs are the detour periphery of some connected graph*?

A graph G is *vertex-traceable* if every vertex of G is the initial vertex of a Hamiltonian path of G. Although every Hamiltonian graph is vertex-traceable, the converse is not true. For example, the Peterson graph is vertex-traceable.

Theorem 2.10 [4] *If G is a graph in which every component of G is vertex-traceable, then G is the detour periphery of some connected graph*
.

There are several infinite classes of graphs that are not the detour periphery of any graph. Some of these classes are described below.

Theorem 2.11[4] *If T is a tree of order* $n \ge 3$ *with* $\Delta(T) \ge n/2$, *then T is not the detour periphery of any connected graph.*

For a tree T of order $n \ge 3$, let

$$S_T = \{v \in V(T) : \deg v \ge 2\} \text{ and } \sigma_T = \sum_{v \in S_T} (\deg v - 2).$$

Theorem 2.12 [4] *Let T be a tree of order* $n \ge 3$. *If* $\Delta(T) + diam(T) + \sigma_T \ge n+3$, *then T is not the detour periphery of any connected graph.*

Theorems 2.11 and 2.12 give only a small indication of what is believed to be true.

Conjecture 2.13 *No tree of order 3 or more is the detour periphery of any connected graph*
.

A result similar to Theorem 2.11 holds for graphs that are not necessarily trees.

Theorem 2.14 *Let G be a graph of order* $n \ge 3$. *If G contains a vertex u such that* $\deg u \ge (n+1)/2$ *and the neighborhood of u is an independent set in G, then G is not the detour periphery of any connected graph.*

Among all connected graphs of order n and size m with $n-1 \le m \le \binom{n}{2}$, there exists a graph G of order n and size m containing a Hamiltonian path, that is, $diam_D(G) = n-1$. Thus, among all connected graphs of order n and size m with $n-1 \le m \le \binom{n}{2}$, the maximum detour diameter is $n-1$. But what can be said about the minimum detour diameter?

Problem 2.15 [5] *For a given pair n, m of integers with* $n \geq 3$ *and* $n-1 \leq m \leq \binom{n}{2}$, *what is the minimum detour diameter* $D_{\min}$ *among all connected graphs of order n and size m?*

We now consider conditions on *n* and *m* for which $D_{\min} = k$ for small values of *k*.

Theorem 2.16 [5] *Let n and m be integers*
.

(1) If $m = n-1$ and $n \geq 3$, then $D_{\min} = 2$.
(2) If $m = n \geq 4$, then $D_{\min} = 3$.
(3) If $n+1 \leq m \leq 2n-3$ and $n \geq 5$, then $D_{\min} = 4$.
(4) If $m = 2n-2$, where $n \geq 6$, then $D_{\min} = 5$.
(5) If $2n-1 \leq m \leq 3n-6$, where $n \geq 7$, then $D_{\min} = 6$.
(6) If $m = 3n-5$ and $n \geq 8$, then $D_{\min} = 7$.

As a consequence of Theorem 2.16(6), we know that $D_{\min} = 7$ for $n = 8$ and $m = 19$. Based on this evidence and Theorem 2.16, one might very well expect that $D_{\min} = 8$ for $n = 9$ and $m = 3n-4 = 23$. However, the graph *G* obtained by adding two new vertices x and y to K_7 and joining *x* and *y* to a common vertex of K_7 has detour diameter 7. Therefore, $D_{\min} = 7$ for $n = 9$ and $m = 23$. As indicated in [5], whether $D_{\min} = 8$ for $m = 3n-4$ and $\text{n} \geq 10$ remains open.

It is well-known that every graph of order *n* and size $m \geq \binom{n-1}{2}+1$ contains a Hamiltonian path, which implies that the detour diameter of every such graph is $n-1$.

Observation 2.17 [5] *Let n and m be positive integers. If* $m \geq \binom{n-1}{2}+1$, *then* $D_{\min} = n-1$.

The following results give sufficient conditions for $D_{\min}$ to be large.

Theorem 2.18 [5] *Let n and k be integers with* $2 \leq k \leq n-2$. *If G is a connected graph of order n and size* $m > k(n-1)/2$, *then* $diam_D(G) \geq k+1$.

Theorem 2.19 [5] *Let c be a nonnegative integer. If G is a connected graph of order* $n \geq 2c+5$ *and size* $\binom{n-1}{2} - c$, *then* $diam_D(G) = n-1$.

Theorem 2.20 [5] *Let c be a number with* $0 \leq c < \frac{1}{2}$. *If G is a connected graph of order* $n \geq \frac{5}{1-2c}$ *and size* $\binom{n-1}{2} - cn$, *then* $diam_D(G) = n-1$.

Problem 2.21 *Find the largest value of m (as a function of n) such that* $D_{\min} = n-2$.

3 Detour Distance and Domination

In the standard definition of domination in a graph, a vertex v dominates itself and each neighbor of v. By a *neighbor* of v, we mean a vertex that is adjacent to v. A neighbor of v can also be interpreted as a vertex distinct from v and whose distance from v is minimum. Necessarily then, if u is a vertex distinct from v whose distance from v is minimum, then $d(u,v)=1$.

For each vertex v in a nontrivial connected graph G, define

$$d^-(v)=\min\{d(u,v):u\in V(G)-\{v\}\}.$$

A vertex $u(\neq v)$ is called a neighbor of v if $d(u,v)=d^-(v)$. A vertex v is said to *dominate* a vertex u if $u=v$ or u is a neighbor of v. Since $d^-(v)=1$ for all $v\in V(G)$, this is equivalent to the standard definition of neighbor and the standard definition of domination.

Again, let G be a nontrivial connected graph and this time let the distance under consideration be the detour distance D. For a vertex v in G, define

$$D^-(v)=\min\{D(u,v):u\in V(G)-\{v\}\}.$$

A vertex $u(\neq v)$ is called a *detour neighbor* of v if $D(u,v)=D^-(v)$. If u is a detour neighbor of v, then v is not necessarily a detour neighbor of u. For example, in each of the two graphs in Figure 3, the vertex u is a detour neighbor of the vertex v, but v is not a detour neighbor of u.

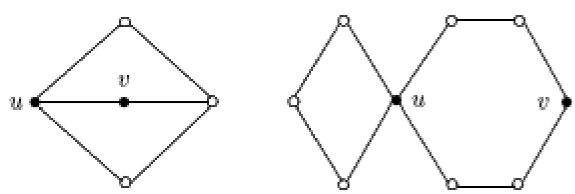

Figure 3: Illustrating detour neighbors

A vertex v is said to *detour dominate* a vertex u if $u=v$ or u is detour neighbor of v. A set S of vertices of G is called a detour dominating set if every vertex of G is detour dominated by some vertex in S. A detour dominating set of G of minimum cardinality is a *minimum detour dominating set* and this cardinality is the *detour domination number* $\gamma_D(G)$. These concepts were introduced and studied in [6]. The following observations are immediate.

Observation 3.1 [6] *If G is a tree, then* $\gamma_D(G)=\gamma(G)$.

Observation 3.2 [6] *If G is Hamiltonian-connected, then* $\gamma_D(G)=1$.

The converse of Observation 3.2 is not true. For example, if $G \cong 2K_2 + K_1$, then $\gamma_D(G) = 1$, but G is not Hamiltonian-connected. The detour domination numbers of several well-known classes of graphs, including the *n*-cube Q_n, cylces, and complete bipartite graphs, were determined in [6]. An upper bound for the detour domination number of a connected graph in terms of its order was established.

Theorem 3.3 [6] *If G is a connected graph of order* $n \geq 3$, *then* $\gamma_D(G) \leq n-2$.

Proposition 3.4 [6] *For every pair k, n of integers with* $1 \leq k \leq n-2$, *there exists a connected graph G of order n such that* $\gamma_D(G) = k$.

For the graphs G and H in Figure 4, $\gamma(G) = 2$ and $\gamma(G) = 3$, where $\{u, v, w\}$ is a minimum detour dominating set in *G*. On the other hand, $\gamma_D(H) = 2$ and $\gamma(H) = 3$, where $\{u, v\}$ is a minimum detour dominating set in *H*. In fact, there is no relationship between the detour domination number and the domination number of a connected graph. Therefore, knowing the value of one of these parameters for a connected graph supplies no information about the value of the other parameter.

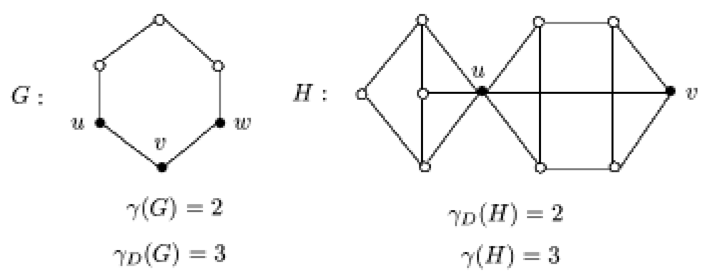

Figure 4: Illustrating the domination number and detour domination number

Theorem 3.5 [6] *For each pair a, b of positive integers, there is a connected graph G such that* $\gamma(G) = a$ and $\gamma_D(G) = b$.

4 Detour Distance and Coloring

In a standard coloring of a graph *G*, each vertex of *G* is assigned a color (a positive integer) such that adjacent vertices are assigned distinct colors. There is no condition on colors that are assigned to nonadjacent vertices. The *chromatic number* $\chi(G)$ of *G* is defined as the minimum number of colors used in a coloring of *G*. The chromatic number of a graph can be defined in another, yet equivalent, manner. Let ***N*** denote the set of positive integers. A *coloring* of a graph *G* is a function $c : V(G) \to \mathbf{N}$ having the property that $c(u) \neq c(v)$ for every pair *u*, *v* of adjacent vertices of *G*. The *value* of a coloring *c* of *G* is

$$\chi(c) = \max\{c(v) : v \in V(G)\}.$$

Then

$$\chi(G) = \min\{\chi(c) : c \text{ is a coloring of } G\}.$$

The reason that these two definitions are equivalent is that if $\chi(G) = k$, then every coloring of *G* whose value is *k* uses precisely the colors $1, 2, \ldots, k$.

Equivalently, a standard *coloring* of a graph G can be defined as a mapping $c: V(G) \to \mathbf{N}$ such that if $d(u,v)=1$ then $|c(u)-c(v)| \geq 1$ and if $d(u,v) \geq 2$ then $|c(u)-c(v)| \geq 0$. These two conditions can be described in terms of a single inequality, namely,

$$d(u,v) + |c(u)-c(v)| \geq 2 \qquad (1)$$

for every two distinct vertices u and v of G. This concept was extended in [7, 8, 9]. For a nontrivial connected graph G of order $n \geq 2$ and diameter d and each integer k with $1 \leq k \leq d$, a *radio k-coloring* of G is a mapping $c: V(G) \to \mathbf{N}$ such that

$$d(u,v) + |c(u)-c(v)| \geq 1+k \qquad (2)$$

for every two distinct vertices u and v of G. Therefore, radio 1-colorings are standard colorings and so radio k-colorings produce a generalization of ordinary colorings of G.

As described in [17], these concepts were inspired by the (FM radio) Channel Assignment Problem. Radio stations located within a certain proximity of one another must be assigned distinct channels, where the nearer two stations are to each other, the greater the difference is in the channels assigned to them. The task of efficiently allocating channels to transmitters is called the *Channel Assignment Problem*. The use of graph theory to study the Channel Assignment Problem and related problems dates back at least to 1970 (see Metzger [22]). In 1980 Hale [19] modeled the Channel Assignment Problem as both a frequency-distance constrained and frequency constrained optimization problem and discussed applications to important real world problems. Since then, a number of different coloring associated with the Channel Assignment Problem have been developed (see [8]).

In a radio $(d-1)$-coloring of a connected graph of diameter d, the colors assigned to adjacent vertices must differ by at least $d-1$, the colors assigned to two vertices whose distance is 2 must differ by at least $d-2$, etc., up to antipodal vertices, whose colors are permitted to be the same. For this reason, radio $(d-1)$-colorings are also called *antipodal colorings*. In an antipodal coloring c of the path P_n of order $n \geq 2$, only its two end-vertices are permitted to be colored the same. If u and v are any two distinct vertices of P_n and $d(u,v)=i$ then $|c(u)-c(v)| \geq n-1-i$. Since P_n is a tree, not only is i the length of a shortest $u-v$ path, it is the length of the only $u-v$ path in P_n and, in particular, the length of a longest $u-v$ path in P_n, the detour distance $D(u,v)$ between u and v. Thus, if c is an antipodal coloring of P_n, then (2) is equivalent to $D(u,v) + |c(u)-c(v)| \geq n-1$. We now consider colorings c that satisfy $D(u,v) + |c(u)-c(v)| \geq n-1$ for arbitrary connected graphs of order $n \geq 3$.

Let G be a connected graph of order n. A *Hamiltonian coloring* of G is an assignment c of colors (positive integers) to the vertices of G such that

$$D(u,v) + |c(u)-c(v)| \geq n-1$$

for every two distinct vertices *u* and *v* of *G*. The *value* $hc(c)$ of a Hamiltonian coloring *c* of *G* is the largest color assigned to a vertex of *G*, that is ,

$$hc(c) = \max\{c(v) : v \in V(G)\}.$$

The *Hamiltonian chromatic number* $hc(G)$ of *G* is the minimum value of a Hamiltonian coloring of *G*, that is,

$$hc(G) = \min\{hc(c) : c \text{ is a Hamiltonian coloring of } G\}.$$

A Hamiltonian coloring *c* of *G* is a *minimum Hamiltonian coloring* if $hc(c) = hc(G)$. Figure 5 shows minimum Hamiltonian colorings of two graphs. These concepts were introduced and studied in [14] and studied further in [13, 15, 16, 23, 24]. An extensive survey on this topic was given in [17].

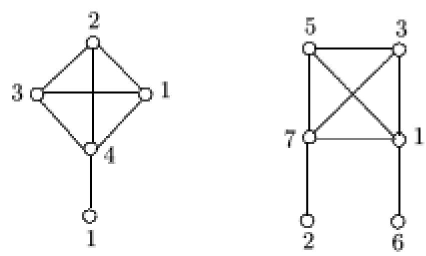

Figure 5: Illustrating Hamiltonian coloring of graphs

Observe that a connected graph *G* has Hamiltonian chromatic number 1 if and only if *G* is Hamiltonian-connected. Thus the Hamiltonian chromatic number of a connected graph *G* can be thought of as a measure of how close *G* is to being Hamiltonian-connected - the closer the Hamiltonian chromatic number is to 1, the closer *G* is to being Hamiltonian-connected. By this measure, the graphs shown in Figure 6 are "close" to being Hamiltonian-connected.

If *G* is a graph of order $n \geq 3$ that is not Hamiltonian, then *G* is not Hamiltonian-connected since for every pair *u*, *v* of adjacent vertices, *G* does not contain a Hamiltonian

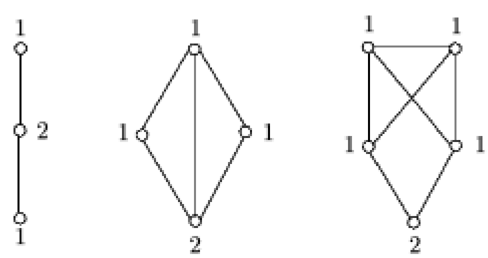

Figure 6: Graphs that are "close" to being Hamiltonian-connected

$u-v$ path. Of course, if *u* and *v* are not adjacent, then *G* might contain a Hamiltonian $u-v$ path. If *G* is a graph of order $n \geq 3$ that is not Hamiltonian, then $D(u,v) \leq n-2$ if *u* and *v* are adjacent and $D(u,v) \leq n-1$ if *u* and *v* are not adjacent. For this reason, in [15] a connected graph *G* of order $n \geq 3$ is called *semi-Hamiltonian-connected* if

$$D(u,v) = \begin{cases} n-2 & \text{if } uv \in E(G) \\ n-1 & \text{if } uv \notin E(G). \end{cases}$$

Since a Hamiltonian coloring c of a graph G of order $n \geq 3$ satisfies $D(u,v) + |c(u) - c(v)| \geq n-1$ for every two distinct vertices u and v of G, every Hamiltonian coloring of a semi-Hamiltonian-connected graph G of order $n \geq 3$ is an ordinary coloring of G.

Proposition 4.1 [15] *If G is a semi-Hamiltonian-connected graph of order* $n \geq 3$ *, then* $hc(G) = \chi(G)$
.

The graph P_3 and the Petersen graph are semi-Hamiltonian-connected and so their Hamiltonian chromatic number equals their chromatic number, which is 2 and 3 respectively (see Figure 7).

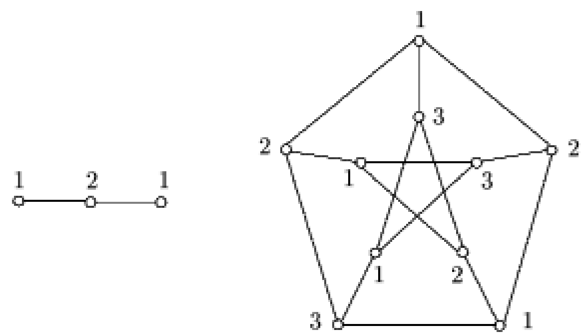

Figure 7: Two semi-Hamiltonian-connected graphs

There is a sharp upper bound for the Hamiltonian chromatic number of a Hamiltonian graph in terms of its order.

Theorem 4.2 [14] *If G is a Hamiltonian graph of order* $n \geq 3$ *, then* $hc(G) \leq n-2$.

Moreover, for every integer k with $1 \leq k \leq n-2$, there is a Hamiltonian graph G_k of order n such that $hc(G_k) = k$.

The *circumference* $cir(G)$ of a connected graph G is the length of a longest cycle in G. If G is a tree, then $cir(G) = 0$. In general, the Hamiltonian chromatic number and the circumference cannot both be small.

Theorem 4.3 [15] *Let G be a connected graph of order* $n \geq 4$ *. If* $2 \leq hc(G) \leq n-1$ *, then* $cir(G) + hc(G) \geq n+2$.

Corollary 4.4 *Let G be a connected graph of order* $n \geq 4$.

(a) If $hc(G) = 2$, then G is Hamiltonian
(b) If $hc(G) = 3$, then $cir(G) \geq n-1$.

Theorem 4.5 [15] *If G is a connected graph of order* $n \geq 5$, *then* $hc(G) \leq (n-2)^2 + 1$.

For $n \geq 6$, let $S_n = K_{1,n-1}$ be the star of order *n*, let S'_n be the tree of order *n* obtained by subdividing an edge of S_{n-1}, and let S''_n denote the unique double star of order *n* containing a vertex of degree 3.

Theorem 4.6 [16] *Let G be a connected graph of order* $n \geq 6$. *Then*

(1) $hc(G) = (n-2)^2 + 1$ if and only if $G = S_n$,
(2) $hc(G) = (n-2)^2 - 1$ if $G \neq S_n$,
(3) $hc(G) = (n-2)^2 - 1$ if and only if $G = S'_n$,
(4) $hc(G) \leq (n-2)^2 - 3$ if $G \neq S_n$ and $G \neq S'_n$,
(5) $hc(G) = (n-2)^2 - 3$ if $G = S''_n$.

5 Detour Distance and Geodesics

The path P_n is the unique graph of order *n* that contains a spanning geodesic. In particular, if *G* is any connected graphs that is not a path, then for every two vertices *u* and *v*, no $u - v$ geodesic of *G* contains every vertex of *G*. On the other hand, it may occur that *G* contains two vertices *u* and *v* such that every vertex of *G* lies on some $u - v$ geodesic of *G*. For example, every vertex in the 3-cube Q_3 of Figure 8 lies on some $u - v$ geodesic of Q_3. Even if *G* does not contain two such vertices, *G* surely contains a set *S* of three or more vertices such that for every vertex *x* of *G*, there exists two vertices *u* and *v* of *S* such that *x* lies on a $u - v$ geodesic in *G*. A set *S* with this property is a *geodetic set*. The minimum cardinality of a geodetic set is the *geodetic number* of *G*, denoted by $g(G)$. A geodetic set of *G* with cardinality $g(G)$ is a minimum geodetic set in *G*. The geodetic number has been studied extensively (see [2, 10, 20], for example).

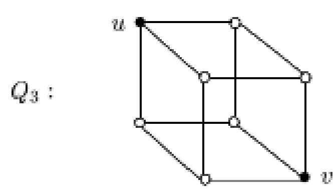

Figure 8: Illustrating $u - v$ geodesics in a graph

There are many graphs *G* having the property that there is a path in *G* containing every vertex of *G*. Of course such a path is a Hamiltonian path. On the other hand a graph having no Hamiltonian path might contain two vertices *u* and *v* such that every vertex of *G* lies on a $u - v$ detour in *G*. For example, every vertex in the graph *G* of Figure 9 lies on some $u - v$ detour of *G*.

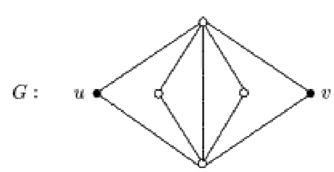

Figure 9: Illustrating $u - v$ detours in a graph

Even if a connected graph G does not contain two vertices u and v such that every vertex of G lies on a $u-v$ detour, G contains a set S of three or more vertices such that for every vertex x of G, there exist two vertices u and v of S such that x lies on a $u-v$ detour in G. A set S with this property is a *detour set* and the minimum cardinality of a detour set is the *detour number* of G, denoted by $dn(G)$. A detour set of G with cardinality $dn(G)$ is a minimum detour set in G. For example, the graph G of Figure 10 has geodesic number 3 and detour number 2, where $\{u, v, w\}$ is a minimum geodetic set; while $\{v, w\}$ is a minimum detour set of G.

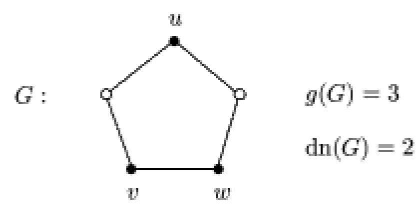

Figure 10: Illustrating the geodesic and detour numbers of a graph

These concepts were introduced and studied in [11, 12]. Relationships between minimum detour sets and minimum geodetic sets in a graph were studied in [12]. Let G be a connected graph and let S and T be two subsets of $V(G)$. The *distance between S and T* is defined as $d(S,T) = \min\{d(s,t) : s \in S, t \in T\}$.

Proposition 5.1 [12] *For every three positive integers k, a, b with $2 \le a \le b$, there exists a connected graph G with $dn(G) = a$ and $g(G) = b$ such that for every minimum geodetic set S of G and every minimum detour set T of G, $d(S,T) = k$.*

We have seen that $g(G) = 3$ and $dn(G) = 2$ for the graph G of Figure 10. Figure 11 shows a graph H with $g(H) = 2$ and $dn(H) = 3$, where $\{u, v\}$ is a minimum geodetic set; while $\{u, v, w\}$ is a minimum detour set of G. Indeed, the geodetic number and detour number are independent parameters, that is , knowing the value of one of these parameters provides no information about the value of the other.

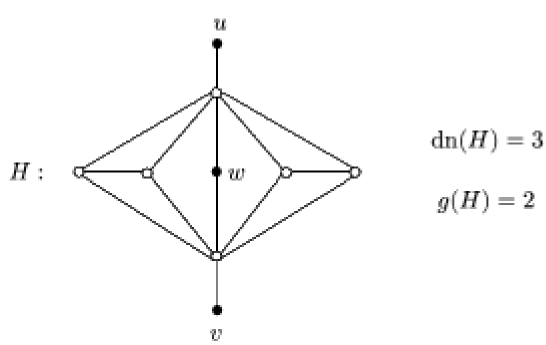

Figure 11: A graph H with $dn(H) > g(H)$

Theorem 5.2 [12] *For every pair $a, b \ge 2$ of integers, there exists a connected graph G with $dn(G) = a$ and $g(G) = b$.*

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