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k-Factor and *l*-Closure in Graphs

Keiko KOTANI
Department of Mathematics, Tokyo University of Science, Shinjuku-ku, Tokyo, 162-8601 Japan
e-mail: kkotani@rs.kagu.tus.ac.jp

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Abstract

The l-closure $cl_l(G)$ of a graph G, as defined by Bondy and Chvátal, is the graph obtained from G by recursively joining nonadjacent vertices with degree-sum at least l.

Let $k \geq 1$ be an integer and G be a graph of order $n > \max\{(3k^2+2k+3)/8, 2k-1+\sqrt{3k^2-6k+3}\}$ with minimum degree at least k and kn is even. We prove that if $cl_{n+k-2}(G)$ is complete, then G has a k-factor.

Keywords: graph, regular factor, closure

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1 Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. We denote by $\delta(G)$ the minimum degree of G. For subsets A and B of $V(G)$, we let $E_G(A,B)$ denote the set of edges joining A and B, and let $e_G(A,B)$ denote the cardinality of $E_G(A,B)$. A vertex x is often identified with $\{x\}$, for example, we write $E_G(x,B)$ for $E_G(\{x\},B)$. For $x \in V(G)$, we denote by $\deg_G(x)$ the degree of x in G. For a subset M of $V(G)$, we denote by $G[M]$ the induced subgraph of G on M. Let $k \geq 1$ be an integer. A k-factor of graph G is a spanning k-regular subgraph of G.

Let G be a graph of order n and l be a nonnegative integer. Among all the graphs H of order n such that $E(G) \subset E(H)$ and $\deg_H(u)+\deg_H(v) < l$ for all $uv \notin E(H)$, there is a unique smallest one. Bondy and Chvátal [1] have called this graph the l-closure of G. Obviously the l-closure of G can be obtained from G by recursively joining nonadjacent vertices with degree-sum at least l. We denote by $cl_l(G)$ the l-closure of G. Concerning the relation between the existence of a k-factor and the l-closure of G, we will show the following theorem.

Theorem 1 *Let $k \ge 1$ and $n \ge 1$ be integers such that kn is even. Let G be a graph of order $n > \max\{(3k^2+2k+3)/8,\ 2k-1+\sqrt{3k^2-6k+3}\}$ such that $\delta(G) \ge k$. Suppose that $cl_{n+k-2}(G) = K_n$. Then G has a k-factor.*

Theorem 1 is best possible in the sense that we cannot replace $n+k-2$ by $n+k-3$, which is shown in the following example:

Let $k \ge 1$ and $n \ge k^2+3$ be integers with n even. First we consider the case where $k \ge 2$. Let $M, L_1, ..., L_{k-1}$ be graphs defined as follows:

$M = K_{\lfloor (n-k)/2 \rfloor}$; for each integer i with $1 \le i \le k-1$, $L_i = K_{1,i}$, and let L_0 be a null graph having $\left(n - |V(M)| - \left|\cup_{i=1}^{k-1} V(L_i)\right|\right)$ vertices. Fix $a_0 \in V(L_0)$ and $a_1 \in V(L_1)$ and, for each i with $2 \le i \le k-2$, let a_i denote the central vertex of L_i.

We define G as follows:

$V(G) = V(M) \cup \left(\cup_{i=0}^{k-1} V(L_i)\right)$, $E(G) = E(M) \cup \left(\cup_{i=1}^{k-1} E(L_i)\right) \cup \{uv \mid u \in V(M), v \in \cup_{i=0}^{k-1} V(L_i)\} \cup F$, where $F = \phi$ if k is even, and $F = \{a_{2j}a_{2j+1} \mid 0 \le j \le (k-3)/2\}$ if k is odd. Then $cl_{n+k-3}(G) = K_n$, but G has no k-factor. Now we consider the case where $k = 1$ and n is an even integer. Let M be a complete graph having $(n-2)/2$ vertices, and L be a null graph having $(n+2)/2$ vertices. We define G as follows: $V(G) = V(M) \cup V(L)$, $E(G) = E(M) \cup \{uv \mid u \in V(M), v \in V(L)\}$.
Then also $cl_{n-2}(G) = K_n$, but G has no 1-factor.

2 Proof of Theorem 1

Let G be a graph. In order to prove Theorem 1, for disjoint subsets S and T of $V(G)$, we define $\delta_G(S,T)$ by

$$\delta_G(S,T) := k|S| + \sum_{y \in T} (\deg_{G-S}(y) - k) - h(S,T),$$

where $h(S,T)$ denotes the number of components C of $G-S-T$ such that $e_G(T, V(C)) + k|C| \equiv 1 \pmod 2$ (such components are referred to as odd components and the other components are referred to as even components). The following criterion for the existence of a k-factor is essential for our proof:

Theorem A (Tutte[2]) *Let $k \ge 1$ be an integer and G be a graph. Then*

(1) *G has a k-factor if and only if $\delta_G(S,T) \ge 0$ for all disjoint subsets S and T of $V(G)$, and*
(2) $\delta_G(S,T) \equiv 0 \pmod 2$.

Now let k, n and G be as in Theorem 1. Set $n_0 = \max\{(3k^2+2k+3)/8,\ 2k-1+\sqrt{3k^2-6k+3}\}$, thus $n > n_0$. By way of contradiction, suppose that G has no k-factor, and suppose further that, for any nonadjacent vertices u and v of G, $G+uv$ has a k-factor. By Theorem A, there exist disjoint subsets S and T of $V(G)$ satisfying

$\delta_G(S,T) \le -2$. We choose such subsets S and T so that $|S \cup T|$ is maximal. We fix these disjoint subsets S and T of $V(G)$ throughout the rest of the proof of Theorem 1. And we set $U = V(G) - S - T, |S| = s, |T| = t,$ and $w = h(S,T)$.

Claim 1 $\sum_{y \in T} \deg_{G-S}(y) \le k(t-s) + w - 2$.

Proof Since $\delta_G(S,T) \le -2$, we can get this claim immediately. □

Claim 2 *Suppose that* $U \ne \phi$. *Then for each* $z \in U, e_G(z,T) \le k-1$.

Proof Let $z \in U$, and set $S' = S \cup \{z\}$. Then we obtain $\delta_G(S',T) \le \delta_G(S,T) - e_G(z,T) + k + 1$. By the maximality of $|S \cup T|, \delta_G(S',T) \ge 0$, this together with $\delta_G(S,T) \le -2$ implies $e_G(z,T) \le k-1$. □

Claim 3 $w \le |U|/3$.

Proof If $w = 0$, obviously $w \le |U|/3$, then we suppose $w \ge 1$. Let C be an odd component. Let $z \in V(C)$, and set $T' = T \cup \{z\}$. Then we obtain $\delta_G(S,T') \le \delta_G(S,T) + \deg_{G-S}(z) - k + 1$. By the maximality of $|S \cup T|$, $\delta_G(S,T') \ge 0$, this together with $\delta_G(S,T) \le -2$ implies $\deg_{G-S}(z) \ge k+1$. Hence we obtain $|V(C)| \ge \deg_{G-S}(z) + 1 - e_G(z,T) \ge 3$ with Claim 2. Therefore for each odd component C, $|C| \ge 3$, and hence we get Claim 3.

Claim 4 (i) *The number of even components of* $G-S-T$ *is at most* 1.

(ii) *If the number of even components of* $G-S-T$ *is equal to* 1, *then* $w = 0$.

(iii) *Each component of* $G-S-T$ *is complete.*

(iv) *Let* C *be an even component. For each* $u \in T$ *and for each* $v \in V(C), uv \in E(G)$.

(v) *For each* $u \in S$ *and for each* $v \in T \cup V, uv \in E(G)$, *and* $G[S]$ *is complete.*

Proof For any nonadjacent vertices *u* and *v* of $G, G+uv$ has a *k*-factor, and hence

$$\delta_{G+uv}(S,T) \ge 0 \qquad (1)$$

by Theorem A.

(i) We assume there exist two even components of $G-S-T$, say C_1 and C_2. Let $u \in V(C_1)$ and $v \in V(C_2)$, then $uv \notin E(G)$ and $\delta_{G+uv}(S,T) = \delta_G(S,T) \le -2$, which contradicts (1).

(ii) Let C_1 be an even component of $G-S-T$. We assume there exists an odd component of $G-S-T$, say C_2. Let $u \in V(C_1)$ and $v \in V(C_2)$, then $uv \notin E(G)$, and set $G' = G + uv$. Since $G'[V(C_1) \cup V(C_2)]$ is an odd component of $G'-S-T$, $\delta_{G'}(S,T) = \delta_G(S,T) \le -2$, which contradicts (1).

(iii) Let C be a component of $G-S-T$. If there exist nonadjacent vertices $u,v\in V(C)$, then $\delta_{G+uv}(S,T)=\delta_G(S,T)\le -2$, which contradicts (1).

(iv) Let C be an even component of $G-S-T$. Suppose that there exist $u\in T$ and $v\in V(C)$ such that $uv\notin E(G)$, and set $G'=G+uv$. Since $G'[V(C)]$ is an odd component of $G'-S-T$, $\delta_{G'}(S,T)=ks+\sum_{y\in T}(\deg_{G-S}(y)-k)+\ 1-(w+1)=\delta_G(S,T)\le -2$ which contradicts (1).

(v) If there exist $v\in S$ and $u\in S\cup T\cup U$ with $v\ne u$ such that $vu\notin E(G)$, then $\delta_{G+uv}(S,T)=\delta_G(S,T)\le -2$, which contradicts (1). □

Now we define the sequence of graphs $G=G_0,G_1,\dots,G_{p_0},G_{p_o+1}=K_n$ and subsets X_i,Y_i and Z_i of $V(G)$ with $X_i,Y_i\subset T, Z_i\subset U$ and $X_i\cap Y_i=\phi\ (i=0,1,\dots,p_0)$ as follows.

I. $X_0=Y_0=Z_0=\phi$.

II. If $U=\phi$, then $G_1=G_0$, and if $U\ne\phi$, then G_1 is the graph obtained from G_0 by recursively joining nonadjacent vertices u and v of U with degree-sum at least $n+k-2$.

III. For each i with $1\le i\le p_0$, let u_i and v_i be nonadjacent vertices of G_i such that $\{u_i,v_i\}\subset(T-Y_{i-1})\cup(U-Z_{i-1})$, $\{u_i,v_i\}\cap\ (T-(X_{i-1}\cup Y_{i-1}))\ne\phi$ and $\deg_{G_i}(u_i)+\deg_{G_i}(v_i)\ge n+k-2$. If $\{u_i,v_i\}\subset T-(X_{i-1}\cup Y_{i-1})$, then $X_i=\{u_i,v_i\}\cup X_{i-1}, Y_i=Y_{i-1}$ and $Z_i=Z_{i-1}$; if $\left|\{u_i,v_i\}\cap(U-Z_{i-1}\right|=1$, say $v_i\in U-Z_{i-1}$ and $u_i\in T-(X_{i-1}\cup Y_{i-1})$, then $X_i=\{u_i\}\cup X_{i-1}, Y_i=Y_{i-1}$ and $Z_i=\{v_i\}\cup Z_{i-1}$; if $\left|\{u_i,v_i\}\cap X_{i-1}\right|=1$, say $v_i\in X_{i-1}$ and $u_i\in T-(X_{i-1}\cup Y_{i-1})$, then $X_i=\{u_i\}\cup X_{i-1}-\{v_i\}, Y_i=\{v_i\}\cup Y_{i-1}$ and $Z_i=Z_{i-1}$. And G_{i+1} is the graph obtained from $G_i+u_iv_i$ by recursively joining nonadjacent vertices u and v with degree-sum at least $n+k-2$ such that one of the following holds:

(i) $u,v\in U$;
(ii) $u\in X_i, v\in U\cup X_i\cup Y_i$; or
(iii) $u\in Y_i\cup Z_i, v\in T\cup U$.

Claim 5 $T\ne\phi$.

Proof We assume $T=\phi$. Since $\delta_G(S,T)=\delta_G(S,\phi)=ks-w$, $w\ge ks+2$ and hence $w\ge 2$. Since $G_{p_0}=K_n$, there exists a graph H with $V(H)=V(G), E(H)\supset E(G)$, and $H[U]=G[U]$ for which there exist components C_1,C_2 of $G[U]$ and vertices $z_1\in V(C_1)$ and $z_2\in V(C_2)$ such that $\deg_H(z_1)+\deg_H(z_2)\ge n+k-2$. On the other hand, $\deg_H(z_1)+\deg_H(z_2)\le |C_1|+|C_2|-2+2s\le n+s-w\le n+s(1-k)-2\le n-2$. Hence we get a contradiction. From this, we get $T\ne\phi$. □

Claim 6 $G_1\ne K_n$. *Thus* $p_0\ge 1$.

Proof We assume $G_1 = K_n$. Then $G_1[T] = G[T]$ is complete, and for each vertex $y \in T$ and for each vertex $z \in U, yz \in E(G)$. Hence $\sum_{y \in T} \deg_{G-S}(y) = t(n-s-1)$, and $t \le k-1$ by Claim 2. On the other hand, by Claims 1 and 3,
$\sum_{y \in T} \deg_{G-S}(y) \le k(t-s) + w - 2 \le k(t-s) + |U|/3 - 2 = k(t-s) + (n-s-t)/3 - 2$.
Therefore $k(t-s) + (n-s-t)/3 - 2 > t(n-s-1)$, and hence $kt + (n-t)/3 - 2 \ge t(n-1)$ for $t \le k-1$. By Claim 5, $t \ge 1$, and $n > k + 2/3$, hence $k - (2n)/3 - 4/3 \ge 0$, which contradicts $n > n_0$. □

By Claim 6, throughout the rest of the proof of Theorem 1, we let $p_0 \ge 1$.

Claim 7 *If* $t - s \ge k - w + 1$, *then* $G_1 = G_0$.

Proof If $|U| \le 1$, $G_1 = G_0$ by the definition of G_1. Then we suppose $|U| \ge 2$. Let $u, v \in U$. And suppose that u and v are nonadjacent vertices of G_0. By Claim 4(i) and (ii), there exist odd components C_1 and C_2 such that $u \in C_1$ and $v \in C_2$. By Claim 2, $e_G(\{u,v\}, T) \le 2(k-1)$. Hence

$$\begin{aligned}
\deg_G(u) + \deg_G(v) &\le \deg_{G-S}(u) + \deg_{G-S}(v) + 2s \\
&\le e_G(\{u,v\}, T) + |C_1| + |C_2| - 2 + 2s \\
&\le 2(k-1) + |U| - (w-2) - 2 + 2s \\
&\le |U| + k + (k - w + 1) - 3 + 2s \\
&\le |U| + k + t - s - 3 + 2s \\
&= n + k - 3.
\end{aligned}$$

Consequently we get $G_1 = G_0$. □

Claim 8 *For each integer i with* $1 \le i \le p_0$, *one of the following three situations occurs:*

(i) $u_i, v_i \in T - (X_{i-1} \cup Y_{i-1})$;
(ii) $u_i \in T - (X_{i-1} \cup Y_{i-1})$ *and* $v_i \in U - Z_{i-1}$; *or*
(iii) $u_i \in T - (X_{i-1} \cup Y_{i-1})$ *and* $v_i \in X_{i-1}$.

Note that $\{u_i, v_i\} \subset (X_i - X_{i-1}) \cup (Y_i - Y_{i-1}) \cup (Z_i - Z_{i-1})$.

Proof By the definition of G_i, we can get this claim immediately. □

Claim 9 *For each integer i with* $1 \le i \le p_0$, *the following hold.*

(i) *Suppose that Claim 8 (i) holds. Then*

$$\sum_{u \in X_i - X_{i-1}} \deg_{G-S}(u) \ge |U| + t - s + k - 2 - 2|Y_i \cup Z_i| + e_G(X_i - X_{i-1}, Y_{i-1} \cup Z_{i-1}).$$

(ii) *Suppose that Claim 8 (ii) or (iii) holds. Then*

$$\sum_{u\in X_i-X_{i-1}}\deg_{G-S}(u)\ge t-s+k+1-|Y_i\cup Z_i|-|X_i\cup Y_i|+e_G(X_i-X_{i-1},Y_{i-1}\cup Z_{i-1})$$
$$-e_G\big((Y_i\cup Z_i)-(Y_{i-1}\cup Z_{i-1}),\ T-(X_{i-1}\cup Y_{i-1})\big).$$

Suppose further that $t-s\ge k-w+1$. *Then*

$$\sum_{u\in X_1}\deg_{G-S}(u)\ge t-s+k-2+w-e_G(Z_1,T).$$

Proof First we suppose Claim 8 (i) holds. Since $|Y_{i-1}\cup Z_{i-1}|=|Y_i\cup Z_i|$,

$$\sum_{u\in X_i-X_{i-1}}\deg_{G-S}(u)=\deg_{G-S}(u_i)+\deg_{G-S}(v_i)$$
$$\ge\deg_{G_i}(u_i)+\deg_{G_i}(v_i)-2s-2|Y_{i-1}\cup Z_{i-1}|+e_G\big(\{u_i,v_i\},Y_{i-1}\cup Z_{i-1}\big)$$
$$\ge n+k-2-2s-2|Y_i\cup Z_i|+e_G\big(\{u_i,v_i\},Y_{i-1}\cup Z_{i-1}\big)$$
$$=|U|+t-s+k-2-2|Y_i\cup Z_i|+e_G(X_i-X_{i-1},Y_{i-1}\cup Z_{i-1}),$$

and we get (i).

Throughout the rest of the proof, we suppose Claim 8 (ii) or (iii) holds. Note that $|Y_{i-1}\cup Z_{i-1}|=|Y_i\cup Z_i|-1$ and $|X_{i-1}\cup Y_{i-1}|=|X_i\cup Y_i|-1$. In the case where Claim 8 (ii) holds,

$$\deg_{G_i-S}(v_i)\le\deg_{G-S}(v_i)+|X_{i-1}\cup Y_{i-1}|-e_G(v_i,X_{i-1}\cup Y_{i-1})+|U|-1-\deg_{G[U]}(v_i)$$
$$\le|X_i\cup Y_i|+e_G(v_i,T-(X_{i-1}\cup Y_{i-1}))+|U|-2.$$

In the case where Claim 8 (iii) holds, $v_i\in X_{i-1}\cup Y_{i-1}$ and

$$\deg_{G_i-S}(v_i)\le\deg_{G-S}(v_i)+|X_{i-1}\cup Y_{i-1}|-1-e_G(v_i,X_{i-1}\cup Y_{i-1})+|U|-e_G(v_i,U)$$
$$\le|X_i\cup Y_i|+e_G(v_i,T-(X_{i-1}\cup Y_{i-1}))+|U|-2.$$

Hence

$$\sum_{u\in X_i-X_{i-1}}\deg_{G-S}(u)=\deg_{G-S}(u_i)$$
$$\ge\deg_{G_i}(u_i)+\deg_{G_i}(v_i)-2s-|Y_{i-1}\cup Z_{i-1}|+e_G(u_i,Y_{i-1}\cup Z_{i-1})-\deg_{G_i-S}(v_i)$$
$$\ge n+k-2-2s-|Y_i\cup Z_i|+1+e_G(u_i,Y_{i-1}\cup Z_{i-1})-\big(|X_i\cup Y_i|+e_G(v_i,T-(X_{i-1}\cup Y_{i-1}))+|U|-2\big)$$
$$\ge t-s+k+1-|Y_i\cup Z_i|-|X_i\cup Y_i|+e_G(X_i-X_{i-1},Y_{i-1}\cup Z_{i-1})$$
$$-e_G((Y_i\cup Z_i)-(Y_{i-1}\cup Z_{i-1}),T-(X_{i-1}\cup Y_{i-1})).$$

Suppose further that $t-s\ge k-w+1$. By Claim 7, $G_1=G_0$. On the other hand, $u_1\in T,v_1\in U$ and $u_1v_1\notin E(G)$, and there exists an odd component C_1 such that $v_1\in C_1$ by Claim 4(iv). Hence

$$\deg_{G-S}(v_1)\le e_G(v_1,T)+|C_1|-1\le e_G(v_1,T)+|U|-(w-1)-1,\ \text{and}$$

$$\sum_{u\in X_1}\deg_{G-S}(u) \ge \deg_{G_0}(u_i)+\deg_{G_0}(v_i)-2s-\deg_{G_0-S}(v_1)$$
$$\ge n+k-2-2s-\left(e_G(v_1,T)+|U|-(w-1)-1\right)$$
$$= t-s+k-2+w-e_G(Z_1,T).$$

Consequently we get (ii). □

Until we finish the proof of Claim 12, we fix an integer p with $1\le p\le p_0$. We define integers $B_0, B_1,\ldots, B_l, A_1, A_2,\ldots, A_l$ with $B_0\le A_1<B_1<A_2<B_2<\cdots<A_l\le B_l$ so that the following hold :

- $B_0=0$ and $B_l=p$;
- if u_1 and v_1 satisfy (ii) of Claim 8, then $A_1=0$;
- if u_p and v_p satisfy (i) of Claim 8, then $A_l=p$;
- for each i with $B_{i-1}+1\le i\le A_i, u_i$ and v_i satisfy (i) of Claim 8;
- for each i with $A_i+1\le i\le B_i, u_i$ and v_i satisfy (ii) or (iii) of Claim 8.

And we set, for each i with $1\le i\le l, a_i=A_i-B_{i-1}$ and $b_i=B_i-A_i$, and $\sum_{i=1}^{l} b_i=q$.

Note that $0\le q\le p$ and $\sum_{i=1}^{l} a_i=p-q$.

Claim 10

$$2\sum_{j=1}^{l}\left(\sum_{i=B_{j-1}+1}^{A_j}|Y_i\cup Z_i|\right)+\sum_{j=1}^{l}\left(\sum_{i=A_j+1}^{B_j}\left(|Y_i\cup Z_i|+|X_i\cup Y_i|\right)\right)=2pq-q^2+q.$$

Proof If Claim 8 (i) holds, then $|X_i\cup Y_i|=|X_{i-1}\cup Y_{i-1}|+2$ and $|Y_i\cup Z_i|=|Y_{i-1}\cup Z_{i-1}|$, and if Claim 8 (ii) or (iii) holds, then $|X_i\cup Y_i|=|X_{i-1}\cup Y_{i-1}|+1$ and $|Y_i\cup Z_i|=|Y_{i-1}\cup Z_{i-1}|+1$. Hence

$$2\sum_{j=1}^{l}\left(\sum_{i=B_{j-1}+1}^{A_j}|Y_i\cup Z_i|\right)+\sum_{j=1}^{l}\left(\sum_{i=A_j+1}^{B_j}\left(|Y_i\cup Z_i|+|X_i\cup Y_i|\right)\right)$$
$$=2\sum_{j=1}^{l}\left(a_j\sum_{i=1}^{j-1}b_i\right)+\sum_{j=1}^{l}\left(2\left(\sum_{i=1}^{j}a_i+\sum_{i=1}^{j-1}b_i\right)+b_j+1\right)b_j$$
$$=2\sum_{j=1}^{l}\sum_{i=1}^{j-1}a_jb_i+2\sum_{j=1}^{l}\sum_{i=1}^{j}a_ib_j+2\sum_{j=1}^{l}\sum_{i=1}^{j-1}b_ib_j+\sum_{j=1}^{l}b_j^2+\sum_{j=1}^{l}b_j$$
$$=2\sum_{j=1}^{l}b_j\sum_{i=1}^{l}a_i+\left(\sum_{j=1}^{l}b_j\right)^2+\sum_{j=1}^{l}b_j$$
$$=2(p-q)q+q^2+q=2pq-q^2+q.$$
□

Claim 11

$$\sum_{i=1}^{p} e_G(X_i - X_{i-1}, Y_{i-1} \cup Z_{i-1}) \ge \sum_{i=1}^{p} e_G((Y_i \cup Z_i) - (Y_{i-1} \cup Z_{i-1}), (X_p \cup Y_p) - (X_i \cup Y_i)) .$$

Proof Let $uv \in \bigcup_{i=1}^{p} E_G((Y_i \cup Z_i) - (Y_{i-1} \cup Z_{i-1}),\ (X_p \cup Y_p) - (X_i \cup Y_i))$. There exists an integer i_1 with $1 \le i_1 \le p$ such that $u \in (Y_{i_1} \cup Z_{i_1}) - (Y_{i_1-1} \cup Z_{i_1-1})$ and $v \in (X_p \cup Y_p) - (X_{i_1} \cup Y_{i_1})$. If $v \in X_p - (X_{i_1} \cup Y_{i_1})$, there exists an integer i_2 with $i_1 < i_2 \le p$ such that $v \in X_{i_2} - X_{i_2-1}$. If $v \in Y_p - (X_{i_1} \cup Y_{i_1})$, there exists an integer i_3 with $i_1 < i_3 \le p$ such that $v \in Y_{i_3} - Y_{i_3-1}$, and hence there exists an integer i_4 with $i_1 < i_4 < i_3$ such that $v \in X_{i_4} - X_{i_4-1}$. Hence there exists an integer i_0 with $i_1 < i_0 \le p$ such that $v \in X_{i_0} - X_{i_0-1}$. Since $Y_{i_0-1} \cup Z_{i_0-1} \supset Y_{i_1} \cup Z_{i_1}, u \in Y_{i_0-1} \cup Z_{i_0-1}$. Consequently

$$uv \in E_G(X_{i_0} - X_{i_0-1}, Y_{i_0-1} \cup Z_{i_0-1}) \subset \bigcup_{i=1}^{p} E_G(X_i - X_{i-1}, Y_{i-1} \cup Z_{i-1}),$$

and we get Claim 11. □

Claim 12 (i)

$$\sum_{u \in X_p \cup Y_p} \deg_{G-S}(u) + e_G\left(T - (X_p \cup Y_p),\ Y_p \cup Z_p\right) \ge q^2 + q(2 - 2p - |U|) + p\left(t - s + |U| + k - 2\right).$$

(ii) *Suppose that* $t - s \ge k - w + 1$, *and* $u_1 \in T$ *and* $v_1 \in U$. *Then* $q \ge 1$ *and*

$$\sum_{u \in X_p \cup Y_p} \deg_{G-S}(u) + e_G(T - X_p \cup Y_p, Y_p \cup Z_p) \ge q^2 + q\left(2 - 2p - |U|\right) + p\left(t - s + |U| + k - 2\right) + w - 1.$$

Proof By Claims 9, 10 and 11,

$$\begin{aligned}
&\sum_{u \in X_p \cup Y_p} \deg_{G-S}(u) \\
&= \sum_{j=1}^{l} \left(\sum_{i=B_{j-1}+1}^{A_j} \left(\sum_{u \in X_i - X_{i-1}} \deg_{G-S}(u) \right) + \sum_{i=A_j+1}^{B_j} \left(\sum_{u \in X_i - X_{i-1}} \deg_{G-S}(u) \right) \right) \\
&\ge \sum_{j=1}^{l} \left(\sum_{i=B_{j-1}+1}^{A_j} \left(|U| + t - s + k - 2 - 2|Y_i \cup Z_i| + e_G\left(X_i - X_{i-1}, Y_{i-1} \cup Z_{i-1}\right)\right) \right) \\
&\quad + \sum_{j=1}^{l} \left(\sum_{i=A_j+1}^{B_j} \left(t - s + k + 1 - |Y_i \cup Z_i| - |X_i \cup Y_i| \right.\right. \\
&\quad \left.\left. + e_G(X_i - X_{i-1}, Y_{i-1} \cup Z_{i-1}) - e_G\left((Y_i \cup Z_i) - (Y_{i-1} \cup Z_{i-1}), T - (X_{i-1} \cup Y_{i-1})\right)\right) \right)
\end{aligned}$$

$$= \sum_{j=1}^{l}\left(\sum_{i=B_{j-1}+1}^{A_j}|U|+t-s+k-2\right)+\sum_{j=1}^{l}\left(\sum_{i=A_j+1}^{B_j}t-s+k+1\right)$$
$$-\left(2\sum_{j=1}^{l}\left(\sum_{i=A_j+1}^{B_j}|Y_i\cup Z_i|\right)+\sum_{j=1}^{l}\left(\sum_{i=A_j+1}^{B_j}|Y_i\cup Z_i|+|X_i\cup Y_i|\right)\right)$$
$$+\sum_{j=1}^{l}\left(\sum_{i=B_{j-1}+1}^{A_j}e_G(X_i-X_{i-1},Y_{i-1}\cup Z_{i-1})\right)+\sum_{j=1}^{l}\left(\sum_{i=A_j+1}^{B_j}e_G(X_i-X_{i-1},Y_{i-1}\cup Z_{i-1})\right)$$
$$-\sum_{j=1}^{l}\left(\sum_{i=A_j+1}^{B_j}e_G((Y_i\cup Z_i)-(Y_{i-1}\cup Z_{i-1}),T-(X_{i-1}\cup Y_{i-1}))\right)$$
$$=(p-q)(|U|+t-s+k-2)+q(t-s+k+1)-(2pq-q^2+q)$$
$$+\sum_{i=1}^{p}e_G(X_i-X_{i-1},Y_{i-1}\cup Z_{i-1})-\sum_{i=1}^{p}e_G((Y_i\cup Z_i)-(Y_{i-1}\cup Z_{i-1}),(X_p\cup Y_p)-(X_i\cup Y_i))$$
$$-\sum_{i=1}^{p}e_G((Y_i\cup Z_i)-(Y_{i-1}\cup Z_{i-1}),T-(X_p\cup Y_p))$$
$$\geq q^2+q(2-2p-|U|)+p(k+|U|+t-s-2)-e_G(Y_p\cup Z_p,T-(X_p\cup Y_p)),$$

and we get (i).

Suppose that $t-s\geq k-w+1$, and $u_1\in T$ and $v_1\in U$. Evidently $q\geq 1$. By Claim 9(ii), $\sum_{u\in X_1}\deg_{G-S}(u)\geq t-s+k-2+w-e_G(Z_1,T)$. Hence

$$\sum_{j=1}^{l}\left(\sum_{i=A_j+1}^{B_j}\left(\sum_{u\in X_i-X_{i-1}}\deg_{G-S}(u)\right)\right)$$
$$=\sum_{u\in X_1}\deg_{G-S}(u)+\sum_{i=2}^{B_1}\left(\sum_{u\in X_i-X_{i-1}}\deg_{G-S}(u)\right)+\sum_{j=2}^{l}\left(\sum_{i=A_j+1}^{B_j}\left(\sum_{u\in X_i-X_{i-1}}\deg_{G-S}(u)\right)\right)$$
$$\geq\sum_{j=1}^{l}\left(\sum_{i=A_j+1}^{B_j}\Big(t-s+k+1-|Y_i\cup Z_i|-|X_i\cup Y_i|\right.$$
$$\left.+e_G(X_i-X_{i-1},Y_{i-1}\cup Z_{i-1})-e_G((Y_i\cup Z_i)-(Y_{i-1}\cup Z_{i-1}),T-(X_{i-1}\cup Y_{i-1})\Big)\right)+w-1,$$

and we can get (ii) by arguing as in the preceding paragraph. □

Now we divide the proof into the following two cases.

Case 1 $p_0\geq\lfloor(t-s+k)/2\rfloor$.

We set $p_1=\max\{\lfloor(t-s+k)/2\rfloor,\ 1\}$ and $\alpha_1(p_1)=\sum_{u\in X_{p_1}\cup Y_{p_1}}\deg_{G-S}(u)+e_G(T-(X_{p_1}\cup Y_{p_1}),Y_{p_1}\cup Z_{p_1})$. For each integer x

with $0 \le x \le p_1$, we set $f_1(x) = x^2 + x\left(2 - 2p_1 - |U|\right) + p_1\left(t - s + |U| + k - 2\right)$.

Since $\alpha_1(p_1) \le \sum_{y \in T} \deg_{G-S}(y)$,

$$\alpha_1(p_1) \le k(t-s) + w - 2, \qquad (2)$$

by Claim 1. Define q as in the paragraph preceding Claim 10 with $p = p_1$. Now we divide the proof further into two subcases.

Subcase 1.1 $U \ne \phi$ and $t - s \ge k - w + 1$.

First we consider the case where $u_1 \in T$ and $v_1 \in U$. In this case, $1 \le q \le p_1$. By Claim 12(ii), $\alpha_1(p_1) \ge f_1(q) + w - 1$. If $|U| \ge 2$, then $p_1 \le -1/2\left(2 - 2p_1 - |U|\right)$, and if $|U| = 1$, then $-1/2\left(2 - 2p_1 - |U|\right) = p_1 - 1/2$. Hence, from the assumption that $U \ne \phi$, it follows that for any integer x with $x \le p_1$, $f_1(x) \ge f_1(p_1)$. In particular, $f_1(q) \ge f_1(p_1)$. Moreover, if $t - s + k \ge 2$, then

$$\begin{aligned} f_1(p_1) &= -p_1^2 + p_1(t - s + k) \\ &\ge -\frac{(t-s+k-1)^2}{4} + \frac{t-s+k-1}{2}.(t-s+k) \\ &= \frac{(t-s-k)^2}{4} - \frac{1}{4} + k(t-s) \\ &\ge -\frac{1}{4} + k(t-s); \end{aligned}$$

if $t - s + k \le 1$, then $f_1(p_1) = t - s + k - 1 \ge k(t-s)$. Thus in either case, $f_1(p_1) \ge -1/4 + k(t-s)$. Consequently we get $\alpha_1(p_1) \ge k(t-s) + w - 5/4$, which contradicts (2).

Now we consider the remaining case, i.e. the case where $u_1, v_1 \in T$. In this case, $0 \le q \le p_1 - 1$. By Claim 12(i), $\alpha_1(p_1) \ge f_1(q)$. Since $p_1 - 1 \le -1/2\left(2 - 2p_1 - |U|\right)$, $f_1(q) \ge f_1(p_1 - 1)$. Moreover, if $t - s + k \ge 2$, then

$$\begin{aligned} f_1(p_1 - 1) &= -p_1^2 + p_1(t - s + k) + |U| - 1 \\ &\ge -\frac{(t-s+k-1)^2}{4} + \frac{t-s+k-1}{2}.(t-s+k) + |U| - 1 \\ &= \frac{(t-s-k)^2}{4} + |U| - \frac{5}{4} + k(t-s) \\ &\ge k(t-s) + |U| - \frac{5}{4}; \end{aligned}$$

if $t - s + k \le 1$, then $f_1(p_1 - 1) = t - s + |U| + k - 2 \ge k(t-s) + |U| - 1$. Thus in either case, $f_1(p_1 - 1) \ge k(t-s) + |U| - (5/4)$.

Therefore, since $|U| \geq w$, $f_1(p_1-1) \geq k(t-s)+w-(5/4)$.
Consequently we get $\alpha_1(p_1) \geq k(t-s)+w-5/4$, which also contradicts (2).

Subcase 1.2 $U=\phi$ or $t-s \leq k-w$.

By Claim 12(i), $\alpha_1(p_1) \geq f_1(q)$.

First we suppose $U=\phi$. Then $q \leq p_1-1$, and hence $f_1(q) \geq f_1(p_1-1)$. Moreover $f_1(p_1-1) \geq k(t-s)-(5/4)$ by the same argument as in the second paragraph of Subcase 1.1, and hence $\alpha_1(p_1) \geq k(t-s)-(5/4)$, which contradicts (2).

Now we suppose $U \neq \phi$ and $t-s \leq k-w$. In this case, we show that $t-s+k \geq 2$. Suppose that $t-s+k \leq 1$. Then $s-t \geq k-1 \geq 0$, and hence $k(s-t) \geq s-t+k-1$. Since $k(s-t) \leq w-2$ by (2), this implies $s-t+k \leq w-1$, which contradicts the assumption that $t-s \leq k-w$. Thus $t-s+k \geq 2$. By the same argument as in the first paragraph of Subcase 1.1, $f_1(q) \geq f_1(p_1)$ and $f_1(p_1) \geq \frac{(t-s-k)^2}{4}-\frac{1}{4}+k(t-s)$.

Moreover, since $t-s-k \leq -w \leq 0$, $\frac{(t-s-k)^2}{4} \geq \frac{w^2}{4}$, and hence

$f_1(p_1) \geq \frac{w^2}{4}-\frac{1}{4}+k(t-s)$.

Therefore, since $\frac{w^2}{4}-\frac{1}{4} > w-2$, $f_1(p_1) > k(t-s)+w-2$.
Con-sequently we get $\alpha_1(p_1) > k(t-s)+w-2$, which also contradicts (2).

Case 2 $p_0 \leq \lfloor (t-s+k)/2 \rfloor - 1$.

In order to proof case 2, we show the following three claims.

Claim 13 $k-s \leq t \leq n-s$

Proof Obviously, $t \leq n-s$ holds.

First we suppose that $w \geq 2$. Since $G_{p_0+1}=K_n$, there exists a graph H with $V(H)=V(G), E(H) \supset E(G)$ and $H[U]=G[U]$ for which there exist components C_1, C_2 of $G[U]$ and vertices $z_1 \in V(C_1)$ and $z_2 \in V(C_2)$ such that
$\deg_H(z_1)+\deg_H(z_2) \geq n+k-2$. On the other hand,
$\deg_H(z_1)+\deg_H(z_2) \leq |C_1|+|C_2|-2+2s+2t \leq n-s-t-2+2s+2t = n+s+t-2$.
Hence we get $k-s \leq t$.

Now we suppose that $w \leq 1$. Since $\delta(G) \geq k, \sum_{y \in T} \deg_{G-S}(y) \geq t(k-s)$. On the other hand, $\sum_{y \in T} \deg_{G-S}(y) \leq k(t-s)+w-2 \leq k(t-s)-1$. Hence we get

$$ts \geq ks+1. \tag{3}$$

If $s=0$, then (3) implies $0 \ge 1$, which is absurd. Hence $s \ge 1$, and this together with (3) implies $t \ge k$, so $t \ge k-s$. Consequently, we get Claim 13. □

We set

$$\alpha_2(p_0) = \sum_{u \in X_{p_0} \cup Y_{p_0}} \deg_{G-S}(u) + e_G(T-(X_{p_0} \cup Y_{p_0}), Y_{p_0} \cup Z_{p_0}) + \sum_{u \in T-(X_{p_0} \cup Y_{p_0})} \deg_{G[T-(X_{p_0} \cup Y_{p_0})]}(u) + e_G(T-(X_{p_0} \cup Y_{p_0}), U - Z_{p_0}).$$

Since $\alpha_2(p_0) \le \sum_{y \in T} \deg_{G-S}(y)$,

$$\alpha_2(p_0) \le k(t-s)+w-2 \qquad (4)$$

by Claim 1. Let q be as before with $p = p_0$.

Claim 14

$$\sum_{u \in T-(X_{p_0} \cup Y_{p_0})} \deg_{G[T-(X_{p_0} \cup Y_{p_0})]}(u) + e_G(T-(X_{p_0} \cup Y_{p_0}), U - Z_{p_0}) \ge (t-2p_0+q)(n-s-2p_0-1).$$

Proof Since $G_{p_0+1} = K_n$, for each vertex $y \in T-(X_{p_0} \cup Y_{p_0})$, $yz \in E(G)$ for any vertex $z \in (T-\{y\}) \cup U - (X_{p_0} \cup Y_{p_0} \cup Z_{p_0})$. Since $|T-(X_{p_0} \cup Y_{p_0})| = t-2p_0+q$ and $|(T-\{y\}) \cup U - (X_{p_0} \cup Y_{p_0} \cup Z_{p_0})| \ge n-s-2p_0-1$, we get Claim 14. □

For each real number x, we set

$$f_2(x) = x^2 + x(t-4p_0+1) + 4p_0^2 + p_0(k-n-2t) + t(n-s-1).$$

Claim 15 $f_2(q) > k(t-s)+w-2$.

Proof We set

$$\beta(s,t) = -\frac{t^2}{4} + t\left(\frac{n}{2} - \frac{k}{2} - \frac{1}{6} - s\right) + s\left(\frac{n}{2} + \frac{k}{2} - \frac{2}{3}\right) + \frac{k^2}{2} + \frac{2n}{3} - \frac{kn}{2} - \frac{1}{4}.$$

By Claim 13, $k-s \le t \le n-s$, and hence $\beta(s,t) \ge \min\{\beta(s,k-s), \beta(s,n-s)\}$. Since $n > n_0$, $\beta(s,k-s) = \frac{3}{4}s^2 + \frac{s}{2}(k-1) + \frac{2n}{3} - \frac{k^2}{4} - \frac{k}{6} - \frac{1}{4} > 0$,

and $\beta(s,n-s) = \frac{3}{4}s^2 + \frac{s}{2}(2k-1-n) + \frac{n^2}{4} + \frac{n}{2} - kn + \frac{k^2}{2} - \frac{1}{4} > 0$.

Hence $\beta(s,t) > 0$. On the other hand,

$$
\begin{aligned}
f_2(q) &\geq f_2\left(-(t-4p_0+1)/2\right) \\
&= p_0(2+k-n) - \frac{t^2}{4} + t\left(n-s-\frac{3}{2}\right) - \frac{1}{4} \\
&\geq \frac{t-s+k-2}{2}(2+k-n) - \frac{t^2}{4} + t\left(n-s-\frac{3}{2}\right) - \frac{1}{4} \\
&= -\frac{t^2}{4} + \frac{t}{2}(n+k-1-2s) + \frac{s}{2}(n-k-2) + \frac{k^2}{2} + n - \frac{kn}{2} - \frac{9}{4} \\
&= \beta(s,t) + k(t-s) + (n-s-t)/3 - 2.
\end{aligned}
$$

By Claim 3, $\frac{|U|}{3} = \frac{n-s-t}{3} \geq w$. Consequently, we get Claim 15. □

By Claims 12(i) and 14, for any integer q with $0 \leq q \leq p_0, \alpha_2(p_0) \geq f_2(q)$ (note that $|U| = n-s-t$). This together with Claim 15 implies $\alpha_2(p_0) > k(t-s) + w - 2$, which contradicts (4). □

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