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On minimally 3-connected graphs on a surface

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Abstract

It is well-known that the maximal size of minimally 3-connected graphs of order $n \geq 7$ is $3n-9$. In this paper, we shall prove that if *G* is a minimally 3-connected graph of order *n*, and is embedded in a closed surface with Euler characteristic χ, then *G* contains at most $2n - \min\{2, 2\chi\}$ edges. This bound is best possible for every closed surface.

Keywords: minimally 3-connected graph, closed surface, triangulation

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1 Introduction

A *triangulation* of a closed surface is a simple graph embedded in the surface such that each face is bounded by a triangle. Except for the triangle embedded in the plane, since the neighborhood of each vertex in a triangulation induces a 2-connected subgraph, the triangulation itself is a 3-connected graph. It is a quite natural question to ask how many edges can we delete from a triangulation without destroying the 3-connectivity. Also, this question is closely related to determine the maximal size of minimally 3-connected graphs embedded in a fixed closed surface. A 3-connected graph is said to be *minimally* 3-*connected* if deletion of any edge results in a non-3-connected graph.

It is easy to see from the Euler's formula that a triangulation of a closed surface with Euler characteristic χ has exactly $3n-3\chi$ edges, where *n* is the number of vertices in it. On the other hand, the maximal size of minimally 3-connected graphs of order $n \geq 7$ is determined to be $3n-9$ ([3], see also [4]). However, extremal graphs for this assertion are complete bipartite graphs $K_{3,n-3}$. The genus of them tends to infinity as $n \to \infty$. So, one can expect that the maximal size of minimally 3-connected graphs on a surface is bounded by a smaller function. In fact, we shall prove the following theorem in Section 2.

Theorem 1 *If G is a minimally 3-connected graph of order n, and is embedded in a closed surface with Euler characteristic* χ, *then*

$$|E(G)| \leq \begin{cases} 2n-2 & \text{if } \chi = 2, \\ 2n-2\chi & \text{if } \chi \leq 1. \end{cases}$$

The bounds given in the theorem are best possible. For the planar case, the wheels attain the equality $|E(G)| = 2n-2$. In Section 2, we shall also prove that extremal graphs in the planar case are exactly the wheels.

For the nonplanar case, consider the *radial graph* $R(G)$ of a triangulation G of the surface. The radial graph $R(G)$ is obtained from G by putting a vertex into each face of G, joining it to all vertices lying on the boundary of the face, and deleting the edges of G. Since every edge is incident with a vertex of degree three, $R(G)$ is minimally 3-connected. Since $R(G)$ is a quadrangulation of the surface, the number of edges in $R(G)$ is exactly $2\left|V\left(R(G)\right)\right|-2\chi$.

As the answer to our first question, we can show the following corollary.

Corollary 2 *If G is a triangulation of a closed surface with Euler characteristic* χ *, then we can delete*

$$\begin{cases} |V(G)|-3 \text{ if } \chi=2 \text{ and } |V(G)|\geq 6, \\ |V(G)|-4 \text{ if } \chi=2 \text{ and } |V(G)|\leq 5, \\ |V(G)|-\chi \text{ if } \chi\leq 1 \end{cases}$$

edges from G preserving 3*-connectivity.*

The bounds given in the corollary are best possible. For the planar case, consider the join of P_{n-2}, a path of order $n-2$, and K_2, two vertices mutually adjacent. In order to preserve the 3-connectivity, we can only delete one edge from each internal vertex of P_{n-2}, and the edge consisting of K_2.

For the nonplanar case, consider the union of a triangulation G and its radial graph $R(G)$. The edges in $R(G)$ cannot be removed in order to preserve the degree of the vertices in $V\left(R(G)\right)-V(G)$.

We give the complete proof the corollary in Section 2.

2 Proofs

In this section, we prove Theorem 1 and Corollary 2. In the proofs, we use the notion of contractible edges.

Let G be a 3-connected graph, and e be an edge of G. Then the graph G/e is the multigraph obtained from G by contracting the edge e (So the edge set $E(G/e)$ is identified with $E(G)-\{e\}$). Obviously, if G is embedded in a closed surface, then G/e is also considered to be embedded in the surface. When G/e is still 3-connected, the edge e is called a *contractible edge* of G. There are many researches on contractible edges in 3-connected graphs [1,2,5,6,7]. Among them, we use the following result in the proof of Theorem 1.

Lemma 3 ([2]) *Let G be a minimally 3-connected graph, and let* $e=xy$ *be an edge of G. If* $\deg_G(x)\geq 4$ *and* $\deg_G(y)\geq 4$*, then e is contractible.*

Lemma 4 ([2]) *Let G be a* 3*-connected graph of order at least 5, and let x be a vertex of degree three in G. Then, there exists a contractible edge incident with x in G.*

We also use the following easy observation.

Lemma 5 *Let G be a minimally 3-connected graph, and let e and f be distinct edges G. If $G/e-f$ is still 3-connected, then e and f are incident with a common vertex of degree three in G.*

Proof Since *G* is minimally 3-connected, $G-f$ has a separating set $S \subset V(G)$ with $|S|=2$. Let A be a component of $G-f-S$, and $B=V(G)-S-A$. Since $G/e-f=(G-f)/e$ is 3-connected, one of *A* and *B*, say *A*, must vanish by the contraction of *e*. This means that *A* consists of one vertex, say *x*, and *e* joins *x* to a vertex in *S*. Then, we can easily see that *x* is incident with both *e* and *f* in *G*, and the degree of *x* is exactly three in *G*. □

As an immediate consequence of this lemma, we have the following:

Lemma 6 *Let G be a minimally* 3*-connected graph, and let $e=xy$ be a contractible edge of G.*

[1] *[2]* If $\deg_G(x)\ge 4$ *and* $\deg_G(y)\ge 4$, *then* G/e *is also minimally* 3*-connected.*

[2] If $\deg_G(x)=\deg_G(y)=3$, *then there exists a set of edges* $F\subset E(G/e)$ *with* $|F|\le 1$ *such that* $G/e-F$ *is minimally* 3*-connected.*

[3] If $\deg_G(x)=3, \deg_G(y)\ge 4$ *and x is adjacent to a vertex of degree three, then there exists a set of edges* $F\subset E(G/e)$ *with* $|F|\le 1$ *such that* $G/e-F$ *is minimally* 3*-connected.* □

Proof of Theorem 1 Let *G* be a minimally 3-connected graph of order *n*, embedded in a closed surface with Euler characteristic χ. We fix χ and apply induction on *n* to prove that $|E(G)|\le 2n-c$, where $c=\min\{2,2\chi\}$. If $n=4$, then $|E(G)|=6=2n-2$ $\le 2n-c$, and hence the assertion holds.

Suppose $n\ge 5$. Let *U* be the set of vertices of degree three in *G*, and let $W=V(G)-U$ be the set of vertices of degree at least four in *G*. Suppose first that there is an edge *xy* with $x,y\in W$. Then by Lemmas 3 and 6(1), G/xy is a minimally 3-connected graph. By the induction hypothesis, we have $|E(G/xy)|\le 2(n-1)-c$. Then, $|E(G)|=|E(G/xy)|+1\le 2(n-1)-c+1<2n-c$, and the result follows.

Suppose next that there is an edge *xz* with $x,z\in U$. By Lemma 4, *x* is incident with a contractible edge, say $e=xy$ (possibly $y=z$). By Lemma 6(2) or (3), there exists a set of edges $F\subset E(G/e)$ with $|F|\le 1$ such that $G/e-F$ is a minimally 3-connected graph. Then, by the induction hypothesis, we have

$$|E(G)| = |E(G/xy)| + 1 + |F| \le 2(n-1) - c + 2 \le 2n - c,$$

and the result follows.

We may now assume that each of U and W is an independent set in G. Since G is a bipartite graph embedded in a surface, even if it is not a 2-cell embedding, every face is bounded by at least four edges. Thus, by the Euler's formula, we can easily show the inequality $|E(G)| \le 2n - 2\chi$. This proves the assertion of the theorem. □

In the planar case, we can determine the extremal graphs for Theorem 1.

Theorem 7 *If G is a minimally 3-connected planar graph satisfying that $|E(G)| = 2|V(G)| - 2$, then G is a wheel.*

Proof We trace the proof of Theorem 1, and use induction on $n = |V(G)|$. If $n \le 5$, then every minimally 3-connected graph of order n is a wheel. Thus the assertion holds.

Suppose $n \ge 6$. Let G be a minimally 3-connected planar graph of order n with $|E(G)| = 2n - 2$. We define U and W as in the proof of Theorem 1. If there exists an edge e joining two vertices of W, then G/e is a minimally 3-connected planar graph and $|E(G/e)| = 2|V(G/e)| - 1$. This contradicts the result of Theorem 1. Hence W is independent in G.

If U is also independent, then G is a bipartite planar graph, and hence we have $|E(G)| \le 2n - 4$. This is not the case. Thus there exists an edge xz with $x, z \in U$. As in the proof of Theorem 1, we have a contractible edge $e = xy$, and a set of edges $F \subset E(G/e)$ with $|F| \le 1$ such that $G/e - F$ is a minimally 3-connected graph. By Theorem 1, we have $2(n-1) - 2 \ge |E(G/e - F)| = |E(G)| - 1 - |F| = 2n - 3 - |F|$. Hence $|F| = 1$, say $F = \{f\}$, and $|E(G/e - f)| = 2(n-1) - 2$. By the induction hypothesis, $G/e - f$ is a wheel of order $n-1$.

Let $v_e \in V(G/e)$ denote the vertex obtained by the contraction of e. First consider the case where v_e is the center of the wheel $G/e - f$. By Lemma 5, f is incident with v_e in G/e. Since $G/e - f$ is a wheel with center v_e, there exists an edge f' in $G/e - f$ having the same endvertices as f. We may assume that f is incident with x and f' is incident with y in G. Note here that $\deg_G(x) + \deg_G(y) = \deg_{G/e}(v_e) + 2 = n + 1 \ge 7$. Since $\deg_G(x) = 3$, we have $\deg_G(y) \ge 4$. We now have a 3-connected graph $G/e - f \cong G/e - f'$. However, the common endvertex y of f' and e has degree at least 4. This contradicts Lemma 5.

Suppose that v_e is not the center of $G/e - f$. Let u be the endvertex of f other than v_e. If u is not the center of $G/e - f$, then u has degree four in G, and it is incident with the center of $G/e - f$. This contradicts the fact that W is independent in

G. Thus u is the center of $G/e-f$. Then we can easily see that G is also a wheel. □

Now we can prove Corollary 2.

Proof of Corollary 2 Let G be a triangulation of a closed surface with Euler characteristic χ, and put $n=|V(G)|$. Then by the Euler's formula, we have $|E(G)|=3n-3\chi$. So, the assertion is immediate from Theorem 1 for the case $\chi \le 1$, and for the case where $\chi = 2$ and $n \le 5$. For the planar case, by Theorem 7, it suffices to show that if $n \ge 6$, then G contains a spanning minimally 3-connected subgraph other than a wheel.

We may assume that G contains a spanning wheel H. Let v denote the center of H. Since G is a plane triangulation, there exists an edge $xy \in E(G)-E(H)$, $x, y \ne v$. Then it can be easily checked that $H' = H + xy - vx$ is 3-connected. Since each vertex of H' has degree at most $\max\{4, n-2\} < n-1$ (by $n \ge 6$), H' does not contain a spanning wheel. Thus, G contains a spanning minimally 3-connected subgraph other than a wheel, as desired. □

References

[1] K. Ando, H. Enomoto and A. Saito, Contractible edges in 3-connected graphs, *J. Combin. Theory Ser. B* **42** (1987), 87–93.

[2] R. Halin, Zur Theorie der n-fach zusammenhängenden Graphen, *Abh. Math. Sem. Univ. Hamburg* **33** (1969), 133–164.

[3] W. Mader, Minimale n-fach zusammenhängende Graphen mit maximaler Kantenzahl, *J. reine angew. Math.* **249** (1971), 201–207.

[4] W. Mader, Über minimal n-fach zusammenhängende, unendliche Graphen und ein Extremalproblem, *Arch. Math.* **23** (1972), 553–560.

[5] W. McCuaig and K. Ota, Contractible triples in 3-connected graphs, *J. Combin. Theory Ser. B* **60** (1994), 308–314.

[6] K. Ota, The number of contractible edges in 3-connected graphs, *Graphs Combin.* **4** (1988), 333–354.

[7] W.T. Tutte, A theory of 3-connected graphs, *Indag. Math.* **23** (1961), 441–455.