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Classification of Type II $\mathbf{Z}_6$-Codes of Length 8

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Abstract

We discuss to classify Type II codes of length 8 over $\mathbf{Z}_6$. There exist exactly two Type II codes up to equivalence.

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1 Introduction

Our interest in this note is how many Type II codes over finite rings $\mathbf{Z}_{2k} (= \mathbf{Z}/2k\mathbf{Z})$ exist up to equivalence. For self-dual codes over finite fields $\mathbf{F}_q$ (especially $q = 2,3$), the classification problems are rather classical and well studied ([5], [6], for example). Moreover Type II codes over $\mathbf{Z}_4$ have been also widely studied, and the classifications are known for lengths 8 [2] and 16 [7]. Although the notion of Type II codes has been generalized to codes over $\mathbf{Z}_{2k}$ for any positive integer *k* in [1], and many Type II codes over $\mathbf{Z}_6$ of length 24 are constructed in [4], it seems that there are not so many results about classifications of codes over finite rings $\mathbf{Z}_{2k} (k > 2)$.

In this note, we give the classification of Type II $\mathbf{Z}_6$-codes of length 8, and show that there exist exactly two Type II codes over $\mathbf{Z}_6$ up to equivalence.

2 Definitions and Basic Facts

A *code C of length n over* $\mathbf{Z}_k$ (or a $\mathbf{Z}_k$ *-code of length n)* is a $\mathbf{Z}_k$ -submodule of $\mathbf{Z}_k^n$, where $\mathbf{Z}_k = \mathbf{Z}/k\mathbf{Z} = \{0,1,2,..., k-1\}$ is the ring of integers modulo *k.* An element of *C* is called a *codeword.* A *generator matrix* of *C* is a matrix whose rows generate *C*. The *inner product* of $x, y \in \mathbf{Z}_k^n$ is defined as $\langle x, y\rangle = x_1y_1 + \cdots + x_n y_n (\in \mathbf{Z}_k)$ where $x = (x_1,..., x_n)$ and $y = (y_1,..., y_n)$. The *dual code* $C^\perp$ of *C* is $C^\perp = \{x \in \mathbf{Z}_k^n \mid \langle x, y\rangle = 0$ for all $y \in C\}$. *C* is called *self-dual* if $C = C^\perp$. The *Euclidean weight* $wt_E(x)$

for $x \in \mathbf{Z}_k^n$ is $wt_E(x) = \sum_{i=1}^{n} \min\{x_i^{\,2}, (k - x_i)^2\} (\in \mathbf{Z})$. A *Type II* $\mathbf{Z}_{2k}$-code C is defined as a self-dual code $C \subset \mathbf{Z}_{2k}^n$ with the condition that the Euclidean weight is divisible by 4*k* for every codeword of *C*. It is well-known [1] that there exists a Type II $\mathbf{Z}_{2k}$-code *C* of length *n* if and only if *n* is divisible by 8.

We denote by H_8 the extended Hamming $\mathbf{Z}_2$-code with the generator matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

We denote by T_4 and $T_4 \oplus T_4$ the $\mathbf{Z}_3$-codes with the following generator matrices:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix},$$

respectively.

Two codes *C* and C' are *equivalent* if C' is obtained by applying some permutation and (–1)-multiplication of some coordinates of *C*. We write $C \cong C'$ when these codes are equivalent.

Proposition 1 ([5], [6]) *H_8 is a unique Type II $\mathbf{Z}_2$-code of length* 8 *and $T_4 \oplus T_4$ is a unique self-dual $\mathbf{Z}_3$ - code of length* 8 *up to equivalence.*

We also use the following facts afterwards.

Lemma 2 *The automorphism group of H_8 acts on the set of all the codewords of weight* 4 *in H_8 and the set of all the codewords of weight* 4 *outside of H_8, respectively.*

Proof It is well-known that the automorphism groups of H_8, which is isomorphic to $2^3 : GL_3(2)$, acts 3-transitively on the set $\{1,2,...,8\}$ of the coordinates of the code. Furthermore the stabilizer of 1, 2, 3 is the group of order 4 generated by the permutations (5 6) (7 8), (5 7) (6 8). This proves the assertion immediately. □

Lemma 3 *Every self-dual $\mathbf{Z}_3$-code of length* 4 *can be obtained by (-1)-multiplications of some coordinates of T_4.*

Proof Clearly every self-dual $\mathbf{Z}_3$-code of length 4 contains the codewords $(1, a, b, 0)$ and $(1, c, 0, d)$. By using (-1)-multiplications, we may assume $a = b = d = 1$, and then *c* should be –1. □

3 Classification

Let C be a Type II code of length 8 over $\mathbf{Z}_6$. For $p = 2,3$, we set $C(\text{mod } p) = \{x(\text{mod } p) \mid x \in C\}$, which is a $\mathbf{Z}_p$-code of length 8. Then $C(\text{mod } 2)$ is Type II and $C(\text{mod } 3)$ is self-dual ([3]), and thus, by Proposition 1, $C(\text{mod } 2) \cong H_8$ and $C(\text{mod } 3) \cong T_4 \oplus T_4$. Hence we have the following:

Lemma 4 *Let C be a Type II code of length* 8 *over* $\mathbf{Z}_6$ *and let G be a generator matrix of C. Then*

$$G' = \begin{pmatrix} 3G \\ 2G \end{pmatrix}$$

is also a generator matrix of C. Moreover, we have the following:

(1) *The* $\mathbf{Z}_2$*-code with the generator matrix* $3G(\text{mod } 2)$ *is equivalent to* H_8.
(2) *The* $\mathbf{Z}_3$*-code with the generator matrix* $2G(\text{mod } 3)$ *is equivalent to* $T_4 \oplus T_4$.

By this lemma, we may assume

$$3G = \begin{pmatrix} 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 0 & 3 & 3 \\ 3 & 0 & 3 & 0 & 3 & 0 & 3 & 0 \end{pmatrix}$$

and $2G$ is a matrix obtained by some permutation of

$$\begin{pmatrix} 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 0 & 2 \end{pmatrix}$$

(we denote this matrix by $2(T \oplus T)$).

Definition Let ϕ be a permutation of the coordinates with $\phi(i) = a_i$. Then we denote the code whose generator matrix is

$$\begin{pmatrix} 3G \\ \phi(2(T \oplus T)) \end{pmatrix}$$

by $C((a_1, a_2, a_3, a_4)\ (a_5, a_6, a_7, a_8))$.

The following matrices are the generator matrices of $C((1,2,3,4)\ (5,6,7,8))$ and $C((1,2,3,5)\ (4,6,7,8))$.

$$\left(\begin{array}{cccccccc} 3&3&3&3&0&0&0&0\\ 3&3&0&0&3&3&0&0\\ 3&3&0&0&0&0&3&3\\ 3&0&3&0&3&0&3&0\\ \hline 2&2&2&0&0&0&0&0\\ 2&4&0&2&0&0&0&0\\ 0&0&0&0&2&2&2&0\\ 0&0&0&0&2&4&0&2 \end{array}\right), \left(\begin{array}{cccccccc} 3&3&3&3&0&0&0&0\\ 3&3&0&0&3&3&0&0\\ 3&3&0&0&0&0&3&3\\ 3&0&3&0&3&0&3&0\\ \hline 2&2&2&0&0&0&0&0\\ 2&4&0&0&2&0&0&0\\ 0&0&0&2&0&2&2&0\\ 0&0&0&2&0&4&0&2 \end{array}\right)$$

Lemma 5 (1) $C((a,b,c,d)(e,f,g,h)) \cong C((e,f,g,h)(a,b,c,d))$.

(2) *Let* σ,τ *be permutations on* $\{a,b,c,d\}$ *and* $\{e,f,g,h\}$, *respectively. Then we have* $C((a,b,c,d)(e,f,g,h)) \cong C(\sigma(a,b,c,d)\tau(e,f,g,h))$.

(3) *Let* ϕ *be any permutation, and* μ *be a permutation satisfying* $\mu(3C)=3C$. *Then* $C(\phi(1,2,3,4)\phi(5,6,7,8)) \cong C(\mu\phi(1,2,3,4)\mu\phi(5,6,7,8))$.

Proof (1) is obvious. By Lemma 3, we can obtain $C(\sigma(a,b,c,d)\tau(e,f,g,h))$ by applying (-1)-multiplications of some coordinates of $C((a,b,c,d)(e,f,g,h))$. Since $3H_8$ is invariable by (-1)-multiplications, we have the assertion (2). Finally we will prove (3). $C(\mu\phi(1,2,3,4)\mu\phi(5,6,7,8))$ is equivalent to the code C' with its generator matrix

$$\begin{pmatrix} \mu^{-1}(3G) \\ \phi(2(T\oplus T)) \end{pmatrix}.$$

By the assumption, $\mu^{-1}(3G)$ generates the code $3C$, and this means C' is equivalent to $C(\phi(1,2,3,4)\phi(5,6,7,8))$. □

Theorem 6 *A Type II code of length* 8 *over* $\mathbf{Z}_6$ *is equivalent to exactly one of* $C((1,2,3,4)(5,6,7,8))$ *and* $C((1,2,3,5)(4,6,7,8))$.

Proof By Lemma 5(2), it is sufficient to consider $\frac{1}{2}\times\binom{8}{4}=35$ Type II codes. Furthermore, by Lemma 2 and Lemma 5(3), the following 7 codes

$C((1,2,3,4)(5,6,7,8))$
$C((1,2,5,6)(3,4,7,8))$ $C((1,3,5,7)(2,4,6,8))$ $C((1,4,6,7)(2,3,5,8))$
$C((1,2,7,8)(3,4,5,6))$ $C((1,3,6,8)(2,4,5,7))$ $C((1,4,5,8)(2,3,6,7))$

are equivalent to the code $C((1,2,3,4)(5,6,7,8))$ and the other 28 codes are equivalent to the code $C((1,2,3,5)(4,6,7,8))$. In order to show the non-equivalence of the two codes, we consider the binary vectors

$$v_1 = (1\ 1\ 1\ 1\ 0\ 0\ 0\ 0),\quad v_1' = (0\ 0\ 0\ 0\ 1\ 1\ 1\ 1),$$
$$v_2 = (1\ 1\ 1\ 0\ 1\ 0\ 0\ 0),\quad v_2' = (0\ 0\ 0\ 1\ 0\ 1\ 1\ 1)\ .$$

The set of the vectors $\{v_1, v_1'\}$ (resp. $\{v_2, v_2'\}$) indicate the "support" of the subcodes equivalent to T_4 of $C((1,2,3,4)(5,6,7,8))\ (\bmod 3)$ (resp. $C((1,2,3,5)(4,6,7,8))\ (\bmod 3)$). Hence they are invariant under the automorphisms of the codes, respectively. Hence the assertion follows from the facts

$$v_1, v_1' \in C((1,2,3,4)(5,6,7,8))\ (\bmod 2),$$

$$v_2, v_2' \notin C((1,2,3,5)(4,6,7,8))\ (\bmod 2).$$ □

Remarks (1) The codes $C((1,2,3,4)(5,6,7,8)), C((1,2,3,5)(4,6,7,8))$ are equivalent to the codes with the following generator matrices (in standard form):

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 0 & 4 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 & 3 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 & 3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 & 3 & 2 & 1 \end{pmatrix},$$

respectively.

(2) In [3], a Type II code was already found and its Lee weight enumerator was also calculated. By comparing their generator matrices, it is easy to check that the known code is equivalent to $C((1,2,3,4)(5,6,7,8))$. We note that the code $C((1,2,3,5)(4,6,7,8))$ also has the same Lee weight enumerator.

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