AKCE J. Graphs. Combin., **1,** No. 1 (2004) 63-67

Parallel Transformations of Graphs: Graphs having Unique Elementary Parallel Transformation

Belmannu Devadas Acharya

Department of Science & Technology, Technology Bhawan, New Mehrauli Road, New Delhi - 110 016
e-mail: acharyad@alpha.nic.in

Communicated by: S. Arumugam
Received: June 07, 2004

Abstract

An elementary parallel transformation (ept) of a graph $G=(V,E)$ is defined as a transfer of an edge to join two nonadjacent vertices in *G* which are equidistant from the ends of the given edge. A parallel transformation is then defined as a sequence *P* of such ept's carried out in the graph, and the so transformed graph is denoted $P(G)$. Further, a graph *H* is regarded parallel to *G* if there exists a parallel transformation *P* of *G* such that $P(G) \cong H$. In this paper, we determine the connected graphs which have at most one ept.

Key words: *Graph, parallel transformation, dominating set.*

AMS Classification : 05C

1 Introduction

Throughout this paper, unless mentioned otherwise, by a "graph" we shall mean a finite simple graph without loops as treated in F. Harary [3]. In this paper we present the concept of elementary parallel transformation and characterise connected graphs which have at most one elementary parallel transformation. These concepts seem to be connected to what is now being termed as *"jump distance"* of a graph (e.g., see [2,8]).

2 Parallel Transformations in Graphs

Let $G=(V,E)$ be a graph and uv be a *link* (or, 'edge'), 'linking' its two ends *u* and *v* such that there exist two other nonadjacent vertices u' and v' in *G* at distance *t* each from *u* and *v*, respectively. Then, the *transformation* of *G* into another graph $G(uv \to u'v')$ (or $G(uv \vec{t} u'v')$ if we want to specify the distance t) obtained by deleting the link uv (or, 'delinking' u and v) and linking the vertices u' and v' by a new link $u'v'$ is called an *elementary parallel transformation* (or 'ept' for short, or *t*-ept if *t* is to be specified), the operation itself being called so. Hence, a *parallel transformation* *P* of *G* is a sequence $(P_1, P_2, ..., P_k)$ of epts P_i, $1 \le i \le k$; we may regard *P* as the composition of the maps P_i and hence conventionally the "image" $P(G)$ of *G* under the composite map *P* can also be termed the *parallel transformation* of *G* under *P*.

A graph H is said to be *parallel to a graph G* if there exists a parallel transformation P of G *onto* H (i.e., so that $P(G) \cong H$), and then we write $G \to H$ or $G \; \vec{P} \; H$ if P is to be specified.

Remark 2.1 The relation '$\to$' so defined is neither symmetric nor transitive (consequently, it is not an equivalence relation on the class of all graphs). Further, we define Φ to be the *null sequence* each component of which delinks no pair of adjacent vertices and links up no pair of nonadjacent vertices at the same distance from the ends of the adjacent vertices; we consider it as the *trivial* parallel transformation of *G*.

Observation 2.2 We observe that complete graphs K_n and complete bipartite graphs $K_{m,n}$ do not admit any non-null ept at all. In fact, the following result shows that these are the only such graphs; its proof uses the well-known notion of a *dominating set* in a graph *G*, which is defined as a set $D \subseteq V(G)$ with the property that every vertex *u* in $V(G) - D$ has a vertex $v \in D$ such that $uv \in E(G)$ (*e.g.*, see T. Haynes *et al* [4,5], S.T. Hedetniemi and R. Laskar [6], H.B. Walikar *et al* [7]).

Theorem 2.3 *A connected graph has no link amenable to ept if and only if it is either complete or complete bipartite.*

Proof Let $G = (V, E)$ be a graph. If *G* is complete then it has no pair of nonadjacent vertices, and hence no ept. If *G* is complete bipartite, with bipartition $\{A, B\}$ of its vertex set $V(G)$, then any pair $\{a', b'\}$ of nonadjacent vertices is contained either in *A* or in *B*, while for every link *ab* of *G*, $a \in A$ and $b \in B$ (or, vice versa) so that $d(a, a')$ and $d(b, b')$ are of different parities: therefore, no ept is possible in this case too.

For the *converse*, suppose that *G* has no ept. We shall show that *G* is either complete or complete bipartite. Towards this end, suppose that *G* is not complete. Then, there exists a pair of nonadjacent vertices $a', b' \in V(G)$. Since *G* is connected, a' and b' are joined by at least one path. Let $d(a', b') = r$ and let $P := (a' = a_1, a_2, ..., a_r, a_{r+1} = b')$ be an $a' - b'$ *geodesic* (i.e., a shortest $a' - b'$ path) in *G*. If $r \geq 3$, since $d(a', a_2) = 1 = d(a_4, a_3)$ and a' and a_4 are not adjacent in *G* as *P* is a geodesic. We see that *G* has the ept $(a_2, a_3) \to (a', a_4)$, a contradiction. Therefore, $r \leq 2$ and since *G* has no loops it follows that $r \in \{1,2\}$. In fact, $r = 2$ since a' and b' are not adjacent in *G*.

Thus, we have shown that any two nonadjacent vertices in *G* are at distance 2 from each other.

Let *A* be a maximal independent set in *G* containing a' and b'. Since a_2 is adjacent to both a' and b', $a_2 \in B = V(G) - A$. We next show that *B* is independent. If it is not, there would exist a link xy in *B*. Since *A* is a maximal independent set, by a theorem of C. Berge [1] (see p. 309, Theorem 2), it must be a (minimal) dominating set of *G* and hence there exist vertices x' and y' in *A* such that $xx', yy' \in E(G)$. If $x' \neq y'$ then $(x, y) \to (x', y')$ must be an ept in *G*, a contradiction to the hypothesis. Therefore, $x' = y'$. Then, at least one of a' and b' must be different from x', say a' is so. Then, since a' and x' are nonadjacent in *G* they must be at distance two as shown already. Hence, let $z \in B$ be adjacent to both a' and x'. If $z = x$, or $z = y$, say $z = x$, , then $(z, y) \to (a', x')$ is an ept in

G, a contradiction. Therefore, *z* must be different from *x* and *y*. We claim that a' must be nonadjacent to both *x* and *y; for*, if a' is adjacent to *x*, say, then $(x, y) \to (a', x')$ is an ept in *G* contradicting the hypothesis. Hence, a' must be nonadjacent to both *x* and *y*. But, then $(z, x') \to (a', x)$ is an ept in *G*, again contradicting the hypothesis. Thus, it follows that *B* is an independent set in *G*. But then every pair of vertices in *B* must be at distance two from each other, being a pair of nonadjacent vertices. Also, since $\{A, B\}$ forms a bipartition of $V(G)$ the distance between any two vertices $a \in A$ and $b \in B$ can only be odd. It follows that *G* must be a complete bipartite graph with bipartition $\{A, B\}$. □

The next extreme case is of graphs which have exactly one ept, which is settled in the following result.

Theorem 2.4 *The only connected graphs which have exactly one ept are* P_4 *, the path of length 3, and the join graph* $K_2 + \overline{K}_2 := K_4 - e$.

Proof Suppose that *G* is a (p, q) -graph having exactly one ept and that $p \geq 5$. Let $(a, b) \to (a', b')$ be the ept in *G*. Let $d(a, a') = t = d(b, b')$ and $P_a = (a = a_0, a_1, a_2, ..., a_t = a')$ and $P_b = (b = b_0, b_1, b_2, ..., b_t = b')$ be an a - a' geodesic and b - b' geodesic in *G*, respectively.

Case 1 P_a and P_b are disjoint.

Then, $t = 1$, for otherwise there would be at least one other ept as we shall show. Let $t \geq 2$. Then, for the largest integer r such that $a_r b_r \in E(G)$ we have $r < t$ and $(a_r, b_r) \to (a_{r+1}, b_{r+1})$ to be another ept in *G*, a contradiction to the hypothesis, and hence the claim. Now, *G* has at least one more vertex, say *v*. Since *G* is connected we may choose *v* to be in the neighbourhood of any of the four vertices a, b, a' and b'. Suppose that *v* is in the *open neighbourhood* $N(a) = \{u \in V : au \in E(G)\}$ of *a*. Then, *v* must be adjacent to b', for otherwise $(a, b) \to (v, b')$ would be another ept in *G*, a contradiction. But then $(a, v) \to (a', b')$ is another ept in *G*, again a contradiction to the hypothesis. Therefore, $v \notin N(a)$. Similarly, we can show that $v \notin N(b)$.

Hence, let $v \in N(a')$. Then, $vb \in E(G)$, for otherwise $(a, a') \to (b, v)$ would be another ept. But then $(v, b) \to (a', b')$ would be another ept in *G*, a contradiction to the hypothesis. Similar contradiction would arise if one assumed $v \in N(b')$.

This case is, thus, ruled out.

Case 2 P_a and P_b are not disjoint.

Then, let *i* and *j* be the first indices of vertex labels on P_a and P_b respectively so that $a_i = b_j$. Without loss of generality, let $i \leq j$. Then, either $i = j$ or $i = j - 1$. Suppose $i = 1$. Since $a' \neq b'$, being nonadjacent vertices in a loop-free graph, and P_a and P_b are geodesics there must exist a vertex *v* adjacent to a_1 but not to *a* or *b* on at least one of P_a and P_b . If *v* is on P_b , then $(a, a_1) \to (b, v)$ is another ept in *G*. If *v* is on P_a and if $j = 1$ then $(b, b_1) \to (a, v)$ is another ept in *G*. If *v* is on P_a and if $j = 2$ then $t \geq 3$. Consequently, if $bv \notin E(G)$ then $(b_1, b_2) \to (b, v)$ is another ept in *G;* and if $bv \in E(G)$, then $(b, v) \to (a, a_3)$ is another ept in *G*. In any case, we get a contradiction to the hypothesis. Hence, $i > 1$ whence $(a, a_1) \to (b, a_2)$ is another ept in *G*, again the same contradiction.

Thus, it follows that $p \leq 4$ whence $p = 4$ as *G* has already an ept. But, then, one can easily verify that P_4 and $K_4 - e$ are the only graphs having exactly one ept. □

In general, for a graph *G* there may exist two different parallel transformations P_1 and P_2 such that $P_1(G) \cong P_2(G)$. In simpler terms, two different particular problems suggest themselves here, viz.,

Problem 2.5 *Determine the graphs G such that all its ept graphs* $G(ab \to a'b') \cong G$ *are pairwise isomorphic. We call such graphs uniquely ept graphs.*

An interesting subproblem of Problem 2.5 is the following:

Problem 2.6 *Determine the uniquely ept graphs G for which* $G(ab \to a'b') \cong G$ *for every choice of an ept* $(a, b) \to (a', b')$ *of G. We call such graphs ept invariant graphs.*

The following is a `higher order' version of Problem 2.6.

Definition 2.7 A graph *G* is $pt(r)$-invariant if *r* is the least positive integer such that for every parallel transformation *P* of *G* having length *r* one has $P(G) \cong G$.

For example, every ept-invariant graph is pt(1)-invariant.

Problem 2.8 *Characterise* $pt(r)$ *-invariant graphs.*

Remark 2.9 It may be easily verified that the *pentagon* C_5 (i.e., the cycle of length 5) and the *hexagon* C_6 (i.e., the cycle of length 6) are uniquely ept graphs but not ept invariant. Clearly, *no cycle of length exceeding* 6 *is uniquely* ept. Hence, we shall regard graphs having at most one ept as *trivial* ept graphs.

Proposition 2.10 *The only connected trivial ept graphs are the complete graphs* K_n*, for any positive integer n, complete bipartite graphs* $K_{m,n}$ *for any positive integers m and n,* P_4 *and* $K_4 - e$.

An infinite class of uniquely ept graphs is provided by the complements of *n* copies of K_2 written nK_2, for $n \in \{3,4,5,...\}$.

Lemma 2.11 *If G is a uniquely ept graph then its girth (i.e., the smallest length of a cycle in G)* $g(G)$ *is at most 6.*

Proof Suppose that $g(G) = n \geq 7$ and let $C_n = (u_1, u_2, ..., u_n, u_1)$ be any cycle of length *n* in *G*. Consider the graphs $H = G(u_1 u_2 \to u_n u_3)$ and $K = G(u_1 u_2 \to u_{n-1} u_4)$. We claim that

H and K are nonisomorphic to end the proof. Since each of H and K is obtained by just one ept each, we see that $g(H) = n-2$ and $g(K) = n-4$ since each of these epts does not create a cycle shorter than $n-2$ and $n-4$ respectively, whence H and K cannot be isomorphic. □

Some more examples of uniquely ept graphs are $K_n \bullet K_n$ and $D_n(C_5)$ which is obtained by taking n copies of $C_5 := C_{i5} = (u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}, u_{i1})$, $i \in \{1,2,...,n\}$ and then, for each i, by joining every vertex u_{ir} in the i-th copy C_{i5} to the neighbours $u_{j(r-1)}$ and $u_{j(r+1)}$ of its copy u_{jr} in C_j where the right-hand subscripts are reduced modulo 5, for each $j \neq i$. (In fact, $D_n(G)$ may be defined analogously for any graph G).

References

[1] Berge, C. (1972). Graphs and Hypergraphs. North-Holland, Amsterdam.

[2] G.T. Chartrand, H. Hevia, E.B. Jarrett and M. Schultz, Subgraph distances in graphs defined by edge transfers. *Discrete Mathematics*, **170** (1–3), 1997.

[3] F. Harary, *Graph Theory*, Addison-Wesley. Reading, Massachusettes, 1969.

[4] T. Haynes, S.T. Hedetniemi and P.J. Slater. *Fundamentals of domination in graphs*, Marcel Dekker, New York, 1997.

[5] T. Haynes, S.T. Hedetniemi and P.J. Slater. *Domination in graphs – Advanced topics*, Marcel Dekker, New York, 1997.

[6] S.T. Hedetniemi, and R.C. Laskar (Eds.) Topics on domination, *Discrete Mathematics*, **86** (1–3), 1990.

[7] H.B. Walikar, B.D. Acharya and E. Sampathkumar, Recent developments in the theory of domination in graphs, *MRI Lecture Notes in Mathematics No. 1*, The Mehta Research Institute of Mathematics & Mathematical Physics, Allahabad, 1979.

[8] B. Wu and X. Guo, Diameters of jump graphs and self-complementary jump graphs, *Graph Theory Notes of New York*, **XI** (2001), 31–34.