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TOTAL AND PAIRED-DOMINATION NUMBERS OF A TREE

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Abstract

A set S of vertices is a total dominating set of a graph G if every vertex of G is adjacent to some vertex in S. A paired-dominating set of G is a dominating set whose induced subgraph has a perfect matching. The minimum cardinality of a total dominating set (respectively, a paired-dominating set) is the total domination number $\gamma_t(G)$ (respectively, the paired-domination number $\gamma_{pr}(G)$). We give sharp upper bounds on the total and paired-domination numbers of trees that improve known bounds for some cases. In particular, we show that for a tree T with order $n \geq 3$ and s support vertices, $\gamma_{pr}(T) \leqslant \gamma_t(T) + s - 1$, $\gamma_t(T) \leq (n+s)/2$, and $\gamma_{pr}(T) \leq (n+2s-1)/2$.

Keywords: Total Domination, Paired-domination.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. We begin with some terminology. For a vertex v of a graph G, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of v denoted by $\deg_G(v)$, is the number of vertices adjacent to v in G. A vertex of degree one is called a *leaf* and its neighbor is a *support vertex*. A *double star* is a tree with exactly two support vertices. The *corona* of a graph G is the graph formed from a copy of G by attaching for each $v \in V$, a new vertex v' and edge vv'. In general, the k*-corona* of a graph G is the graph of order $k|V(G)|$ obtained from G by adding a path of length k to each vertex of G so that the resulting paths are vertex disjoint.

The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A subset S of V is a *paired-dominating set* of G abbreviated PDS, if S is a dominating set and the subgraph induced by the vertices of S contains a perfect matching. The *paired-domination number* $\gamma_{pr}(G)$ is the minimum cardinality of a paired-dominating set. If S is a paired-dominating set with a perfect matching M, then two vertices v_j and v_k are said to be *paired in* S if the edge $v_jv_k \in M$. Paired-domination was introduced by Haynes and Slater [10] and is studied, for example, in [1, 3, 4, 6, 11, 12, 13, 14, 15].

A set $S \subseteq V$ is a *total dominating set* abbreviated TDS, if every vertex in V is adjacent to a vertex in S. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. Since every paired-dominating set is a total dominating set, $\gamma(G) \leq \gamma_t(G) \leq \gamma_{pr}(G)$ for every graph with no isolated vertices. We call a paired (respectively, total) dominating set of minimum cardinality a $\gamma_{pr}(G)$-*set* (respectively, $\gamma_t(G)$-*set*). For a comprehensive treatment of domination in graphs, see [7, 8].

The following upper bounds are known.

Theorem 1. (Cockayne, Dawes, and Hedetniemi [5]) *For any connected graph G of order $n \geq 3$, $\gamma_t(G) \leq 2n/3$.*

Theorem 2. (Haynes and Slater [10]) *For any graph G without isolated vertices, $\gamma_{pr}(G) \leq 2\gamma(G)$.*

Brigham, Carrington, and Vitray [2] characterized the connected graphs that attain the bound of Theorem 1 as follows.

Theorem 3. (Brigham, et al. [2]) *Let G be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = 2n/3$ if and only if G is C_3, C_6, or the 2-corona of some connected graph.*

In this note we show for a tree T of order $n \geq 3$, that $\gamma_{pr}(T) \leqslant \gamma_t(T) + s - 1$, $\gamma_t(T) \leq (n+s)/2$, and $\gamma_{pr}(T) \leq (n+2s-1)/2$, where s is the number of support vertices of T. These bounds improve the bounds of Theorems 1 and 2 for some cases.

2. Upper Bounds

Before presenting our main results, we make a couple of straightforward observations.

Observation 4. *If v is a support vertex of a graph G, then v is in every $\gamma_{pr}(G)$-set and in every $\gamma_t(G)$-set.*

Observation 5. *For any connected graph G with diameter at least three, there exists a $\gamma_t(G)$-set that contains no leaves of G.*

For a vertex v in a rooted tree T, we denote by T_v the subtree of T induced by v and its descendants. We now present the upper bounds.

Theorem 6. *If T is a tree of order at least three with s support vertices, then*

$$\gamma_t(T) \leqslant \gamma_{pr}(T) \leqslant \gamma_t(T) + s - 1.$$

Proof. The lower bound follows from the definition. To establish the upper bound, we proceed by induction on the order of T. It is easy to see that the inequality holds for $n \in \{3, 4, 5\}$ establishing the base case .

Let $n \geqslant 6$, and assume that for any tree T' of order $3 \leq n' < n$ having s' support vertices, $\gamma_{pr}(T') \leqslant \gamma_t(T') + s' - 1$. Let T be a tree of order n with s support vertices, and let S and D be a $\gamma_{pr}(T)$-set and a $\gamma_t(T)$-set, respectively.

If T is a star $K_{1,n-1}$ or a double star, then $\gamma_{pr}(T) \leqslant \gamma_t(T) + s - 1$, since $\gamma_{pr}(T) = \gamma_t(T) = 2$ and $s - 1 \geqslant 0$. Hence we may assume $\operatorname{diam}(T) \geq 4$.

If any support vertex, say t, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to t. It is a routine matter to check that $\gamma_{pr}(T) = \gamma_{pr}(T')$, $\gamma_t(T) = \gamma_t(T')$ and $s' = s$. Applying the inductive hypothesis to T', we obtain the desired result. Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf.

We now root the tree at a vertex r of maximum eccentricity $\operatorname{diam}(T) \geq 4$. Let u be a support vertex at maximum distance from r and v be the parent of u in the rooted tree. Then $\deg_T(u) = 2$. Let w be the parent of v and x be the parent of w. By our choice of u, every child of v is either a leaf or a support vertex of degree two. Consider the following three cases:

Case 1. v has a child besides u, say y, that is a support vertex. By Observation 4, u and y are in S. Now at most one of u or y is paired v implying, without loss of generality, that u is paired with its leaf u'. Let $T' = T - T_u$. Since $y \in S$, $S \cap T'$ is a PDS of T', and so $\gamma_{pr}(T') \leqslant \gamma_{pr}(T) - 2$. On the other hand, every $\gamma_{pr}(T')$-set can be extended to a PDS of T by adding the vertices u and u'. Thus, $\gamma_{pr}(T) \leqslant \gamma_{pr}(T') + 2$, and hence, $\gamma_{pr}(T) = \gamma_{pr}(T') + 2$. By Observations 4 and 5, we know that there exists a $\gamma_t(T')$-set D' containing v and y. Hence, $D' \cup \{u\}$ is a TDS of T and $\gamma_t(T) \leq \gamma_t(T') + 1$. Since every $\gamma_t(T)$-set contains v, y, and u, it follows that $D \cap V(T')$ is a TDS of T', and so, $\gamma_t(T') \leq \gamma_t(T) - 1$. Thus, $\gamma_t(T') = \gamma_t(T) - 1$ and $s' = s - 1$. Applying the inductive hypothesis to T', we have $\gamma_{pr}(T') \leqslant \gamma_t(T') + s' - 1$. Therefore, $\gamma_{pr}(T) - 2 \leqslant (\gamma_t(T) - 1) + (s - 1) - 1$, and hence, $\gamma_{pr}(T) \leqslant \gamma_t(T) + s - 1$.

Case 2. v is a support vertex and has no child besides u of degree two. Let $T' = T - T_v$. If T' is reduced to a path P_2, then T is a corona and the result is valid. Thus assume that T' has order at least three. Then $\gamma_{pr}(T) \leqslant \gamma_{pr}(T') + 2$ and $s - 2 \leqslant s' \leqslant s - 1$. By Observation 4, u and v are in D. Now, if $(N(w) - \{v\}) \cap D \neq \emptyset$, then $D - \{u, v\}$ is a TDS of T'. If $(N(w) - \{v\}) \cap D = \emptyset$, then $(D \cup \{x\}) - \{u, v\}$ is a TDS of T' (since x is adjacent to at least one vertex of D). Thus, $\gamma_t(T') \leqslant \gamma_t(T) - 1$. Applying the inductive hypothesis to T', we have $\gamma_{pr}(T') \leqslant \gamma_t(T') + s' - 1$. Hence, $\gamma_{pr}(T) - 2 \leqslant (\gamma_t(T) - 1) + (s - 1) - 1$, and therefore, $\gamma_{pr}(T) \leqslant \gamma_t(T) + s - 1$.

Case 3. v has no child besides u, that is, $\deg_T(v) = 2$. Suppose first that $\deg_T(w) \geqslant 3$, and let $T' = T - T_v$. Then $s' = s - 1$. Any $\gamma_{pr}(T')$-set can be extended to PDS of T by adding u and v, and so, $\gamma_{pr}(T) \leqslant \gamma_{pr}(T') + 2$. Furthermore, by Observations 4 and 5, we may assume that u and $v \in D$. Now, if $(N(w) - \{v\}) \cap D \neq \emptyset$, then $D - \{u, v\}$ is a TDS of T'. If $(N(w) - \{v\}) \cap D = \emptyset$, then $(D \cup \{x\}) - \{u, v\}$ is a TDS of T'. Thus, $\gamma_t(T') \leqslant \gamma_t(T) - 1$. Applying the inductive hypothesis to T', we obtain the desired inequality.

Next suppose that $\deg_T(w) = 2$, and let $T' = T - T_w$. If $T' \in \{P_1, P_2\}$, then $T \in \{P_5, P_6\}$ and the inequality holds. Thus assume that T' has order at least three. Then $s - 1 \leqslant s' \leqslant s$. Since any $\gamma_{pr}(T')$-set can be extended to a $\gamma_{pr}(T)$-set by adding u and v, $\gamma_{pr}(T) \leqslant \gamma_{pr}(T') + 2$. Also by Observations 4 and 5, we may assume that u and v are in D. Furthermore, we can choose D such that $w \notin D$. To see this assume that $w \in D$. If $x \in D$, then the minimality of D implies that

$(N(x) - \{w\}) \cap D = \emptyset$. Hence we can substitute a neighbor of x for w in D. If $x \notin D$, then $(N(x) - \{w\}) \cap D = \emptyset$, for otherwise, $D - \{w\}$ is a TDS of T with cardinality less than $\gamma(T)$, a contradiction. Again we can substitute a neighbor of x for w in D. Hence, $D \cap V(T')$ is a TDS of T', and so $\gamma_t(T') \leq \gamma_t(T) - 2$. Applying the inductive hypothesis to T', we obtain $\gamma_{pr}(T) \leqslant \gamma_t(T) + s - 1$. □

From Theorem 2, we have that $\gamma_{pr}(T) \leqslant 2\gamma(T)$ for any non-trivial tree. This upper bound and the upper bound of Theorem 6 are sharp. For example, both upper bounds are attained by stars and subdivided stars where each edge is subdivided exactly once. The trees T having $\gamma_{pr}(T) = 2\gamma(T)$ are characterized in [9]. We note that $2\gamma(T)$ and $\gamma_t(T) + s - 1$ are incomparable. First, the difference $2\gamma(T) - (\gamma_t(T) + s - 1)$ can be arbitrarily large as can be seen with a caterpillar T, where T has $k \geq 2$ support vertices each adjacent to exactly one leaf and the distance between every pair of consecutive support vertices is four. Then $\gamma(T) = 2k - 1$, $\gamma_t(T) = 2k$, and $s = k$. In fact, the difference $2\gamma(T) - (\gamma_t(T) + s - 1)$ can be arbitrarily large even when $\gamma_{pr}(T) = \gamma_t(T) + s - 1$ as can be seen with the following tree H_k. Let H_k, $k \geq 2$, be the tree formed from a star $K_{1,k+1}$ by subdividing one of its edges exactly once and each of the other $k + 1$ edges thirteen times. (Alternately, H_k is formed from $P_2 \cup kP_{14}$ by adding a new vertex v and $k + 1$ edges such that v is adjacent to a leaf of each of the paths.) For example, the tree H_3 is illustrated in Figure 1. Then $\gamma(H_k) = 5k + 1$, $s = k + 1$, $\gamma_t(H_k) = 7k + 2$, and $\gamma_{pr}(H_k) = 8k + 2 = \gamma_t(H_k) + s - 1 < 2\gamma(H_k)$.

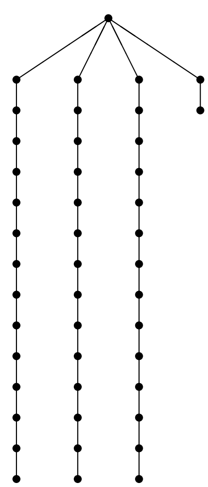

Figure 1: The tree H_3.

On the other hand, the difference $(\gamma_t(T) + s - 1) - 2\gamma(T)$ can also be infinitely large even when $\gamma_{pr}(T) = 2\gamma(T)$. Let T_k for $k \geq 1$ be the tree formed from a path on $3k - 1$ vertices labeled $v_1, v_2, ..., v_{3k-1}$ by attaching a path of length one to each vertex labeled v_i where $i \equiv 1 \ (mod\ 3)$ and a path of length three to each vertex v_j where $j \equiv 2 \ (mod\ 3)$. See Figure 2 for an example of T_4. For T_k,

$\gamma_{pr}(T_k) = 4k = 2\gamma(T_k)$, $\gamma_t(T_k) = 4k$, and $s = 2k$. Thus, $(\gamma_t(T_k) + s - 1) - 2\gamma(T_k) = 2k - 1 < \gamma_t(T_k) + s - 1$.

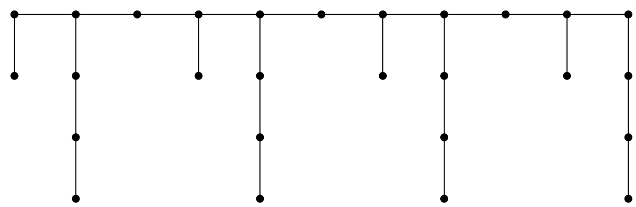

Figure 2: The tree T_4 .

Theorem 7. *If T is a tree of order $n \geqslant 3$ with s support vertices, then*

(*a*) $\gamma_t(T) \leqslant (n + s)/2$,

(*b*) $\gamma_{pr}(T) \leq (n + 2s - 1)/2$,

and these bounds are sharp.

Proof. We proceed by induction on the order n . It is a routine matter to check that the result is valid if $diam(T) \in \{2, 3\}$, establishing the base case.

Assume that every tree T' of order $3 \leq n' < n$ with s' support vertices satisfies $\gamma_t(T') \leqslant (n' + s')/2$ and $\gamma_{pr}(T') \leq (n' + 2s' - 1)/2$. Let T be a tree of order n with s support vertices.

We now root T at a vertex r of maximum eccentricity $diam(T) \geqslant 4$. Let u be a support vertex at maximum distance from r and v its parent in the rooted tree. We consider the following two cases.

Case 1. $\deg_T(v) \geqslant 3$. Then either v is a support vertex of T or v has a child besides u as a support vertex. Let $T' = T - T_u$. Clearly, $n' = n - (|N[u]| - 1) \geqslant 3$, and $s' = s - 1$. Observations 4 and 5 imply that there is a $\gamma_t(T')$ -set S' containing v . Thus, $S' \cup \{u\}$ is a total dominating set of T , implying that $\gamma_t(T) \leqslant \gamma_t(T') + 1$. Also, any $\gamma_{pr}(T')$ -set can be extended to a PDS of T by adding u and an adjacent leaf, so $\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2$. Applying the inductive hypothesis to T' , it follows that

$$\gamma_t(T) \leqslant \gamma_t(T') + 1 \leqslant (n' + s')/2 + 1 \leqslant (n + s)/2$$

and

$$\gamma_{pr}(T) \leqslant \gamma_{pr}(T') + 2 \leqslant (n' + 2s' - 1)/2 + 2 \leqslant (n + 2s - 1)/2.$$

Case 2. $\deg_T(v) = 2$. Since $diam(T) \geqslant 4$, let w be the parent of v in the rooted tree.

First, assume that $\deg_T(w) \geqslant 3$. Let $T' = T - T_v$. Then $n' = n - (|N[u]|) \geqslant 3$, and $s' = s - 1$. Also every $\gamma_t(T')$-set (respectively, $\gamma_{pr}(T')$-set) can be extended to a TDS (respectively, PDS) by adding the vertices u and v, and hence, $\gamma_t(T) \leqslant \gamma_t(T') + 2$ and $\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2$. Applying the inductive hypothesis to T', we obtain

$$\gamma_t(T) \leqslant \gamma_t(T') + 2 \leqslant (n' + s')/2 + 2 \leqslant (n + s)/2$$

and

$$\gamma_{pr}(T) \leqslant \gamma_{pr}(T') + 2 \leqslant (n' + 2s' - 1)/2 + 2 \leqslant (n + 2s - 1)/2.$$

Second, suppose that $\deg_T(w) = 2$. Let $T' = T - T_w$. It can be seen that the result is valid if T' has order one or two. So assume that $n' = n - (|N[v]| + 1) \geqslant 3$. Then $s' \leqslant s$, $\gamma_t(T) \leqslant \gamma_t(T') + 2$, and $\gamma_{pr}(T) \leqslant \gamma_{pr}(T') + 2$. Applying the inductive hypothesis to T', we obtain the desired inequality.

That the bound of (a) is sharp can be seen by the 2-corona of a tree of order k, where $n = 3k$, $s = k$, and $\gamma_t(T) = 2k = (n + s)/2$. The bound of (b) is obtained by the trees T formed from a star $K_{1,k}$ by subdividing each edge of the star exactly five times. Here, $n = 6k + 1$, $s = k$, and $\gamma_{pr}(T) = 4k = (n + 2s - 1)/2$. □

We note the upper bounds on $\gamma_t(T)$ of Theorems 1 and 7(a) are both sharp for the 2-corona of a tree. By Theorem 3, these are the only trees achieving the bound of Theorem 1. Our new bound is sharp for other families and is an improvement over the known bound on the total domination number when $s < n/3$.

Also, note that the bound on $\gamma_{pr}(T)$ of Theorem 7(b) beats the known upper bound of $2\gamma(T)$ of Theorem 2 in some cases. For example, the bound of Theorem 7(b) is sharp for the trees H_k shown in Figure 1 where the difference in the two bounds is arbitrarily large.

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