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FURTHER RESULTS ON DEGREE SETS FOR GRAPHS

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Abstract

The degree set $D(G)$ of a graph G is the set of degrees of G. For a finite, non-empty set S of non-negative integers, we show that there exists a disconnected graph G such that $D(G) = S$, and determine the minimum order of such graphs. In addition, we investigate degree sets for k-connected graphs, k-edge connected graphs, unicyclic graphs and maximal k-degenerate graphs.

Keywords: degree set, k-connected graphs, k-edge connected graphs, unicyclic graphs,maximal k-degenerate graphs .

1.Introduction

All graphs considered here are finite, undirected, without loops and without multiple edges. We denote the vertex-set, and the edge-set of a graph G by $V(G)$ and $E(G)$, respectively. We let $\overline{G}$ denote the complement of G. The *girth* of G is the length of a shortest cycle in G. For any two disjoint graphs G and H, $G \cup H,$ $G + H,$ and $G \circ H$ denote the *union*, the *sum* and the *corona* of G and H, respectively, as defined in [1]. For any connected graph G, we write kG for the graph with k components, each component being isomorphic to G. The *degree set* $D(G)$ of a graph G is the set of degrees of the vertices of G. For a finite, nonempty set S of non-negative integers, we shall write $\mu(S)$ to represent the minimum order of a graph G such that $D(G) = S$. If $S = \{a_1, a_2, \dots, a_n\}$, where $n \geq 1$ and $0 \leq a_1 < a_2 < \dots < a_n$, then it will be convenient to write $\mu(S)$ as simply $\mu(a_1, a_2, \dots, a_n)$. Since every graph which contains a vertex of degree a_n has order at least $a_n + 1$, it follows that $\mu(a_1, a_2, \dots, a_n) \geq a_n + 1$.

Kapoor et al [2] studied degree sets for connected graphs, trees, planar and outerplanar graphs. In this paper, we show that for a given finite nonempty set S of non-negative integers, there exists a disconnected graph G such that $D(G) = S$, and

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also the minimum order of such a graph is determined. Moreover, we study the degree sets for k-connected graphs, k-edge connected graphs, unicyclic graphs and maximal k-degenerate graphs and obtain their minimum orders.

2. Disconnected graphs

Let us recall the following result of [2] for our later use.

Theorem 2.1. *For every set $S = \{a_1, a_2, \ldots, a_n\}, n \geq 1$, of integers with $1 \leq a_1 < a_2 < \ldots < a_n$, there exists a connected graph G of order $a_n + 1$ such that $D(G) = S$.*

On the basis of above result, we introduce the following definition. Let $S = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, be a set of positive integers such that $0 \leq a_1 < a_2 < \ldots < a_n$. Then $\mu_{dc}(S) = \mu_{dc}(a_1, a_2, \ldots, a_n)$ denotes the minimum order of a disconnected graph G for which $D(G) = S$. The value of $\mu_{dc}(S)$ for $n = 1$ is stated as follows:

Theorem 2.2. *Let a be a non-negative integer. Then $\mu_{dc}(a) = 2(a + 1)$.*

Theorem 2.3. *For every set $S = \{a_1, a_2, \ldots, a_n\}, n \geq 2$, of non-negative integers with $a_1 < a_2 < \ldots < a_n$, there exists a disconnected graph G such that $D(G) = S$. Moreover, $\mu_{dc}(S) = a_1 + a_n + 2$.*

Proof. For $n = 2$, we observe that the graph $K_{a_1+1} \cup K_{a_2+1}$ is disconnected of the least order such that its degree set is S, so that $\mu_{dc}(S) = a_1 + a_2 + 2$. For $n = 3$, the graph $F = K_{a_1+1} \cup \{K_{a_2} + \overline{K}_{a_3-a_2+1}\}$ is disconnected and satisfy the required property. Moreover, F has the smallest possible order a_1+a_3+2, so we conclude that $\mu_{dc}(S) = a_1 + a_3 + 2$. For $n \geq 4$, in view of Theorem 2.1, there exists a connected graph H of the minimum order $\mu(a_3 - a_2, a_4 - a_2, \ldots, a_{n-1} - a_2) = a_{n-1} - a_2 + 1$ such that $D(H) = \{a_3 - a_2, a_4 - a_2, \ldots, a_{n-1} - a_2\}$. Consequently, the graph $G = K_{a_1+1} \cup \{K_{a_2} + (\overline{K}_{a_n-a_{n-1}} \cup H)\}$ is disconnected such that $D(G) = S$. Since G has order $a_1 + a_n + 2$ and is the smallest desired graph satisfying the required property, it follows that $\mu_{dc}(a_1, a_2, \ldots, a_n) = a_1 + a_n + 2$. □

3. k-Connected (k-edge connected) graphs

We now consider the important subclasses of graphs, namely k-connected graphs and k-edge connected graphs, and we present results dealing with the degree sets and their minimum orders. Recall that a graph G is *k-connected* (resp. *k-edge connected*) if the removal of less than k vertices (resp. k edges) from G results in a connected graph.

Theorem 3.1. *Let $S = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, be a set of integers with $1 \leq a_1 < a_2 < \ldots < a_n$. Then there exists a k-connected (resp. k-edge connected) graph G with $D(G) = S$ if and only if $a_1 \geq k$. Moreover, if $a_1 \geq k$, then the minimum order of a k-connected (resp. k-edge connected) graph G with $D(G) = S$ is $a_n + 1$.*

Proof. Suppose G is a k-connected (resp. k-edge connected) graph such that $D(G) = S$. Then $\delta(G) \geq k$, and hence $a_1 \geq k$.

Conversely, let $a_1 \geq k$. We give an example of a k-connected (resp. k-edge connected) graph G of order $a_n + 1$ such that $D(G) = S$. To follow the inductive process, we first observe that for $n = 1$, the graph K_{a_1+1} has the required property. Also for $n = 2$, the k-connected and k-edge connected graph $K_{a_1} + \overline{K}_{a_2-a_1+1}$ satisfies again the desired property. For $n \geq 3$, in view of Theorem 2.1, there exists a connected graph H of order $a_{n-1} - a_1 + 1$ such that $D(H) = \{a_2 - a_1, a_3 - a_1, \ldots, a_{n-1} - a_1\}$. Consequently, the graph $G = K_{a_1} + \{\overline{K}_{a_n - a_{n-1}} \cup H\}$ is k-connected and k-edge connected with $D(G) = S$ and has order $a_n + 1$. □

4. Unicyclic graphs

We now turn our attention to the unicyclic graphs (i.e., connected graphs containing precisely one cycle).

Theorem 4.1. *For every set $S = \{a_1, a_2\}$ of integers with $1 \leq a_1 < a_2$, there exists an unicyclic graph G with $D(G) = S$ if and only if $a_1 = 1$ and $a_2 \geq 3$. Furthermore, if $a_1 = 1$, and $a_2 \geq 3$, the minimum order of an unicyclic graph G with $D(G) = S$ is $3(a_2 - 1)$.*

Proof. Obvious and hence it is omitted. □

Theorem 4.2. *Let $S = \{a_1, a_2, \ldots, a_n\}, n \geq 3$, be a set of integers with $1 \leq a_1 < a_2 < \ldots < a_n$. Then there exists an unicyclic graph G with $D(G) = S$ if and only if $a_1 = 1$.*

Proof. Necessary part is obvious.

Sufficiency: Suppose $S = \{a_1, a_2, \ldots, a_n\}, n \geq 3$, is a set of positive integers with $1 = a_1 < a_2 < \ldots < a_n$. If $n = 3$, then we shall construct an unicyclic graph G with the girth 3 and $D(G) = S$ as follows: Consider a triangle K_3 whose vertex set is $\{v_1, v_2, v_3\}$, and then adjoin $a_2 - 2, a_3 - 2$ and $a_2 - 2$ end-edges, respectively, at v_1, v_2 and v_3. Obviously, $\mu_{uc}(S) = 2a_2 + a_3 - 3$. For $n \geq 4$, an unicyclic (p, q) graph G with $D(G) = S$ contains at least one vertex of degree a_i for $2 \leq i \leq n$, and the remaining $p - (n - 1)$ vertices are each of degree at least $1 = a_1$, and we have

$$\begin{aligned} 2p &= 2q \\ &\geq p - (n-1) + \sum_{i=2}^{n} a_i \\ \text{or } p &\geq \sum_{i=1}^{n} (a_i - 1) \end{aligned}$$

Hence, $\mu_{uc}(S) \geq \sum_{i=1}^{n} (a_i - 1)$

We now give our unicyclic graphs G_n having this minimum number of vertices with $D(G_n) = S$. If $n = 4$, then consider a triangle K_3 whose vertex set is

$\{v_1, v_2, v_3\}$, and then adjoin $a_2 - 2, a_3 - 2$ and $a_4 - 2$ end-edges, respectively, at v_1, v_2 and v_3. Since G_4 is the smallest required graph with $D(G_4) = S$, we have

$$\begin{aligned}\mu_{uc}(S) &= 3 + (a_2 - 2) + (a_3 - 2) + (a_4 - 2)\\ &= \sum_{i=1}^{4}(a_i - 1).\end{aligned}$$

Next, we proceed by induction on n. Assume that the result is true for all $n \le m$, where $m \ge 4$. Let $n = m + 1$. Then $S = \{a_1, a_2, \dots, a_{m+1}\} = S' \cup \{a_{m+1}\}$, where $S' = \{a_1, a_2, \dots, a_m\}$. By induction hypothesis, there exists an unicyclic graph G_m with $\mu_{uc}(S') = \sum_{i=1}^{m}(a_i - 1)$ and degree set S'. The unicyclic graph G_{m+1} is obtained from G_m by adjoining $a_{m+1} - 1$ end-edges at a vertex of degree 1 of G_m has order $\sum_{i=1}^{m+1}(a_i - 1)$ and degree set S. Thus the result holds for all n. □

In view of the above theorem, we introduce the following notion. Let $S = \{a_1, a_2, \dots, a_n\},\ n \ge 3$, be a set of integers with $1 = a_1 < a_2 < \dots < a_n$. Then $\mu_{uc}(S)$ denotes the minimum order of an unicyclic graph G for which $D(G) = S$.

In Theorem 4.2, the value of $\mu_{uc}(S)$ is as follows:

$$\mu_{uc}(S) = \begin{cases} 2a_2 + a_n - 3 & \text{for} \quad n = 3 \\ \sum_{i=1}^{n}(a_i - 1) & \text{for} \quad n \ge 4 \end{cases}$$

5. k-Degenerate graphs

In 1970, Lick and White [3] introduced the concept of k-degenerate graphs, $k \ge 1$. A graph G is said to be *k-degenerate* if every induced subgraph H of G satisfies the inequality $\delta(H) \le k$.

In the following, we first state a result of [2] for immediate use.

Theorem 5.1. *Let $S = \{a_1, a_2, \dots, a_n\},\ n \ge 1$, be a set of positive integers with $a_1 < a_2 < \dots < a_n$. Then there exists a planar graph G with $D(G) = S$ if and only if $a_1 \le 5$.*

Theorem 5.2. *Let $S = \{a_1, a_2, \dots, a_n\},\ n \ge 1$, be a set of positive integers with $a_1 < a_2 < \dots < a_n$. Then there exists a connected, k-degenerate graph $G, k \ge 5$, such that $D(G) = S$ if and only if $a_1 \le k$.*

Proof. Necessity is obvious.

Sufficiency: If $a_1 = k$ for $k = 5$, then by Theorem 5.1, there exists a connected planar graph G which is 5-degenerate such that $D(G) = S$. Now, for $a_1 = k = i + 1$ for $i = 5, 6, \dots, k - 1$, we construct a connected, k-degenerate graph G_k with $D(G_k) = S$ by the following recursive method. Let H_i be a connected, i-degenerate graph for $i = 5$ such that $D(H_i) = \{a_1, a_2, \dots, a_n\}$, where $a_1 = i$ and $a_2 > i + 1$. Denote the vertices of degree a_1 in H_i by $u_1, u_2, \dots, u_m,\ m \ge 1$.

Let H_i' be another identical copy of H_i so that the vertex u_i' of H_i' corresponds to the vertex u_i of H_i. Then a connected $(i+1)$-degenerate graph G_{i+1} results by joining u_j and u_j' for each $j = 1, 2, \ldots, m$. Note that the minimum degree among the vertices of G_{i+1} is $a_1 = i + 1$. □

In view of Theorem 5.2, we can make the following definition. Let $S = \{a_1, a_2, \ldots, a_n\},\ n \geq 1$, be a set of positive integers with $a_1 < a_2 < \ldots < a_n$ and $a_1 \leq k$ and $k \geq 5$. Then $\mu_{kd}(S)$ denotes the minimum order of a connected, k- degenerate graph G for which $D(G) = S$. For $n = 1$, $\mu_{kd}(S) = a_1 + 1$ for $a_1 \leq k$, since the (a_1+1)-clique has the degree set $\{a_1\}$. However, for an arbitrary set S of positive integers, it appears to be hard to assertain the value of $\mu_{kd}(S)$.

A k-degenerate graph G is called *maximal k-degenerate*, if for every edge $e \in E(\overline{G}), G + e$ is not k-degenerate. We recall two lemmas of [4] for our later use.

Lemma 5.3. *Let G be a maximal k-degenerate graph of order $p \geq k + 1$. Then G is k-connected and it has $kp - \frac{1}{2}k(k+1)$ edges.*

Lemma 5.4. *Let G be a graph of order at least $k + 1$. Then G is maximal k-degenerate if and only if G contains a vertex v of degree k and $G - v$ is maximal k-degenerate.*

Theorem 5.5. *Let $S = \{a_1, \ldots, a_n\},\ 1 \leq n \leq 2$, be a set of positive integers with $a_1 < a_2$. Then there exists a maximal k-degenerate graph G with $D(G) = S$ if and only if $a_1 = k$. Moreover, if $a_1 = k$, then the minimum order of a maximal k-degenerate graph G with $D(G) = S$ is $a_n + 1$.*

Proof. Necessary part follows from Lemma 5.3 and the definition of k-degenerate graphs.

Sufficiency: Suppose $S = \{a_1, \ldots, a_n\}, 1 \leq n \leq 2$, is a set of positive integers such that $k = a_1 < a_2$. The graphs K_{k+1} and $K_k + \overline{K}_{a_n - k+1}$ are maximal k-degenerate graphs for $n = 1$ and $n = 2$, respectively, and have required property.

Now, suppose G is any maximal k-degenerate graph with $D(G) = \{a_1, \ldots, a_n\}$ with $a_1 = k$. Since $\Delta(G) = a_n$, G has at least $a_n + 1$ vertices. The maximal k-degenerate graphs given above have this minimum number of vertices. □

Theorem 5.6. *Let $S = \{a_1, a_2, \ldots, a_n\},\ n \geq 3$, be a set of positive integers such that $a_1 < a_2 < \ldots < a_n$. Then there exists a maximal k-degenerate graph G with $D(G) = S$ if and only if $a_1 = k$ and $a_2 \geq 2k - 1$. Moreover, if $a_1 = k$ and $a_2 \geq 2k - 1$, then the minimum order of a maximal k-degenerate graph G with $D(G) = S$ is $\sum_{i=1}^{n}(a_i - k) + k + 1$.*

Proof. Suppose G is a maximal k-degenerate graph with $D(G) = S$. Then G contains a vertex of degree k. Hence, $a_1 = k$. With the aid of Lemma 5.4, it is easy to see that a graph $G(a_2) = K_{a_1} + \overline{K}_{a_2 - a_1 + 1}$ having degree set $\{a_1, a_2\}$ is maximal k-degenerate, and contains at least k vertices of degree k if and only if $a_1 = k$ and $a_2 \geq 2k - 1$. We proceed by induction on n. For $n = 3$, we now construct a maximal k-degenerate graph $G(a_3)$ having degree set S by the following recursive method: $G(a_3)$ is constructed from $G(a_2)$ by adding $a_3 - k$ new vertices, and joining each of these to every vertex $u_i, 1 \leq i \leq k$, of degree k in $G(a_2)$. Clearly, all $a_3 - k$ newly added vertices are of degree k in $G(a_3)$ and also $a_3 \geq 2k$. Consequently, there are at least k vertices of degree k in $G(a_3)$. Let $n \geq 4$. Assume that the result is true for all $m < n$. Next, form a maximal k-degenerate graph $G(a_n)$ from $G(a_{n-1})$ by adding $a_n - k$ new vertices, and joining each of these vertices to every vertex $v_i, 1 \leq i \leq k$, of degree k in $G(a_{n-1})$. Hence the result follows by induction for all values of n. Further, the maximal k-degenerate graph $G(a_n)$ given above has the minimum number of vertices $\sum_{i=1}^{n}(a_i - k) + k + 1$. □

References

[1] Harary, F., Graph Theory, Addison-Wesley, Reading Mass, 1969.

[2] Kapoor, S.F., Polimeni, A.D., and Wall, C.W., Degree sets for graphs, *Fundamenta Mathematicae*, XCV (1977) 189–194.

[3] Lick, D.R., and White, A.T., k-degenerate graphs, *Canad. J. Math.*, **22** (1970) 1082–1096.

[4] Mitchem, J., Maximal k-degenerate graphs,*Utilitas Mathematica*, **11** (1977) 101–106.