

## HEAVY CYCLES IN HAMILTONIAN WEIGHTED GRAPHS

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### Abstract

Let  $G$  be a 2-connected weighted graph such that the minimum weighted degree is at least  $d$ . In [1], Bondy and Fan proved that either  $G$  contains a cycle of weight at least  $2d$  or every heaviest cycle in  $G$  is a hamiltonian cycle. If  $G$  is not hamiltonian, this theorem implies the existence of a cycle of weight at least  $2d$ , but in case of  $G$  is hamiltonian we cannot decide whether  $G$  has a heavy cycle or not. In this paper, we prove that if  $G$  is triangle-free, then  $G$  has a cycle of weight at least  $2d$  even in case of  $G$  is hamiltonian.

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### 1. Introduction

We only consider undirected graphs which have no loops or multiple edges. Let  $V(G)$  and  $E(G)$  denote the set of vertices and the set of edges of a graph  $G$ , respectively. A weighted graph is one in which every edge  $e$  is assigned a nonnegative real number  $w(e)$ , called the *weight* of  $e$ . For a subgraph  $H$  of  $G$ , the *weight* of  $H$  is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

For each vertex  $v \in V(G)$ ,  $N_G(v)$  is the set, and  $d_G(v)$  the number, of neighbors of  $v$  in  $G$ . We define the *weighted degree* of  $v$  in  $G$  by

$$d_G^w(v) = \sum_{u \in N_G(v)} w(uv).$$

Let  $x$  and  $y$  be distinct vertices in  $G$ . We define

$$\varepsilon_G(xy) = \begin{cases} 0 & \text{if } xy \notin E(G) \\ 1 & \text{if } xy \in E(G). \end{cases}$$

When no confusion occurs, we denote  $N_G(v)$ ,  $d_G(v)$ ,  $d_G^w(v)$  and  $\varepsilon_G(xy)$  by  $N(v)$ ,  $d(v)$ ,  $d^w(v)$  and  $\varepsilon(xy)$ , respectively.

A cycle which contains all the vertices of a graph is called a *hamiltonian cycle*. If a graph  $G$  has a hamiltonian cycle, then we call  $G$  *hamiltonian*. A *triangle-free* graph is one which contains no cycle of length 3. Moreover, we call a path  $P$  a *longest heaviest path* of  $G$  if

- (i)  $w(P)$  is maximum, and
- (ii)  $P$  is a longest path of  $G$  subject to (i).

In [1], Bondy and Fan proved the following theorem.

**Theorem 0.1 (Bondy and Fan [1]).** *Let  $G$  be a 2-connected weighted graph and let  $d$  be a nonnegative real number. If  $d^w(v) \geq d$  for every vertex  $v$  in  $G$ , then*

- (a)  $G$  has a cycle of weight at least  $2d$ , or
- (b) every heaviest cycle in  $G$  is a hamiltonian cycle.

If we consider the weighted complete graph in which every edge has weight 1, we know that conclusion (b) of Theorem 0.1 cannot be dropped. However, there are a lot of graphs in which both (a) and (b) of Theorem 0.1 hold and we cannot obtain the existence of a heavy cycle by this theorem. In this paper, we prove that if  $G$  is a triangle-free weighted graph, we can find a heavy cycle even if  $G$  is hamiltonian. Our main result is the following.

**Theorem 0.2.** *Let  $G$  be a 2-connected triangle-free weighted graph and let  $d$  be a nonnegative real number. If  $d^w(v) \geq d$  for every vertex  $v$  in  $G$ , then  $G$  has a cycle of weight at least  $2d$ .*

## 2. Proof of Theorem 2

In our proof of Theorem 0.2, the following lemma is essential.

**Lemma 1.** *Let  $G$  be a weighted graph and let  $P$  be a longest heaviest path of  $G$  with endvertices  $x$  and  $y$ . Assume that*

$$d(x) + d(y) - \varepsilon(xy) \leq |E(P)|.$$

*Then*

- if  $xy \notin E(G)$ , then  $P$  has weight at least  $d_G^w(x) + d_G^w(y)$ , and
- if  $xy \in E(G)$ , then the cycle  $xPyx$  has weight at least  $d_G^w(x) + d_G^w(y)$ .

*Proof.* Let  $P = a_1a_2 \dots a_p$  be a longest heaviest path of  $G$  where  $a_1 = x$  and  $a_p = y$ . Then we have  $N(a_1) \subseteq V(P)$  and  $N(a_p) \subseteq V(P)$ . Let

- $N_1 = \{a_i \mid a_i \in N_G(a_1), a_{i-1} \notin N_{G-a_1}(a_p)\}$ ,
- $N_2 = \{a_i \mid a_i \in N_G(a_1), a_{i-1} \in N_{G-a_1}(a_p)\}$ ,

- $N_3 = \{a_i \mid a_i \in N_{G-a_1}(a_p), a_{i+1} \notin N_G(a_1)\}$  and
- $N_4 = \{a_i \mid a_i \in N_{G-a_1}(a_p), a_{i+1} \in N_G(a_1)\}$ .

Moreover, let

$$E_1 = \{a_1v \mid v \in N_1\}, E_2 = \{a_1v \mid v \in N_2\}, E_3 = \{va_p \mid v \in N_3\} \text{ and } E_4 = \{va_p \mid v \in N_4\}.$$

Now we define a mapping  $\varphi_1$  of  $\bigcup_{i=1}^3 E_i$  to  $E(P)$  such that

- for  $e = a_1a_i \in E_1 \cup E_2$ ,  $\varphi_1(e) = a_{i-1}a_i$  and
- for  $e = a_ia_p \in E_3$ ,  $\varphi_1(e) = a_ia_{i+1}$ ,

and let  $F_i = \{\varphi_1(e) \mid e \in E_i\}$  for  $i = 1, 2, 3$ . Now it is easy to see that  $F_1 \cap F_2 = \emptyset$ . And by the definition of  $E_3$ ,  $a_{i+1} \notin N(a_1)$  if  $a_ia_p \in E_3$ , hence  $F_1$ ,  $F_2$  and  $F_3$  are disjoint.

It follows from the fact  $d(x) + d(y) - \varepsilon(xy) \leq |E(P)|$  that

$$\begin{aligned} \sum_{i=1}^3 |F_i| &= |E_1| + |E_2| + |E_3| \\ &= |N_1| + |N_2| + |N_3| \\ &= |N(a_1)| + |N(a_p) \setminus \{a_1\}| - |N_4| \\ &\leq |E(P)| - |N_4| \\ &= |E(P)| - |E_4|. \end{aligned}$$

Thus  $|E(P) \setminus \bigcup_{i=1}^3 F_i| \geq |E_4|$ . Let  $\varphi_2$  be an injection of  $E_4$  to  $E(P) \setminus \bigcup_{i=1}^3 F_i$  and let  $F_4 = \{\varphi_2(e) \mid e \in E_4\}$ . Note that  $F_1, F_2, F_3$  and  $F_4$  are disjoint.

Assume that  $a_1a_i \in E_1$  and  $Q_1 = a_{i-1}a_{i-2} \dots a_1a_ia_{i+1} \dots a_p$ . Then, since  $w(Q_1) \leq w(P)$ ,  $w(a_1a_i) \leq w(\varphi_1(a_1a_i))$ . By the similar argument as above, we have  $w(e) \leq w(\varphi_1(e))$  for all  $e \in E_1 \cup E_3$ . Suppose  $a_j \in N_2$ . Then we have  $a_{j-1}a_p \in E_4$ . Let  $C$  be a cycle  $a_1a_2 \dots a_{j-1}a_p a_{p-1} \dots a_j$  and  $e = \varphi_2(a_{j-1}a_p)$ . Since  $e \in E(C)$ ,  $Q_2 = C - e$  is a path in  $G$ . Then it follows from the fact  $w(Q_2) \leq w(P)$  that  $w(a_1a_j) + w(a_{j-1}a_p) \leq w(\varphi_1(a_1a_j)) + w(\varphi_2(a_{j-1}a_p))$  for all  $a_j \in N_2$ . Therefore, if  $a_1a_p \notin E(G)$ ,

$$\begin{aligned} d^w(a_1) + d^w(a_p) &= \sum_{v \in N_1} w(a_1v) + \sum_{v \in N_2} w(a_1v) + \sum_{v \in N_3} w(va_p) + \sum_{v \in N_4} w(va_p) \\ &= \sum_{e \in E_1} w(e) + \sum_{e \in E_3} w(e) + \sum_{a_j \in N_2} (w(a_1a_j) + w(a_{j-1}a_p)) \\ &\leq \sum_{e \in F_1} w(e) + \sum_{e \in F_3} w(e) + \sum_{e \in F_2} w(e) + \sum_{e \in F_4} w(e) \\ &\leq w(P), \end{aligned}$$

which implies the assertion. And in case of  $a_1a_p \in E(G)$ ,

$$\begin{aligned}
d^w(a_1) + d^w(a_p) &= \sum_{v \in N_1} w(a_1v) + \sum_{v \in N_2} w(a_1v) + \sum_{v \in N_3} w(va_p) + \sum_{v \in N_4} w(va_p) + w(a_1a_p) \\
&= \sum_{e \in E_1} w(e) + \sum_{e \in E_3} w(e) + \sum_{a_j \in N_2} (w(a_1a_j) + w(a_{j-1}a_p)) + w(a_1a_p) \\
&\leq \sum_{e \in F_1} w(e) + \sum_{e \in F_3} w(e) + \sum_{e \in F_2} w(e) + \sum_{e \in F_4} w(e) + w(a_1a_p) \\
&\leq w(P) + w(a_1a_p).
\end{aligned}$$

Hence the cycle  $a_1Pa_p a_1$  has weight at least  $d^w(a_1) + d^w(a_p)$ , which implies the assertion.  $\square$

Now we prove Theorem 0.2 by using Lemma 1 and the following lemma.

**Lemma 2 (Bondy and Fan [2]).** *Let  $G$  be a 2-connected weighted graph and let  $P$  be a heaviest path in  $G$  with endvertices  $x$  and  $y$ . Then there exists a cycle  $C$  in  $G$  such that  $w(C) > w(P)$  or  $w(C) \geq d^w(x) + d^w(y)$ .*

*proof of Theorem 2.* Let  $P$  be a longest heaviest path in  $G$ , and let  $x, y$  be endvertices of  $P$ . Since  $G$  is triangle-free and  $N(x), N(y) \subseteq V(P)$ ,  $|N(x)| \leq |V(P)|/2$  and  $|N(y)| \leq |V(P)|/2$ . Moreover, if  $xy \notin E(G)$ ,  $|N(x)| \leq (|V(P)| - 1)/2$  and  $|N(y)| \leq (|V(P)| - 1)/2$ . Hence, whether  $x$  and  $y$  are adjacent or not, we have  $d(x) + d(y) - \varepsilon(xy) \leq |E(P)|$ . In case of  $xy \in E(G)$ , Lemma 1 implies the existence of a cycle of weight at least  $d^w(x) + d^w(y) \geq 2d$ , which is a required cycle. Thus we may assume  $xy \notin E(G)$ , then Lemma 1 implies that  $w(P) \geq d^w(x) + d^w(y) \geq 2d$ . Now it follows from Lemma 2 that there exists a cycle  $C$  in  $G$  such that  $w(C) > w(P) \geq 2d$  or  $w(C) \geq d^w(x) + d^w(y) \geq 2d$ , which is a required cycle.  $\square$

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