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NOTE ON CONSTRUCTION METHODS OF UPPER BOUND GRAPHS

HIROSHI ERA
Faculty of Information and Communication, Bunkyo University
Chigasaki 253-0007, JAPAN
SHIN-ICHI IWAI, KENJIRO OGAWA AND MORIMASA TSUCHIYA *
Department of Mathematical Sciences, Tokai University
Hiratsuka 259-1292, JAPAN
E-mail: tsuchiya@ss.u-tokai.ac.jp *

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Abstract

In this paper, we consider construction of upper bound graphs. An upper bound graph can be transformed into a nova by contractions and a nova can be transformed into an upper bound graph by splits. By these results, we get a characterization on upper bound graphs.

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1. Introduction

In this paper, we consider finite undirected simple graphs. For a vertex v in G, the *neighborhood* of v is the set of vertices which are adjacent to v, and denoted by $N_G(v)$. $N_G[v] = N_G(v) \cup \{v\}$. For a vertex subset $S \subseteq V(G)$, $< S >_G$ is the induced subgraph of G induced by S. A vertex v is called a *simplicial vertex* if $N_G(v)$ is a complete subgraph.

For a poset $P = (X, \leq)$, the *upper bound graph* (*UB-graph*) of P is the graph $UB(P) = (X, E_{UB(P)})$, where $uv \in E_{UB(P)}$ if and only if $u \neq v$ and there exists $m \in X$ such that $u, v \leq m$. McMorris and Zaslavsky [5] introduced this concept, and they gave a characterization of upper bound graphs in terms of simplicial cliques as follows.

A *clique* in the graph G is the vertex set of a maximal complete subgraph, and a family $\mathcal{C}$ of complete subgraphs *edge covers* G if and only if for each edge $uv \in E(G)$, there exists $C \in \mathcal{C}$ such that $u, v \in C$.

Theorem 1 (McMorris and Zaslavsky [5]). *A graph G is a UB-graph if and only if there exists a family $\mathcal{C}(G) = \{C_1, C_2, ..., C_n\}$ of complete subgraphs of G such that*
(i) $\mathcal{C}(G)$ edge covers G, and
(ii) for each C_i, there is a vertex $v_i \in C_i - (\bigcup_{j \neq i} C_j)$.

Furthermore, such a family $\mathcal{C}(G)$ must consist of cliques of G and is the only such family if G has no isolated vertices. □

For an edge clique cover $\mathcal{C}(G) = \{C_1, C_2, ..., C_n\}$ satisfying the conditions of Theorem 1, a *representation vertex set* $R(\mathcal{C}(G))$ on $\mathcal{C}(G)$ is a vertex subset $\{v_1, v_2, ..., v_n\}$ such that $v_i \in C_i - (\bigcup_{j \neq i} C_j)$ for each $i = 1, ..., n$. Each vertex $v_i \in R(\mathcal{C}(G))$ is a simplicial vertex. Furthermore $\mathcal{C}(G)$ is said to be the *simplicial edge clique cover* of G. In [5] they also use a concept of canonical posets. For a UB-graph G, the *canonical poset* of G is $can(G) = (V(G), \leq_{can(G)})$, where $x \leq_{can(G)} y$ if and only if (1) $y \in R(\mathcal{C}(G))$ and $xy \in E(G)$ or (2) $x = y$. Then the UB-graph of $can(G)$ is G and $can(G)$ is the minimum poset of $\mathcal{P}_{UB}(G)$, where $\mathcal{P}_{UB}(G)$ is the set of posets with the same upper bound graph G, and fixed $R(\mathcal{C}(G))$ as the maximal elements set.

In [6] and [7] we dealt with families of posets in $\mathcal{P}_{UB}(G)$. Especially we considered properties of distances between such posets and the diameter of $\mathcal{P}_{UB}(G)$, and also a method for constructing posets in $\mathcal{P}_{UB}(G)$. That is, each poset in $\mathcal{P}_{UB}(G)$ can be obtained from the canonical poset of G by poset operations. In [3] and [8] we also consider similar constructions on double bound graphs.

In [4] Lundgren and Maybee obtained another characterization of upper bound graphs in terms of ordered edge covers. Using their results, Bergstrand and Jones [1] showed other characterizations of upper bound graphs. They obtained some properties on transformations of upper bound graphs as follows:

Proposition 2 (Bergstrand and Jones [1]). *If G is a connected graph with $\omega(G) \leq 4$, and G is an upper bound graph, then the graph obtained by successively identifying all pairs of adjacent vertices of degree at least $\omega(G)$ and deleting their common neighbors is a complete graph or a $\mathcal{B}$-nova.* □

For a graph G, $\omega(G)$ is the number of vertices in a maximum clique of G. A $\mathcal{B}$-*nova* is defined as follows. Let $\mathcal{B}$ be the family which consists of $K_2, K_3, K_4, K_4 - e$, and the graph G_1 obtained by adding a vertex of degree one to a K_3. A $\mathcal{B}$-nova is a graph obtained from a star $K_{1,n}$ $(n \geq 2)$ by replacing each edge with a graph from $\mathcal{B}$, the vertex of identification in each graph is either a non-simplicial vertex or a simplicial vertex that has a simplicial neighbor. A *nova* is a graph obtained from a star $K_{1,n}$ $(n \geq 1)$ by replacing each edge with a complete graph with at least two vertices.

In this paper we consider transformation methods between upper bound graphs and novas.

2. Construction of upper bound graphs

For an edge $e = uv$, a *contraction* of the edge e means removing e and identifying u and v in such a way that the resulting vertex w is adjacent to vertices, except u and v, which were originally adjacent to u or v. The notation for the graph obtained by contraction of e in G is G/e. By the definition of contractions we obtain the following.

Proposition 3. *Let G be a connected UB-graph and $e = uv$ be an edge of G such that u and v are non-simplicial vertices. Then G/e is also a UB-graph.*

Proof. Let $\mathcal{C}(G) = \{C_1, C_2, ..., C_n\}$ be a simplicial edge clique cover of G and w be the vertex in G/e corresponding to e. We modify $\mathcal{C}(G)$ to $\mathcal{C}(G/e) =$

$\{C'_1, C'_2, ..., C'_n\}$, where

$$C'_i = \begin{cases} C_i & \text{if } u \notin C_i \text{ and } v \notin C_i \\ (C_i - \{u, v\}) \cup \{w\} & \text{if } u \in C_i \text{ or } v \in C_i. \end{cases}$$

Since u and v are non-simplicial vertices, each C'_i is a clique of G/e with simplicial vertices. Thus $\mathcal{C}(G/e)$ is a simplicial edge clique cover of G/e and G/e is a UB-graph. □

Lemma 4. *Let T be the set of non-simplicial vertices of a connected UB-graph G. Then the induced subgraph $< T >_G$ of G is connected.*

Proof. Let $can(G)$ be the canonical poset of G. Since G is connected, $can(G)$ is a connected poset. Each element of T is a minimal element of $can(G)$. So the union of T and the set of upper bound elements of T induces a connected subposet and $< T >_G$ is a connected subgraph. □

Using these facts we obtain the following results.

Theorem 5. *Let G be a UB-graph. Then the graph H obtained by successive contractions of all pairs of adjacent non-simplicial vertices in G is a nova.*

Proof. By Proposition 3, H is a UB-graph. Further, Lemma 4 implies that all non-simplicial vertices in G will collapse into one vertex in H. Thus H is a nova. □

For a graph G and a vertex w, let $C_G(w) = \{C_i; C_i$ is a clique of G and $w \in C_i\}$ and $C_G(w) - w = \{C_i - \{w\}; C_i \in C_G(w)\}$. Next we consider the following operation. A *split* of a non-simplicial vertex w in a graph G is a graph H obtained from G by replacing w by two adjacent vertices u, v such that u is adjacent to each vertex in $\cup_{C_i \in C(u)} C_i - \{w\}$ and v is adjacent to each vertex in $\cup_{C_i \in C(v)} C_i - \{w\}$, where $C(u)$ and $C(v)$ are sets of cliques in G satisfying the following conditions: $C(u) \cup C(v) = C_G(w)$, $|C(u)| \geq 2$, $|C(v)| \geq 2$ and $C(u) \cap C(v) \neq \emptyset$. Thus a split of a non-simplicial vertex w replaces w by two adjacent non-simplicial vertices u and v. $G \circ w$ denotes an arbitrary graph obtained from G by a split of w in G. We obtain the following results on splits.

Proposition 6. *Let G be a connected UB-graph and w be a non-simplicial vertex of G. Then $G \circ w$ is a UB-graph.*

Proof. Let $\mathcal{C}(G) = \{C_1, C_2, ..., C_n\}$ be a simplicial edge clique cover of G. We modify $\mathcal{C}(G)$ to $\mathcal{C}(G \circ w) = \{C'_1, C'_2, ..., C'_n\}$, where

$$C'_i = \begin{cases} C_i & \text{if } C_i \notin C(u) \cup C(v) \\ (C_i - \{w\}) \cup \{u\} & \text{if } C_i \in C(u) - C(v) \\ (C_i - \{w\}) \cup \{v\} & \text{if } C_i \in C(v) - C(u) \\ (C_i - \{w\}) \cup \{u, v\} & \text{if } C_i \in C(u) \cap C(v). \end{cases}$$

Since w is a non-simplicial vertex, its removal retains each C'_i as a clique with simplicial vertices. By $C(u) \cap C(v) \neq \emptyset$, there exists a clique Cf_i which contains u and v. Thus $\mathcal{C}(G \circ w)$ is a simplicial edge clique cover of $G \circ w$, and $G \circ w$ is a UB-graph. □

Proposition 7. *Let G be a connected UB-graph, $e = uv$ be an edge of G such that u and v are non-simplicial vertices and w be the vertex in G/e corresponding to e. Then there exists a $(G/e) \circ w$ such that $(G/e) \circ w = G$.*

Proof. Since u and v are non-simplicial vertices in G, $|C_G(u)| \geq 2$, $|C_G(v)| \geq 2$. In addition there exists a clique containing both u and v, and w is a non-simplicial vertex in G/e. Then $C(u)$ and $C(v)$ can be chosen as $C(u) = (C_G(u) - \{u, v\}) \cup \{w\}$ and $C(v) = (C_G(v) - \{u, v\}) \cup \{w\}$. Thus the split operation is applicable to w in G/e and we obtain a $(G/e) \circ w$ such that $(G/e) \circ w = G$. □

This result means that for an edge $e = uv$ in a graph G such that u and v are adjacent to the same simplicial vertex, the contraction on e is an inverse operation of the split on the vertex w in G/e corresponding to e. In general non UB-graphs can be transformed into UB-graphs by contractions. However, UB-graphs cannot be transformed into non UB-graphs by splits mentioned above. When a graph G is given we can show if it is a UB-graph or not by successive contractions. This method gives another characterization for a UB-graph. The next result suggests how to select edges for contractions in the processes.

Theorem 8 (Bergstrand and Jones [1]). *A graph G is a UB-graph if and only if for every pair of adjacent non-simplicial vertices u and v, there exists a simplicial vertex s such that $N_G(s)$ contains both u and v.* □

Furthermore we have the following results on adjacent non-simplicial vertices.

Lemma 9. *Let u and v be non-simplicial adjacent vertices in G such that there exists a simplicial vertex adjacent to u and v. Let x and y be non-simplicial adjacent vertices in G such that there exist no simplicial vertices adjacent to x and y. Then there exists a simplicial vertex of G/uv adjacent to x and y in G/uv if and only if there exists a non-simplicial vertex s in G satisfying the following conditions:*

(1) $< N_G[s] - \{u, v\} >_G$ is a complete subgraph of G,

(2) for each vertex $z \in N_G[s] - \{u, v\}$, z is adjacent to u or v,

(3) $x, y \in N_G[s]$.

Proof. Let w be the corresponding vertex of the edge uv in G/uv. First we consider the sufficiency. The conditions (1) and (2) mean that s is a simplicial vertex of G/uv adjacent to w. The condition (3) means that s is adjacent to x and y.

Next we consider the necessity. Let s be a simplicial vertex of G/uv adjacent to x and y in G/uv. s must be a non-simplicial vertex of G. So s is adjacent to w in G/uv, and $N_G[s] - \{u, v\} = N_{G/uv}[s] - \{w\}$. Thus each vertex $z \in N_G[s] - \{u, v\}$ is adjacent to u or v. Therefore $< N_G[s] - \{u, v\} >_G$ is a complete subgraph of G, and $x, y \in N_G[s]$. □

Remark 10. *For a simplicial vertex s of G/uv and non-simplicial adjacent vertices x, y satisfying the conditions of Lemma 9, $s, x, y \in N_G[u] \cup N_G[v]$.* □

By Theorem 8, if G is a non UB-graph, there exist non-simplicial adjacent vertices x and y such that all simplicial vertices of G are not adjacent to both x and y. If vertices u and v are non-simplicial adjacent vertices in G which are adjacent to a simplicial vertex of G and G/uv is a UB-graph, we have the following two cases:

Case 1. there exists a simplicial vertex of G/uv which is adjacent to x and y and is not simplicial vertex of $G,$ that is, G/uv has a new simplicial vertex.

Case 2. without loss of generality $y = v,\ xu \in E(G)$ and there exists a simplicial vertex of G adjacent to both x and $u,$ that is, xy identifies with $xu.$

We formalize these facts below.

Proposition 11. *Let G be a non UB-graph, and u and v be non-simplicial adjacent vertices in G which are adjacent to a simplicial vertex of G. Then G/uv is a UB-graph if and only if for each pair of non-simplicial adjacent vertices x and y which are not adjacent to simplicial vertices of $G,$ x and y satisfy the conditions NS or $IE.$*

NS *: there exists a non-simplicial vertex s in G satisfying the following conditions:*

(i) $< N_G[s] - \{u, v\} >_G$ is a complete subgraph of $G,$

(ii) for each vertex $z \in N_G[s] - \{u, v\},$ z is adjacent to u or $v,$

(iii) $x, y \in N_G[s].$

IE *: $y = v,\ xu \in E(G)$ and there exists a simplicial vertex of G adjacent to both x and u in $G.$* □

Since novas are UB-graphs, we now obtain the following result.

Proposition 12. *Let G be a connected graph. G is a UB-graph if and only if the graph obtained by successive contractions of adjacent non-simplicial vertices u and v satisfying the following conditions is a nova:*
(1) u and v are adjacent to a simplicial vertex of $G,$
(2) there exist no pair of non-simplicial adjacent vertices x and y which are not adjacent to simplicial vertices of G satisfying the conditions NS or $IE.$

Proof. By Theorem 5, the necessity holds and we have a sequence:

$$G \overset{contraction}{\longrightarrow} \cdots \overset{contraction}{\longrightarrow} \text{nova}.$$

By Proposition 11, if G is a non UB-graph, G/uv is a UB-graph, where u and v satisfy the conditions of Proposition 11. Since novas are UB-graphs, we can not obtain a nova from a non UB-graph G by successive contractions of non-simplicial adjacent vertices in G satisfying the conditions of Proposition 12. □

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