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11-DOILIES WITH VERTEX SETS OF SIZES $275, 286, \ldots, 462$

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Abstract

In this paper, we close the gap between the results of [8] and [9] by showing that there are 11-doilies for vertex sets V of size $|V| = 275, 286, \ldots, 462$. In [8] it is shown that there are 11-doilies with $|V| = 462, 473, \ldots, 1001$, and in [9] it is shown that there are such diagrams with $|V| = 231, 242, \ldots, 352$. This is the third paper in a series intended to show that 11-doilies exist for any possible size of vertex set, answering a question of Grünbaum. We continue to extend the method that is developed in [7], [8], and [9]. The crucial step in the method of those papers is based on saturated chain decompositions of planar spanning subgraphs of the p-hypercube, and on edge-disjoint path decompositions of planar spanning subgraphs of the p-hypercube. We continue to study these types of decompositions. In constructing the doilies we use a basic structure called the Venn graph of a doodle of the doily. Using the new Venn graph that we create here we show the existence of the above mentioned diagrams. In fact, without checking all the details we show that there are at least 2^{17} (non-isomorphic) 11-doilies with these sizes of vertex sets.

Keywords: Venn diagram, doily,Venn graph, doodle.

1. Introduction

First, we briefly repeat the necessary definitions and notations. They can be found in more detail in [3], [4], [5], [1], [7], and [8].

An n-*Venn diagram* consists of a family $\mathcal{F}$ of n simple closed Jordan curves in the plane so that all possible intersections (2^n many) of the interiors and the exteriors of these curves are nonempty and connected.

We note that each of the 2^n cells can be described by an n-tuple of zeros and ones where the i th coordinate is 0 if X_i is the unbounded exterior of C_i, is 1 otherwise for $i = 1, 2, \dots, n$.

It is clear that there is a one-to-one correspondence between the 2^n sets of a Venn diagram and the vertices of the n-dimensional hypercube. If $A = X_1 \bigcap X_2 \bigcap \cdots \bigcap X_n$ is a set in a Venn diagram, then the corresponding n-tuple in the hypercube is the *description* of A.

A Venn diagram is *simple* if at most two curves intersect (transversely) at any point in the plane. Among the nonsimple Venn diagrams, we will consider only those in which any two curves meet (not necessarily transversely) in isolated points and not in segments of curves.

Two Venn diagrams are *isomorphic* if, by a continuous transformation of the plane, one of them can be changed into the other or its mirror image.

A Venn diagram with n curves is said to be *symmetric* if there is a fixed center about which rotations through $360/n$ degrees map the family of curves into itself so that the diagram is not changed by the rotation. A symmetric Venn diagram will be called a *doily*. Similarly to the Venn diagram, a doily with n curves will be called an n*-doily*.

Several interesting properties of Venn diagrams and Venn graphs were derived in [1] and in [8]. Here we simply state those properties we need and refer the reader to [1] and [8] for more details.

With each Venn diagram one can associate two graphs. The Venn diagram itself can be viewed as a planar graph $V(\mathcal{F})$ where all the intersection points of the curves in $\mathcal{F}$ are the vertices of $V(\mathcal{F})$ and the segments of the curves with vertices as endpoints are the edges of $V(\mathcal{F})$. In proper context, confusion rarely arises from also calling this graph a Venn diagram. In the rest of this paper the notations $\mathcal{F}$ and $V(\mathcal{F})$ are freely interchanged when it is not important that the Venn diagram is a graph. The Venn diagram $V(\mathcal{F})$ may have multiple edges. The graph $V(\mathcal{F})$ is a planar graph. Note that the Venn diagram depends upon its drawing in the plane. The planar graph dual to $V(\mathcal{F})$ will be called the *Venn graph*, denoted by $D(\mathcal{F})$.

2. Some preliminary arguments

Recently, considerable attention has been given to symmetric Venn diagrams, [6]. The concept was introduced by Henderson [11]. The simple Venn diagrams with one, two, or three circles can obviously be represented as symmetric Venn diagrams. Henderson provided two examples of non-simple symmetric Venn diagrams with five curves, using five pentagons and five triangles. Grünbaum conjectured in [5] that symmetric p-Venn diagrams exist for all prime numbers p. For more detailed history of this conjecture, see [7].

Henderson argued that doilies could not exist if the number of the curves is a composite number. His argument is based on the fact that in a doily, the cells having exactly k many 1 's ($k \neq 0, n$) and $n-k$ many 0 's in their description have a rotational symmetry. The number $\binom{n}{k}$ of such faces must be divisible by n for all positive integers $k < n.$ A theorem of Leibnitz says that $\binom{n}{k}$ is divisible by n for all positive $k < n$ if and only if n is a prime number ([10]). Thus symmetric Venn diagrams can exist only with a prime number of curves. For the rest of this paper, p is always a prime number.

In [7] it is shown that the possible numbers of vertices $|V|$ for a p-doily are those numbers $|V|,$ for which $\lceil \frac{2^p-2}{p-1} \rceil \leq |V| \leq 2^p - 2,$ is divisible by $p;$ and thus if $p = 11,$ then $|V| = 209, 220, \ldots, 2046.$

The creation of a p-doily requires finding a curve and a center of rotation. This curve must be successively rotated p times through an angle of $360/p$ degrees. The difficulty of this process is finding a suitable curve. In [7] a different approach is established. Instead of rotating a curve, rotate a portion of the Venn graph of a symmetric Venn diagram. This portion of the Venn graph, together with a set of transformations, is called a *doodle*. The graph of the doodle is inside a sector of the circle with central angle $360/p$ degrees. The center of rotation is always the center of the circle. The transformations of a doodle create the Venn graph of a doily. Finally, using the method of Construction 2 (to follow), a doily is created.

In [8] a new approach is introduced. It starts with the definition of a graph with special properties and a labeling of its edges. This graph is a modification of the Venn graph as defined above, but we will still refer to it as the *Venn graph of a Venn diagram*. This definition fits our purpose better.

Definition 1. *A 2-connected, planar, spanning, labeled subgraph G^* of the n-hypercube together with an edge numbering $\mathcal{N}$ of the edges of $G^*,$ is called a* Venn graph of a Venn diagram (more specifically, an n-Venn diagram) *iff*

1. *to each edge e of G^* is assignedone of the numbers $1, 2, \ldots, n$ (called the* edge number*) corresponding to the coordinate where the descriptions of the two end-vertices of e differ;*
2. *any two faces of G^* share at most one edge with a given edge number; and*
3. *an edge number that appears in a face of G^* appears exactly twice in that face.*

In a case when each face of G^* is a 4-face and the graph G^* is 3-connected, G^* is the *Venn graph of a simple n-Venn diagram*. Indeed, by Construction 2, a 2ℓ-face of the Venn graph coresponds to a vertex of the Venn diagram, where exactly ℓ curves meet.

Since the hypercube is a bipartite graph, each cycle is an even cycle, so each face of any 2-connected, spanning subgraph has an even number of edges. Two edges that are adjacent in the cube cannot share the same edge number, would correspond to parallel edges. Furthermore, any curve of a Venn diagram $\mathcal{F}$ is not self-intersecting, thus two faces of the Venn graph G^* cannot share the same edge number twice.

Construction 2. *If G^* is the Venn graph of an n-Venn diagram and $\mathcal{N}$ is an edge numbering with the above properties, then a planar diagram (multigraph) $\mathcal{F}$ can be constructed in the following way. Consider a planar drawing of G^*, and construct its planar dual $D^*(G^*)$, as in Step 1. Then in Step 2 use the edge numbering $\mathcal{N}$ to number the edges (Jordan arcs) of $D^*(G^*)$.*

> **Step 1**. *Create a graph $D^*(G^*)$ by placing a vertex x_F in the interior of each face F of G^* and joining it to each edge of F by a simple Jordan arc, such that*
>
> **(i)** *The arcs inside in a face F meet only at the new vertex x_F, and*
>
> **(ii)** *In every edge of G^* the arcs (exactly two of them) meet in the same (interior) point.*
>
> **Step 2**. *Assign to each simple Jordan arc the edge number of the edge it meets. This identifies each simple closed Jordan curve in the Venn diagram.*
>
> **Step 3**. *The set of vertices $\{x_F \colon F \text{ is a face of } G^*\}$ and the closed Jordan curves identified in Step 2 form the n-Venn diagram $\mathcal{F}$. $\mathcal{F}$ can also be considered as a planar drawing of a multigraph and we usually do not distuingish between the two points of view.*

It is easy to see that the diagram $\mathcal{F}$ thus obtained is a Venn diagram. It is also clear that if $D(\mathcal{F})$ is a Venn graph (in the usual sense) of a Venn diagram $\mathcal{F},$ that is, if it is a subgraph of the n-hypercube, then using the method described above, starting with $G^* = D(\mathcal{F}),$ the obtained multigraph $\mathcal{F}^*$ is graph-isomorphic to the Venn diagram $\mathcal{F}.$

Suppose that a symmetric p-Venn diagram has been constructed by rotating one simple Jordan curve through an angle of $360/p$ degrees p times about an appropriate center. We observe that labeling the curves $1, \ldots, p$ clockwise, say, induces a unique labeling of each region by a binary p-tuple having 0 or 1 in the ith coordinate $(i = 1, 2, \ldots, p)$ according to whether the region is outside or inside curve $i.$ Letting $\mathbf{a} =< a_1, a_2, ..., a_p >,$ where a_i is 0 or $1,$ we define the *shift* s of $\mathbf{a}$ by $s(\mathbf{a}) =< a_p, a_1, a_2, ..., a_{p-1} >;$ a *rotation* is a composition of shifts.

Definition 3. *Let $\mathcal{B}$ be a set of binary p-tuples. A binary p-tuple $\mathbf{a}$ (not necessarily in $\mathcal{B}$) is called* independent *from $\mathcal{B}$ iff $\mathbf{a}$ cannot be obtained by a shift or a rotation of any other element of $\mathcal{B}$. A set of binary p-tuples $\mathcal{B}$ is called independent iff every element $\mathbf{a}$ of $\mathcal{B}$ is independent from $\mathcal{B}$. The* weight $w(\mathbf{a})$ *of a binary p-tuple $\mathbf{a}$ is the number of 1's in the tuple.*

Definition 4. *A* generator $\mathcal{G}$ *is a maximal independent set of binary* p*-tuples* $\mathbf{a}$ *such that*

$$1 \leq w(\mathbf{a}) \leq p - 1.$$

If only those binary p-tuples are considered in which there are both a 0 and a 1 bit, then $1 \leq w(\mathbf{a}) \leq p - 1$ immediately follows.

Obviously, for every number p there is a generator $\mathcal{G}$, and in a generator $\mathcal{G}$ there are exactly

$$\frac{\binom{p}{k}}{p}$$

many elements with weight $k,$ for each $1 \leq k \leq p - 1,$ and thus, there are

$$\sum_{k=1}^{p-1} \frac{\binom{p}{k}}{p} = \frac{2^p - 2}{p}$$

many elements. If $p = 11,$ then in a generator $\mathcal{G}$ there are exactly 186 elements.

Definition 5. *A* p-doodle *is a pair of sets* $(G, \mathcal{S})$ *where* G *is a planar subgraph of the p-hypercube (called the* Venn graph of a doodle*) and* $\mathcal{S} = \{s_1, s_2, \ldots, s_p\}$ *(called the* transformation set*), is a*

sequence of transformations, *(shifts and/or rotations) with the following properties:*

1. *The set* $\mathcal{G}$ *of descriptions of the vertex set* G *is a generator set,*
2. $\{s(\mathbf{a}) \mid \mathbf{a} \in \mathcal{G}, s \in \mathcal{S}\} \cup \{<0, 0, \ldots, 0>, <1, 1, \ldots, 1>\}$ *is a set of descriptions of a 2-connected, spanning, planar subgraph of the* p*-hypercube,*
3. *Sets* $s(\mathcal{G})$ *for* $s \in \mathcal{S}$ *are pairwise disjoint.*

Definition 5 will be applied with $s_j = s_1^j$. It is easy to see that the following holds:

1. If $\mathbf{a} \in \mathcal{G},$ then for every $1 \leq j \leq p,$ $s_j(\mathbf{a}) = s_1 \circ s_1 \cdots \circ s_1(\mathbf{a}),$ where shift s_1 is applied j times, and,
2. $s_p(\mathcal{G}) = \mathcal{G}.$

Recall that the vertices, edges, and faces of a Venn graph correspond to the faces, edges, and vertices of the Venn diagram, respectively. Note that some of the pairs of vertices in a graph of a doodle may be connected by an edge without violating the conditions of Definition 1, obtaining a new graph of a doodle. Every time a set of new edges is added to the graph of a doodle, the number of edges of the corresponding Venn graph is increased by a multiple of p. This increases the vertices of the corresponding doily by a multiple of $p.$ Note also that in the Venn graph of a doily, sometimes p-many symmetrical pairs of vertices from different images of G_1 can be connected by p-many edges without violating the conditions of Definition 1. Adding these p-many edges, a new Venn graph of a doily is created. This also increases the number of vertices of the corresponding doily by p.

Definition 6. *A doodle or a Venn graph is called* extendable *if some pairs of the vertices of the graph of the doodle can be connected by an edge, and/or p-many symmetrical pairs of vertices from different images of G_1 of the corresponding Venn graph can be connected by p-many edges without violating the conditions of Definition 1. It is* fully extendable *if it is extendable and the resulting Venn graph of a doily is a 3-connected graph having all 4-faces.*

The process of finding the Venn graph of the doily with the appropriate generator is the decisive step in the construction. In [7] we used the following:

Let $\mathcal{P}(X)$ denote the poset of all subsets of a set X, where $|X| = n$. The level k $(0 \leq k \leq n)$ in the poset $\mathcal{P}(X)$ is the set of all subsets of X with exactly k many elements. A chain $\mathcal{C} = A_1 \subset A_2 \subset \cdots \subset A_k$ in $\mathcal{P}(X)$ is a *saturated* chain iff it intersects every level between levels l and $l + k$ for some integers l and k. Using the characteristic functions of the subsets of the set X, a saturated chain in $\mathcal{P}(X)$ corresponds to a set of binary codes of the hypercube which is a path from level l $(0 \leq l \leq n)$ to level $l + k$ $(0 \leq l + k \leq n)$ for some integers l and k. A saturated chain $A_1 \subset A_2 \subset \cdots \subset A_k$ in $\mathcal{P}(X)$ is called *symmetric* if $|A_1| = n - |A_k|$. Thus a saturated chain is symmetric if it goes from level m to level $k = n - m$.

In [7] and [8], the Venn graph of a non-simple doily is a planar subgraph of the p-hypercube in which there are as many saturated chains as vertices that we want to construct in the doily. In the case of the Venn graph of the doodle the number of chains is $1/p$ th of the number of vertices of the corresponding doily. If one can find these saturated chains of the Venn graph of the doodle (in which the binary p-tuples are from a generator), then the process is done. It was shown in [2] (see also [12]) that

Theorem 7. *$\mathcal{P}(X)$ can be partitioned into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ many edge disjoint symmetric chains.*

In [7] and [8], Theorem 7 helped us. The Venn graph of the doodle of a p-doily is created by first partitioning the p-hypercube into

$$\binom{p}{\frac{p-1}{2}}$$

many symmetric chains. If there are

$$\frac{\binom{p}{\frac{p-1}{2}}}{p}$$

many chains (in which the binary p-tuples are from a generator) that form the required Venn graph of the doodle, then the process is done. If not, then some of the chains combined (and not individual binary p-tuples) have to rotate either completely or partially into different positions in which they fit together in the required way. If this is possible, then the process is done. If these cannot be achieved, then some of the symmetric chains have to extend into saturated, but not necessarily symmetric, chains. If these chains form the required graph, then the process is done again.

This method in [7] and in [8] helped us to find a suitable graphical structure of the Venn graph of the doodle of the 11-doily, if there are at least

$$\frac{\binom{p}{\frac{p-1}{2}}}{p} = \frac{\binom{11}{\frac{11-1}{2}}}{11} = 42$$

many saturated chains in the Venn graph of the doodle. Using this graphical structure it was shown in [8] that there are p-doilies for all possible size vertex sets if $p \leq 7$, and that there are 11-doilies with vertex sets $462, 473, \ldots, 1001$.

Unfortunately, this is not the case if the vertex set of a 11-doily is less than 462. In any chain, the number of 1's of the codes of vertices is increasing from the top down to the bottom. (In the theory of finite sets, usually chains and the hypercube are oriented with all 0's at the bottom (empty set) and all 1's at the top (full set). We do this here upside-down because all 1's means that the face is inside of all the curves.) In a chain the number of 1's can be changed at most p-times. If there are enough chains, at least $42,$ in the Venn graph of the doodle, then the 186 vertices can be absorbed in them by changing the numbers of 1's. If this number is less than $42,$ then instead of saturated chains we need to create edge-disjoint paths in order to absorb all of the 186 vertices. The number of 1's in some of the paths in some places may decrease. Thus we use edge-disjoint decompositions of planar, spanning subgraphs of the p-hypercube in creating the Venn graph of the doily. In Tables 2, 3, and in Figures 2, 8, the edge numbers are bold faced iff they are generated by two codes for which the number of 1's is decreased in the path.

There are many such decompositions that we described above. It is not difficult to find some of them. It becomes difficult when one wants to find one of them with the possible minimum number of paths, or if one wants to find a Venn graph of a doodle that can be extended with many new edges. The Venn graph of the doodle of this paper is produced by many complete or partial rotations of some of the paths, by trial-and-error. Although we tried to find a Venn graph of a doodle with the minimum number of paths and which is extendable to a Venn graph of a doodle with at least 42 paths, we failed. Instead we show a Venn graphs of a doodle with $25,$ paths –six more than the minimum number– which successfully closes the gap up of the results of papers [7] and [8]. We prove the following result.

Theorem 8. *There are 11-doilies with the vertex sets of 275, 286, …, 462, 473, …, 1001.*

We believe that using Theorem 7 and the method developed in [7], one can prove that there are p-doilies for all prime numbers p larger than 11. We do not expect that using the method presented in this paper one can prove that there are p-doilies for all possible vertex sets and for all prime numbers p larger than 11, or even that there are p-doilies for all prime numbers p larger than 11. There are two reasons. First, in general, the characterization of those numbers $|V|$ divisible by p between $\lceil \frac{2^p-2}{p-1} \rceil$ and $2^p - 2$ remains unknown. There are two known cases, $2^p - 2$ and $\binom{p}{\frac{p-1}{2}}$, the first is by Fermat's little theorem, ($2^{p-1} \equiv 1 \pmod p$)). We think, in these cases, that saturated chains are more useful than edge-disjoint paths. Second, no similar result to Theorem 7 is known for edge-disjoint paths decompositions of planar, spanning subgraphs of the p-hypercube. The edge-disjoint path method is simply computational. With a lot of patience and some luck, one can find a desired Venn graph of a doodle for a given not-too-big prime number p.

3. 11-doily

Now we choose a set of binary 11-tuples, a generator. It is tedious but not hard to check that the set of 11-tuples in Table 1 satisfies the conditions of a generator; each is numbered for use in Figure 1. Table 2 shows the edge numbers of the Venn graph of the doodle. These are generated by the binary codes of Table 1. From the generator we create the Venn graph of doodle 1.

The Venn graph of the doodle is in Figure 1. The edge numbering of this Venn graph of the doodle is in Figure 2. The Venn graph of the doodle of the 11-doily is in Figure 3. The 11-doily with the vertex set of size 275 is in Figure 4. Finally, the Jordan curve that creates this doily is in Figure 7.

All of the figures of a Venn graph of a doily (or a Venn graph of a doodle) have some vertices that are incident to (dashed or not dashed) edges, with the understanding that in the exterior they are all incident with the single vertex $< 0, 0, \ldots, 0 >$.

Note that the graph of this doodle is extendable. Table 3 shows the vertices of the graph of the doodle which can be connected without violating Definition 1. In addition there are two pairs of vertices which can be connected between two consecutive copies of the doodle. The table shows those pairs of vertices as well. It also shows all the edge numbers created by these new edges. In this table there are 17 pairs of vertices. This is the maximum number of new edges that can be added to the graph of the doodle without violating Definition 1. In Table 3, the numbers in the second column indicates the pair of vertices that are connected. These numbers are from Table 1. The third and fourth columns show the codes of these vertices. Finally, the fifth and the last columns show the edge numbers of the new edges. The last two rows of Table 3 show the edge connecting a pair of vertices from two consecutive copies of Venn graph of the doodle in the Venn graph of the doily. The numbers 191 and 193 indicate the vertex obtained by the shift of the vertex 5 and 7 in Table 1, that is, $s(< 10011110000 >) =< 01001111000 >$ and $s(< 10111110100 >) =< 01011111010 >$. These vertices are in the second copy of the graph of the doodle in the Venn graph. They are in the left most column of the second copy of the Venn graph of the doodle in rows 5 and 7.

These edges can be added one by one. One can also add any set of these edges to the graph of the doodle. 17 steps, each adding one new edge, create 18 different 11-doilies with vertex sets $275, 286, \ldots, 462$. Adding any subset of this edge set to the graph of the doodle creates at least 2^{17} different non-isomorphic graphs of the doodle. It can be more than 2^{17}; this comes from the fact that after adding some new edges, some original edges can be deleted from the graph without violating Definition 1. For instance, after adding the new edges connecting the vertices 30 and 68, and 67 and 77 in the Table 1, the edge connecting the vertices 68 and 69 can be deleted from the Venn graph of the doodle in Figure 7. This creates some additional new graphs of doodles.

With limited space for publication, we only show here the Venn graph of the doodle in which all of the possible pairs of vertices are connected without violating Definition 1. This is in Figure 7. Figure 8 shows the Venn graph of the doily. Figure 10 shows the 11-doily which is created by this Venn graph. Finally, Figure 11 shows the Jordan curve. The rotation of this curve over $360/11$ degrees creates the non-simple 11-doily with 462 vertices. Similarly to the case $p = 7$ of [8], if we followed a step-by-step extension, then it would show how little one curve changes from one case to the next one.

In Figures 4 and 9 there are 2048 faces, 275 or 462 vertices, and 2048 or 2508 edges, respectively. It is difficult to show a decent drawing of such diagrams in a paper of limited size. Therefore, making the graphics clearer in Figures 5 and 10 we show $1/11$ th portion of Figures 4 and 9, respectively.

Table 1: A generator for a 11-doily with vertex sets $275, 286, \ldots, 462$.

Path 1	#
100 000 000 00	1
100 010 000 00	2
100 010 100 00	3
100 110 100 00	4
100 111 100 00	5
101 111 100 00	6
101 111 101 00	7
101 111 111 00	8
101 111 111 10	9

Path 2	#
110 000 000 00	10
110 000 000 10	11
110 001 000 10	12
010 001 000 10	13
011 001 000 10	14
011 011 000 10	15
011 011 100 10	16
011 011 101 10	17
011 011 111 10	18
011 111 111 10	19
011 111 111 11	20

Path 3	#
110 001 010 10	21
100 001 010 10	22
100 001 011 10	23
101 001 011 10	24
101 011 011 10	25
101 011 011 11	26
001 011 011 11	27
001 111 011 11	28
001 111 011 01	29
001 111 111 01	30
001 110 111 01	31
011 110 111 01	32
011 110 111 11	33

Path 4	#
111 001 000 10	34
111 001 000 00	35
011 001 000 00	36
011 011 000 00	37
011 011 001 00	38
011 011 001 01	39
011 011 011 01	40
011 111 011 01	41

Aux Path 5	#
011 111 001 01	42

Path 6	#
101 001 000 00	43
101 001 010 00	44
101 101 010 00	45
101 101 011 00	46
001 101 011 00	47
001 101 011 01	48

Aux Path 7	#
011 101 011 00	49
011 101 011 01	50

Path 8	#
110 001 000 00	51
110 101 000 00	52
110 101 000 01	53
110 101 010 01	54
110 101 110 01	55
010 101 110 01	56
011 101 110 01	57
011 101 111 01	58
001 101 111 01	59

Aux Path 9	#
001 101 111 00	60

Path 10	#
110 011 000 00	61
110 011 100 00	62
110 011 101 00	63
110 011 111 00	64
111 011 111 00	65
011 011 111 00	66
011 111 111 00	67
001 111 111 00	68
001 110 111 00	69

Path 11	#
110 100 000 10	70
111 100 000 10	71
111 100 000 00	72
111 100 010 00	73
111 100 011 00	74
111 100 111 00	75
111 110 111 00	76
011 110 111 00	77

Path 12	#
110 100 000 00	78
110 100 010 00	79
110 100 011 00	80
100 100 011 00	81
000 100 011 00	82
000 100 111 00	83
000 110 111 00	84
000 110 111 01	85

Path 13	#
100 100 000 00	86
100 100 100 00	87
100 100 110 00	88
100 100 111 00	89
100 100 111 10	90
100 000 111 10	91
100 000 111 11	92
101 000 111 11	93
101 000 011 11	94
101 010 011 11	95
101 010 001 11	96
101 011 001 11	97
111 011 001 11	98
011 011 001 11	99
011 011 011 11	100
011 011 111 11	101

Aux Path 14	#
000 100 110 00	102

Path 15	#
101 000 000 00	103
101 010 000 00	104
101 010 001 00	105
101 010 001 10	106
001 010 001 10	107
001 010 001 11	108
001 011 001 11	109

Path 16	#	Aux Path 20	#
101 011 001 00	110	101 000 011 00	152
111 011 001 00	111	101 100 011 00	153
111 011 001 01	112	**Path 21**	**#**
111 011 000 01	113	101 010 010 00	154
110 011 000 01	114	101 010 010 10	155
110 011 010 01	115	101 010 110 10	156
110 001 010 01	116	001 010 110 10	157
110 001 110 01	117	001 011 110 10	158
110 001 110 11	118	001 001 110 10	159
010 001 110 11	119	000 001 110 10	160
010 101 110 11	120	000 001 110 11	161
011 101 110 11	121	000 101 110 11	162
011 101 111 11	122	**Aux Path 22**	**#**
Path 17	**#**	001 010 110 00	163
001 010 001 00	123	**Path 23**	**#**
001 011 001 00	124	101 110 011 00	164
001 011 001 10	125	100 110 011 00	165
001 111 001 10	126	110 110 011 00	166
001 111 101 10	127	110 110 111 00	167
001 111 100 10	128	010 110 111 00	168
011 111 100 10	129	010 110 110 00	169
010 111 100 10	130	010 110 110 10	170
010 111 100 11	131	010 100 110 10	171
010 101 100 11	132	010 100 110 11	172
Path 18	**#**	**Path 24**	**#**
101 000 001 00	133	111 000 000 00	173
101 100 001 00	134	111 000 100 00	174
101 100 101 00	135	111 000 100 10	175
111 100 101 00	136	111 000 110 10	176
111 100 100 00	137	011 000 110 10	177
110 100 100 00	138	011 000 110 11	178
110 100 100 10	139	010 000 110 11	179
010 100 100 10	140	**Path 25**	**#**
010 100 100 11	141	100 001 000 00	180
Path 19	**#**	100 001 100 00	181
101 010 011 00	142	110 001 100 00	182
001 010 011 00	143	110 001 110 00	183
001 010 111 00	144	010 001 110 00	184
000 010 111 00	145	010 001 110 10	185
000 011 111 00	146	010 101 110 10	186
010 011 111 00	147		
010 111 111 00	148		
010 101 111 00	149		
010 101 111 01	150		
010 101 111 11	151		

Table 2: The edge numbering of the Venn graph of a Venn graph of a doodle of a non-simple 11 -doily with 275 vertices.

P 1	#
1	
5	1
7	2
4	3
6	4
3	5
9	6
8	7
10	8
2	9
P 2	#
2	
10	10
6	11
1	12
3	13
5	14
7	15
9	16
8	17
4	18
11	19
1	20
P 3	#
8	
2	21
9	22
3	23
5	24
11	25
1	26
4	27
10	28
7	29
6	30
2	31
10	32
6	33

P 4	#
3	
10	34
1	35
5	36
9	37
11	38
8	39
4	40
2	41
XP 5	#
4	
8	42
P 6	#
2	
8	43
4	44
9	45
1	46
11	47
5	48
XP 7	#
2	
11	49
5	50
P 8	#
10	
4	51
11	52
8	53
7	54
1	55
3	56
9	57
2	58
5	59

XP 9	#
7	
11	60
P 10	#
5	
7	61
9	62
8	63
3	64
1	65
4	66
2	67
6	68
11	69
P 11	#
4	
3	70
10	71
8	72
9	73
7	74
5	75
1	76
2	77
P 12	#
4	
8	78
9	79
2	80
1	81
7	82
5	83
11	84
3	85

P 13	#
4	
7	86
8	87
9	88
10	89
4	90
11	91
3	92
7	93
5	94
8	95
6	96
2	97
1	98
8	99
7	100
4	101
XP 14	#
1	
9	102
P 15	#
3	
5	103
9	104
10	105
1	106
11	107
6	108
2	109

P 16	#
6	
2	110
11	111
9	112
3	113
8	114
5	115
7	116
10	117
1	118
4	119
3	120
9	121
5	122
P 17	#
1	
6	123
10	124
4	125
7	126
9	127
2	128
3	129
11	130
5	131
8	132
P 18	#
5	
4	133
7	134
2	135
9	136
3	137
10	138
1	139
11	140
6	141

P 19	#
8	
1	142
7	143
3	144
6	145
2	146
4	147
5	148
11	149
10	150
9	151
XP 20	#
5	
4	152
8	153
P 21	#
9	
10	154
7	155
1	156
6	157
5	158
3	159
11	160
4	161
2	162
XP 22	#
9	
10	163

P 23	#
4	
3	164
2	165
7	166
1	167
9	168
10	169
5	170
11	171
6	172
P 24	#
2	
7	173
10	174
8	175
1	176
11	177
3	178
4	179
P 25	#
6	
7	180
2	181
8	182
1	183
10	184
4	185
11	186

Table 3: The new edges added to the Venn graph of the doodle of Figure 1. The pair of numbers in the parentheses in the second column are from Table 1. The edge numbers are in the fifth column.

#	edge	code	code	e #
1	(30,68)	001 111 111 01	001 111 111 00	11
2	(35,51)	111 001 000 00	110 001 000 00	3
3	(48,50)	001 101 011 01	011 101 011 01	2
4	(48,59)	001 101 011 01	001 101 111 01	7
5	(67,77)	011 111 111 00	011 110 111 00	6
6	(69,84)	001 110 111 00	000 110 111 00	3
7	(70,78)	110 100 000 10	110 100 000 00	10
8	(74,80)	111 100 011 00	110 100 011 00	3
9	(78,86)	110 100 000 00	100 100 000 00	2
10	(83,89)	000 100 111 00	100 100 111 00	1
11	(96,106)	101 010 001 11	101 010 001 10	11
12	(96,108)	101 010 001 11	001 010 001 11	1
13	(97,109)	101 011 001 11	001 011 001 11	1
14	(110,124)	101 011 001 00	001 011 001 00	1
15	(133,152)	101 000 001 00	101 000 011 00	8
16	(184,190)	010 001 110 00	010 011 110 00	5
17	(186,192)	010 101 110 10	010 111 110 10	5

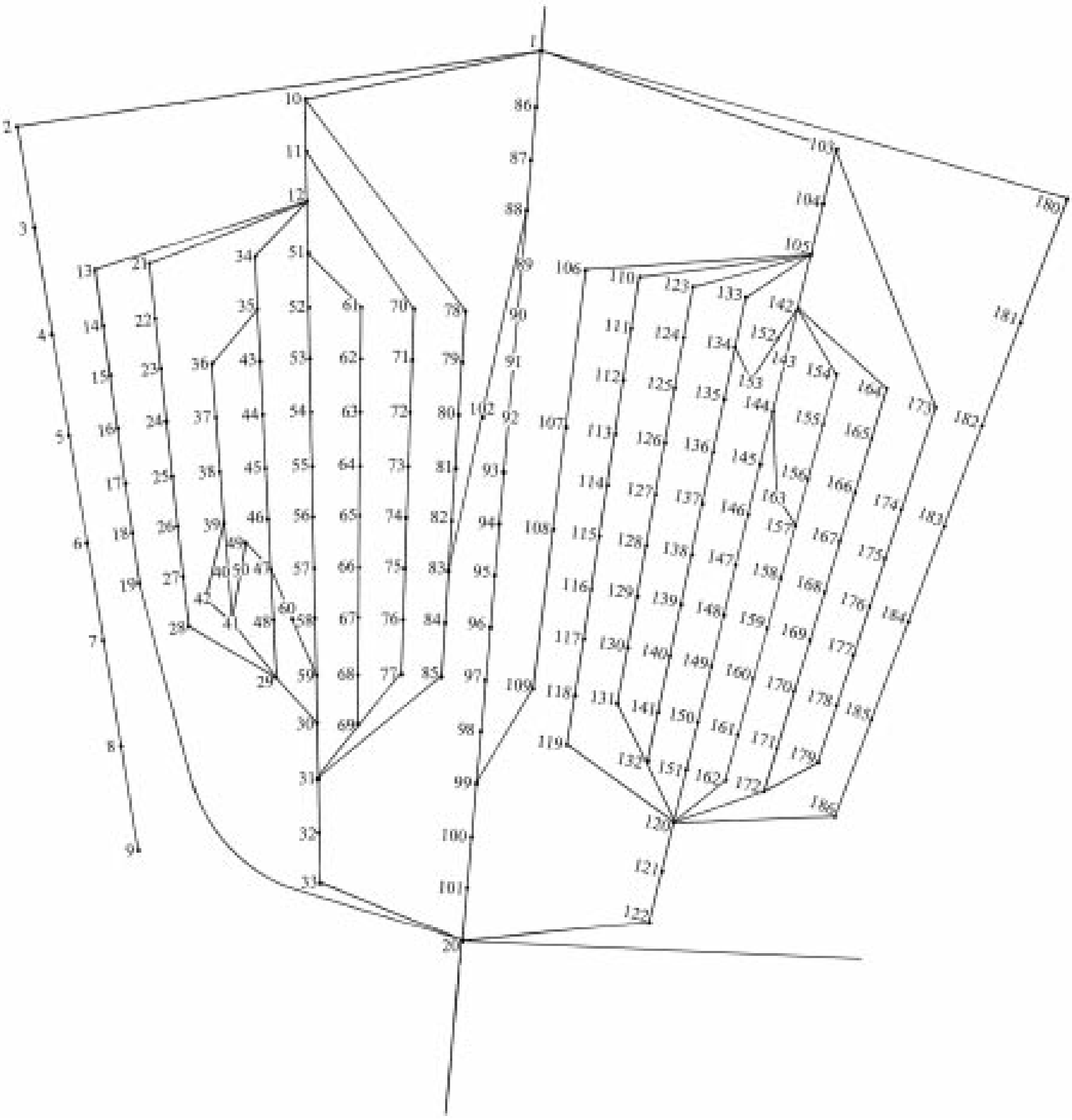

Figure 1: The Venn graph of a doodle of a non-simple 11 -doily with 275 vertices. The numbers are from Table 1. The corresponding binary codes can also be found in Table 1.

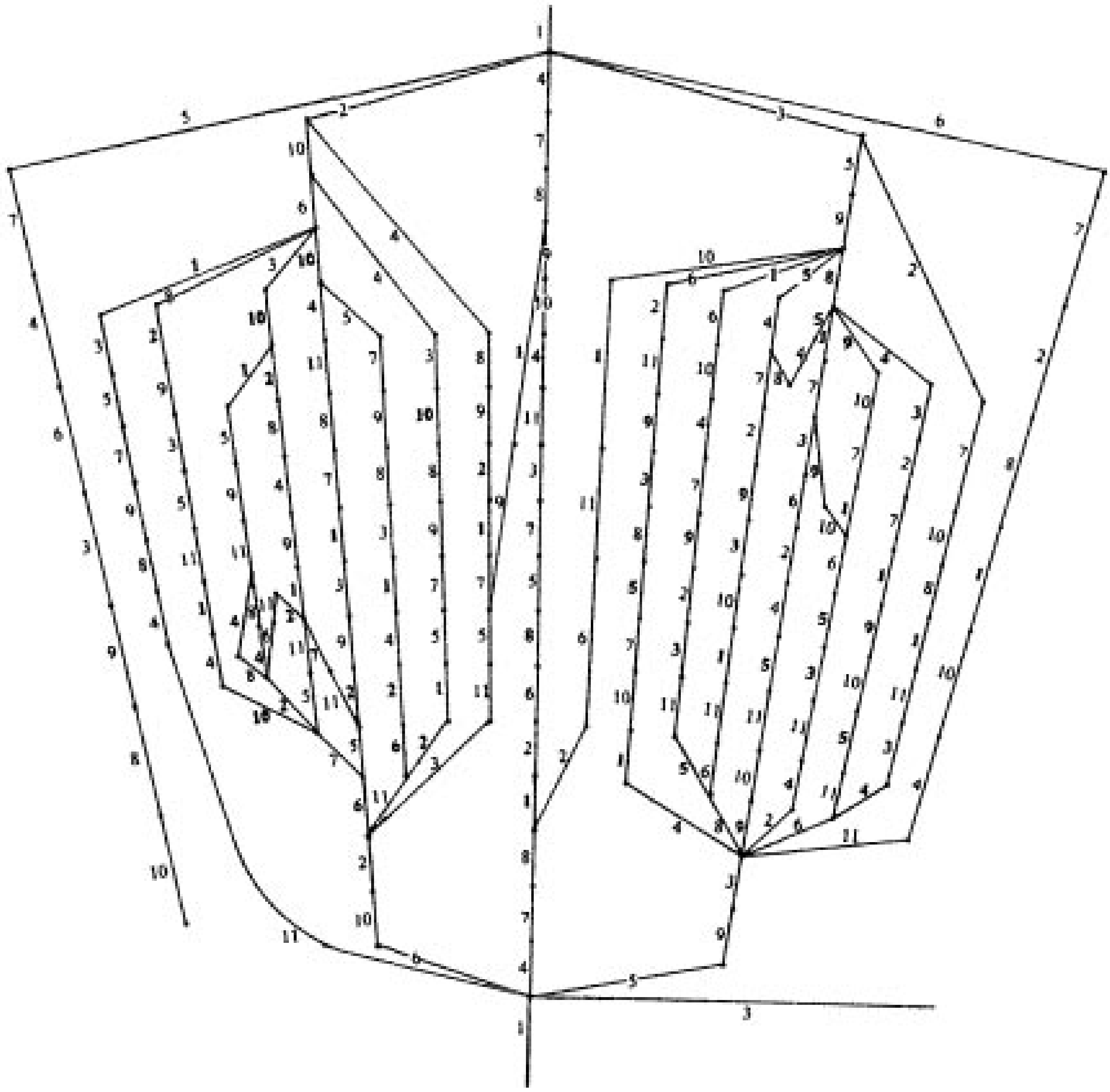

Figure 2: The edge numbering of the Venn graph of a doodle of a non-simple 11 -doily with 275 vertices. The numbers are from Table 2.

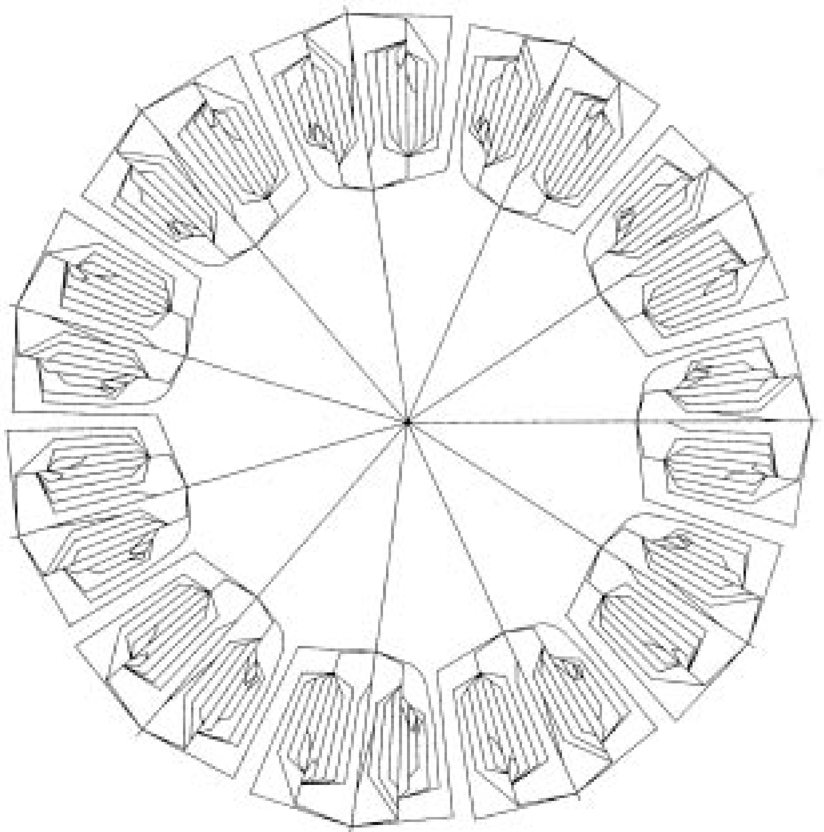

Figure 3: The Venn graph of a non-simple 11 -doily with 275 vertices.

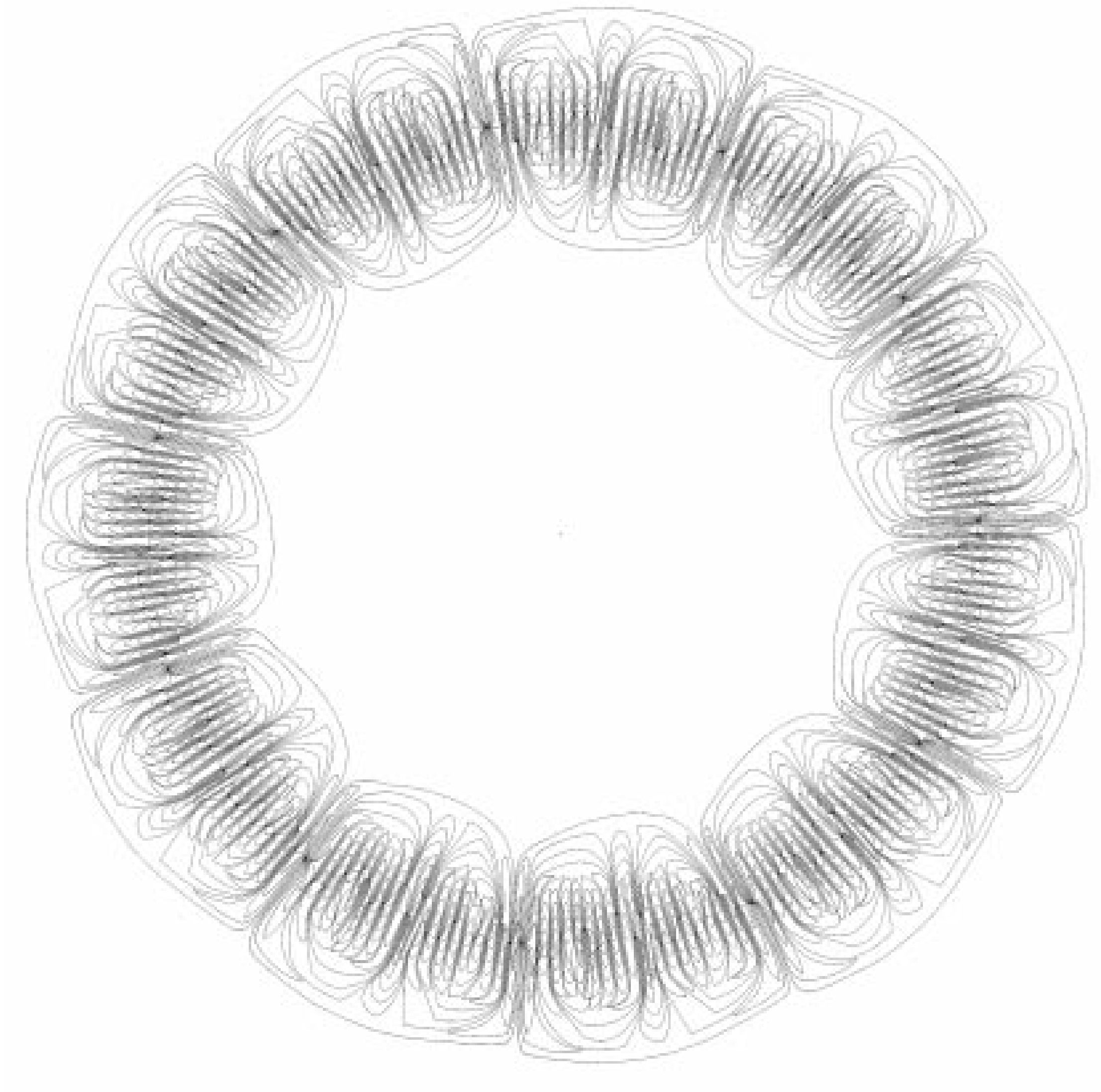

Figure 4: The non-simple 11-doily with 275 vertices.

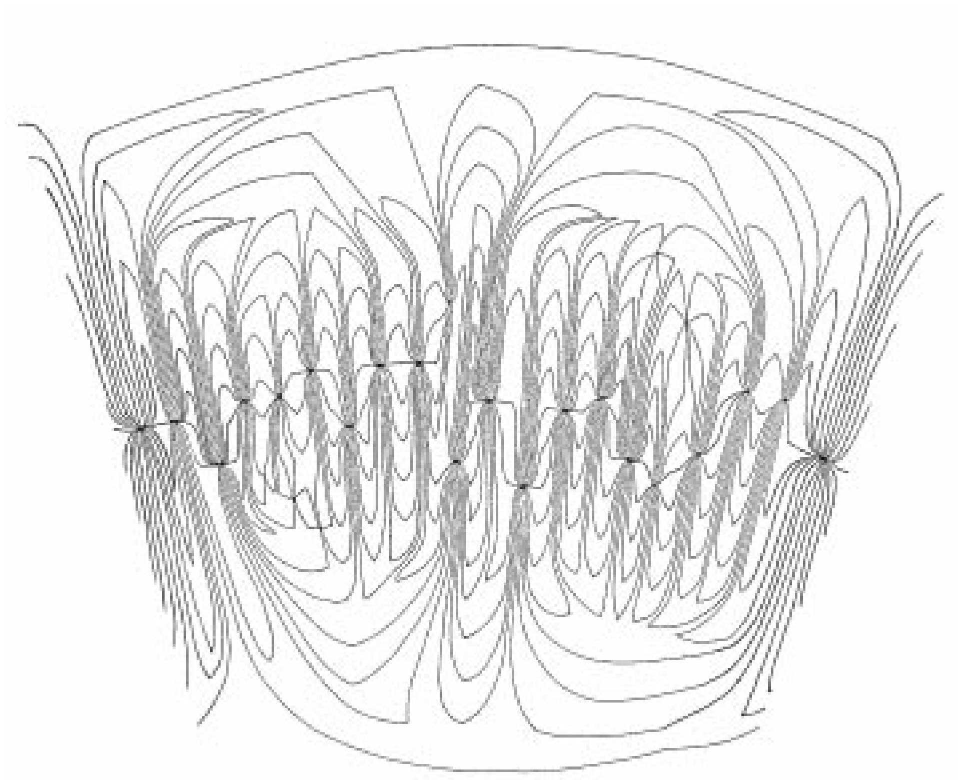

Figure 5: $1/11$ th of a non-simple 11-doily with 275 vertices.

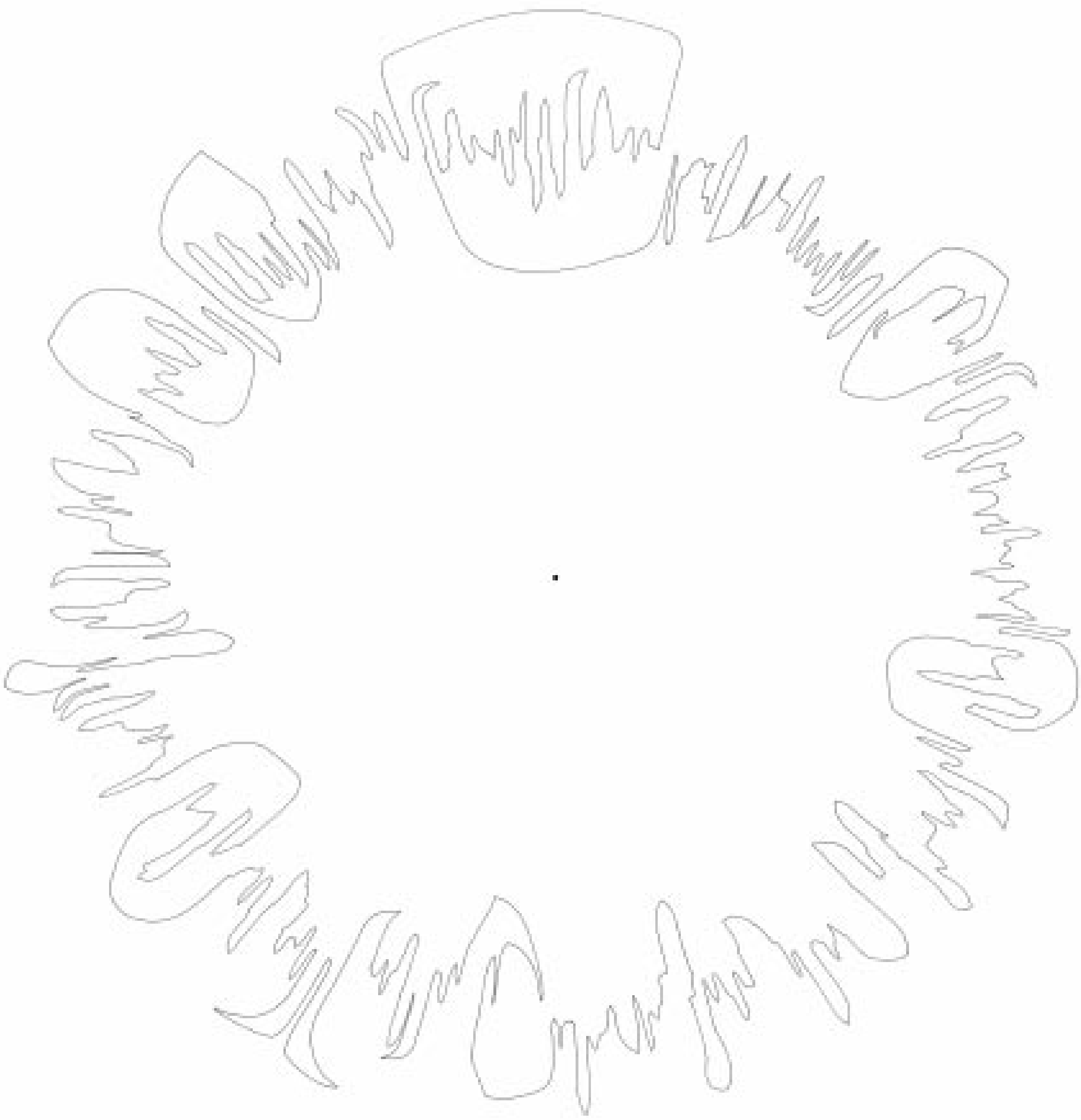

Figure 6: The Jordan curve, and the center of rotation. The rotation of this curve 11 times over $360/11$ degrees creates the non-simple 11-doily with 275 vertices in Figure 4.

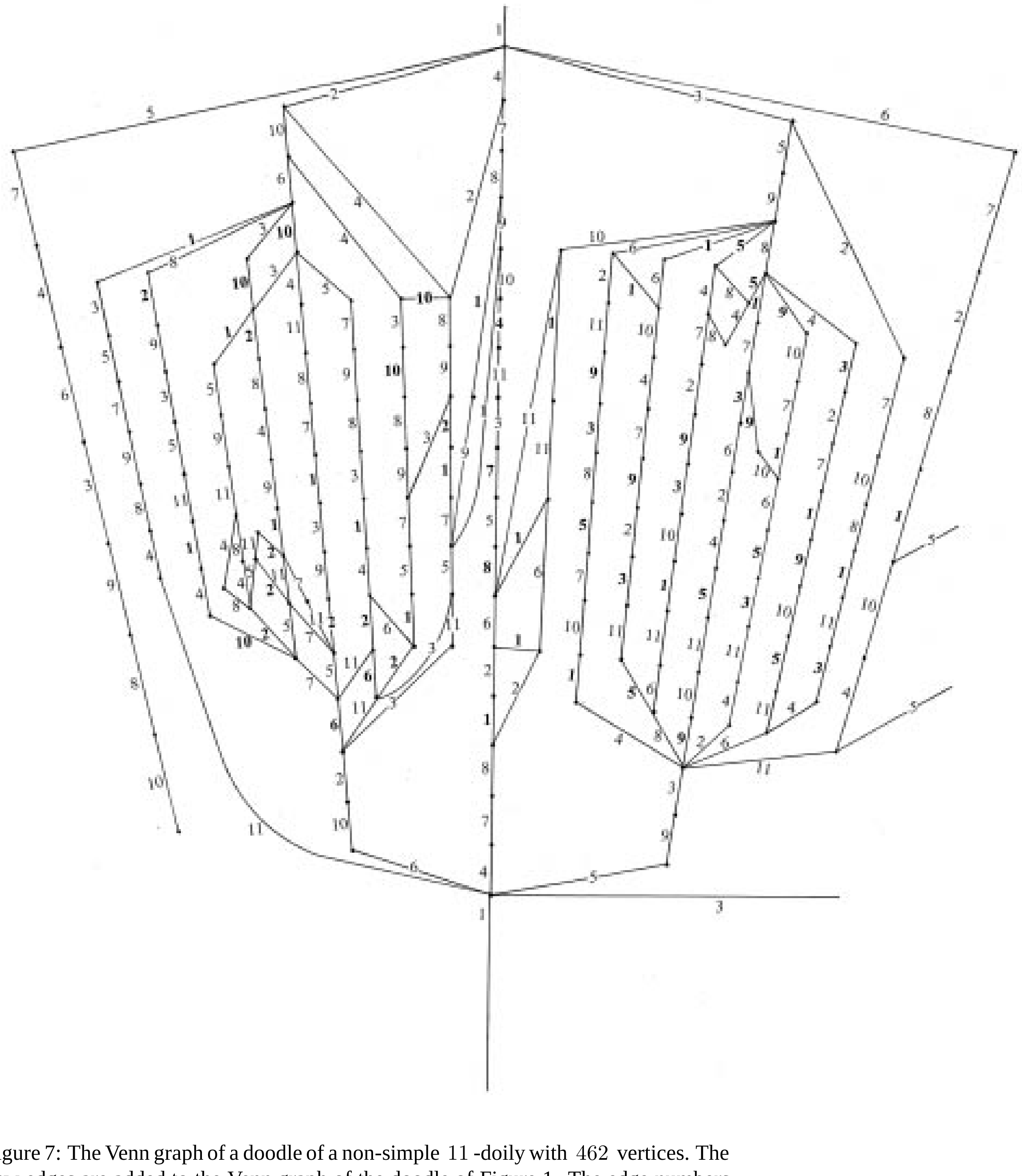

Figure 7: The Venn graph of a doodle of a non-simple 11 -doily with 462 vertices. The new edges are added to the Venn graph of the doodle of Figure 1. The edge numbers are from Tables 2 and 3.

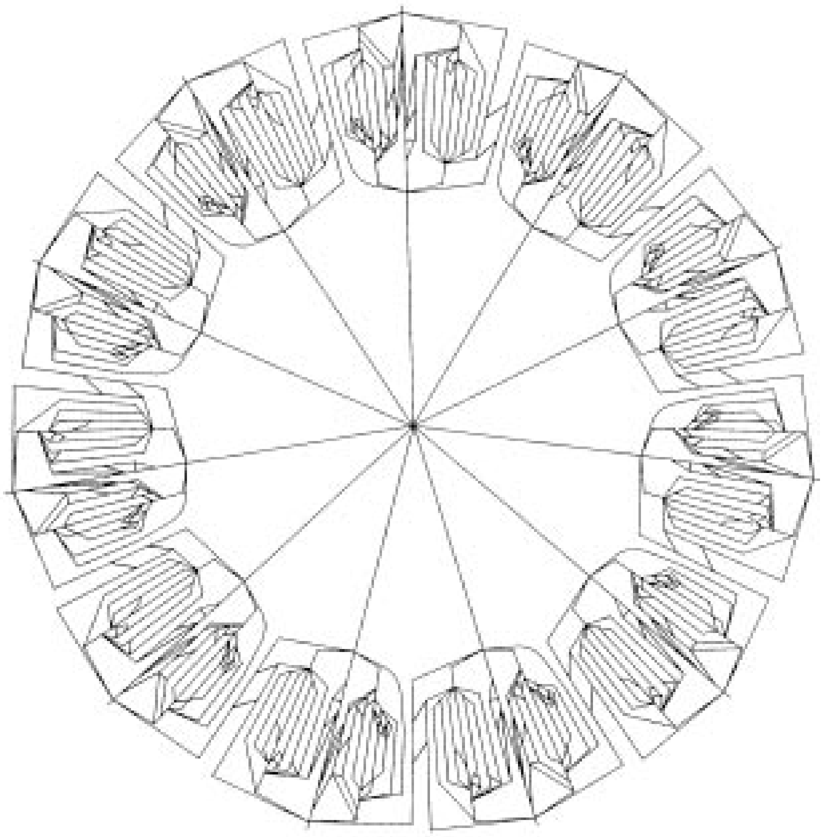

Figure 8: The Venn graph of a non-simple 11 -doily with 462 vertices.

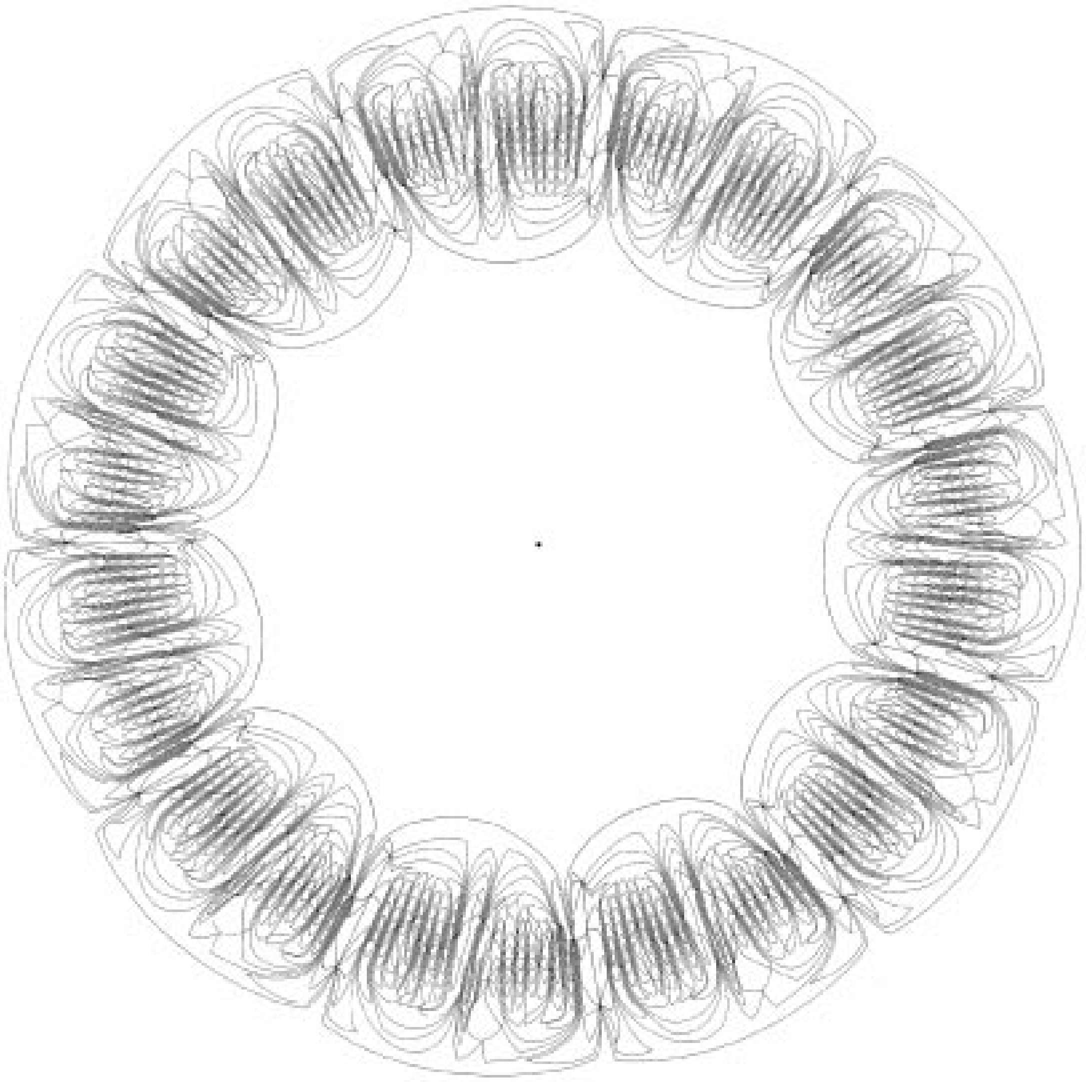

Figure 9: The non-simple 11-doily with 462 vertices.

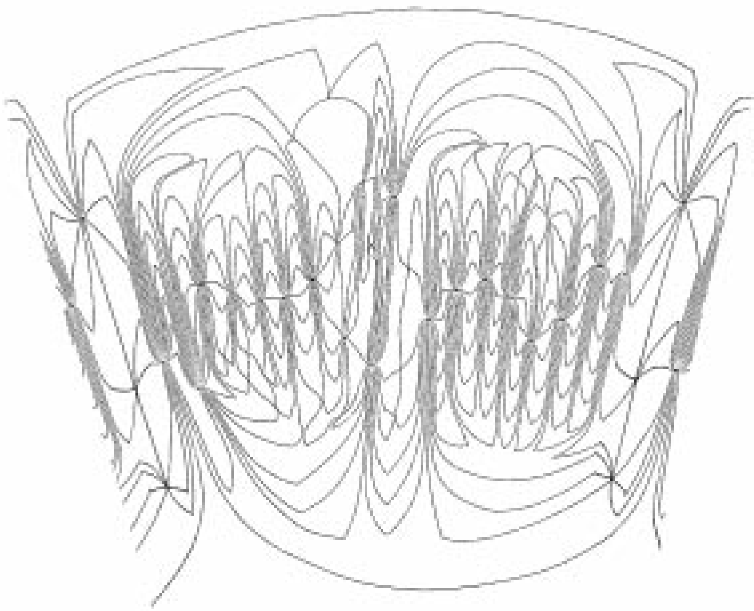

Figure 10: $1/11$ th of a non-simple 11 -doily with 462 vertices.

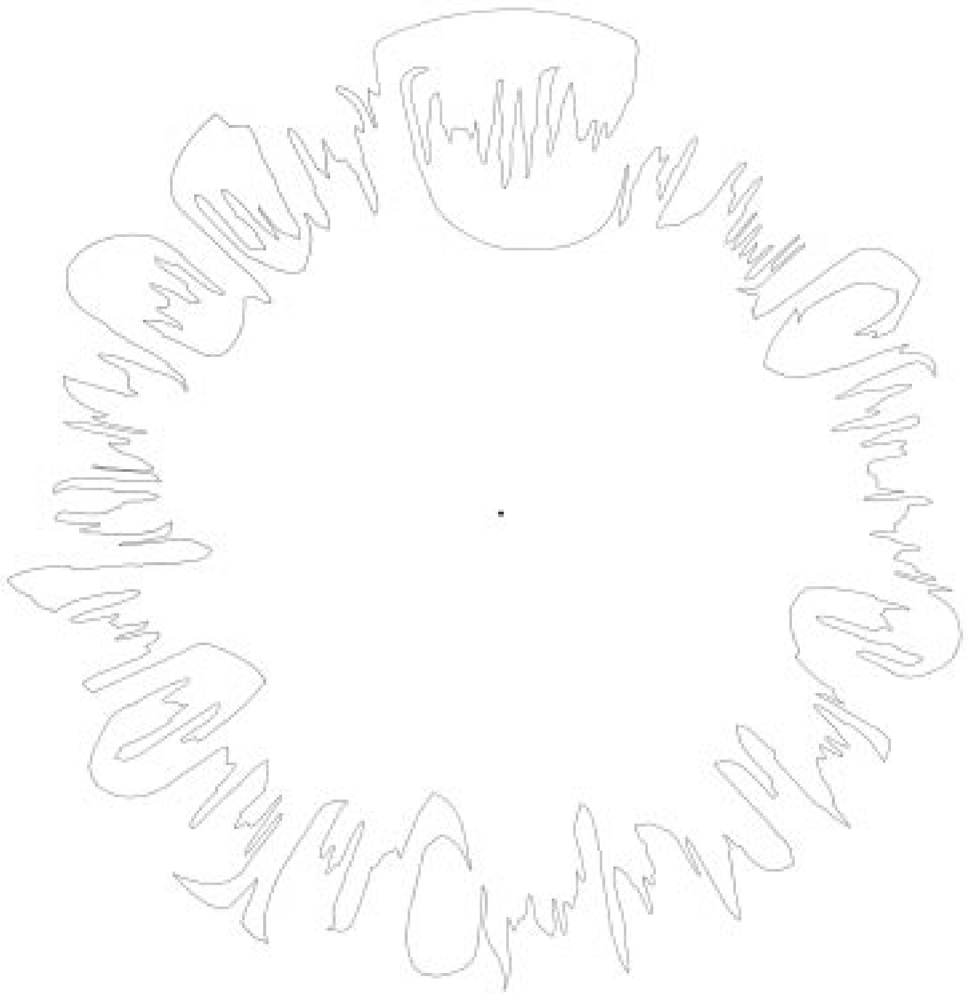

Figure 11: The Jordan curve, and the center of rotation. The rotation of this curve 11 times over $360/11$ degrees creates the non-simple 11-doily with 462 vertices in Figure 10.

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